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Ferguson-Dirichlet 過程的隨機函數及其貝氏應用

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中文摘要

具有類似傳統特徵函數的單維度c-特徵函數已被證實為相當有用的另一種特徵函數，特別是在傳統特徵函數難以解決的問題上，這種新的特徵函數常可顯示它的實用性，同時Jiang, Dickey, and Kuo (2004)將這種單維度的c-特徵函數更近一步推廣到多變量的c-特徵函數。很可惜地，由於缺少反演公式，它的實用性受到一些限制，特別是在尋找Ferguson-Dirichlet過程的隨機函數方面。本文將提供單維度c-特徵函數的反演公式，同時也將提供例子以說明反演公式的使用及實用性。

關鍵詞：反演公式，單維度c-特徵函數，Ferguson-Dirichlet過程，隨機函數

Abstract

It was shown that the univariate c-characteristic function, which has many properties similar to those of traditional characteristic function, is a useful alternative characteristic function, especially when the traditional characteristic function is hard to deal with. The c-characteristic function was further extended to multivariate c-characteristic function by Jiang, Dickey, and Kuo (2004). Unfortunately, its usefulness has been some what limited by the lack of the inverse formula, especially on finding the random functionals of the Ferguson-Dirichlet process. In this paper, we shall give an inverse formula of the univariate c-characteristic function. An application shall also be given to illustrate this inverse formula and to show its usefulness.

Keywords: inverse formula, univariate c-characteristic function, Ferguson-Dirichlet process, random functional

1 Introduction

The c-characteristic function, which was first given by Jiang (1988), has been shown to have the similar properties (eg. uniqueness and convergence) as those of the traditional characteristic function. This univariate c-characteristic function is therefore a very useful alternative to the traditional characteristic function, especially for those cases that are hard to deal with traditional characteristic function. For example, the Dirichlet distribution has a very complicated traditional characteristic function. However, it has a very simple c-characteristic function. Some applications of this univariate c-characteristic function can be seen in Jiang (1988, 1991), Jiang, Dickey, and Kuo (2004) further extend it to the multivariate c-characteristic function. Since the univariate c-characteristic function has no inverse formula so far, its utility is somewhat limited. In this paper, we shall study a tool, which is an inverse formula of the univariate c-characteristic function, to find the random functionals of the Ferguson-Dirichlet process.

We first give some notations and properties of the univariate c-characteristic function in Section 2. A derivation of an inversion formula is given in Section 3. An example is also given to illustrate this new inversion formula in this section. Finally, conclusions are given in Section 4.

2 Definition and properties of univariate c-characteristic function

Before reviewing the properties of the univariate c-characteristic function, we state its definition first.

Definition 1 (Jiang, 1988) *For any random variable X with support in $[-a, a]$, we define the c-characteristic functiom of X as*

$$g(t; X, c) = E[(1 - itX)^{-c}],\ \ |t| < a^{-1},\ \ c > 0,$$

where a is a positive real number.

With the above definition, it can be shown that the c-characteristic function of the weighted average $W = \sum_{i=1}^{I} x_i u_i$ of a Dirichlet vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_I)$ with parameters $\boldsymbol{b} = (b_1, b_2, \ldots, b_I)$ (see Jiang, 1988, for notations) has the following simple expression

$$g(t; W, c) = \prod_{i=1}^{I} (1 - itx_i)^{-b_i},$$

where $c = \sum_{i=1}^{I} b_i$ and $x_1, x_2, \ldots, x_I$ are real numbers. However the traditional characteristic function of W is very complicated. For the rest of this section, we shall present two more properties of the c-characteristic function.

The following property says that there is one-to-one correspondence between X and $g(t; X, c)$ for any positive real number c.

Lemma 2 (Jiang, 1988) *For any random variables X, Y with supports in $[-a, a]$ and any positive real number c, if for all $|t| < a^{-1}$*

$$g(t; X, c) = g(t; Y, c),$$

then X and Y have the same distribution.

The following convergence theorem is analogous to that for the traditional characteristic function.

Theorem 3 (Jiang, 1988) *Given $c > 0$, assume that random variables $X, X_1, X_2, \dots$ (with supports in $[-a, a]$) correspond to $g(t; , X, c)$, $g(t; X_1, c)$, $g(t; X_2, c), \dots$, respectively. If for all $|t| < a^{-1}$*

$$g(t; X_n, c) \to g(t; X, c), \text{ as } n \to \infty,$$

then X_n converges in distribution to X.

If there is a random variable X with corresponding c-characteristic function, $g(t; X, c)$ and $\lim_{n\to\infty} g(t; X_n, c) = g(t; X, c)$, for all $|t| < a^{-1}$, by Lemma 2 and Theorem 3, then X_n converges in distribution to X. However, knowing c-characteristic function $g(t; X, c)$, it is not easy to guess what the random variable X is. In order to provide the limiting distribution, we shall present a method to find the probability density function of X when its c-characteristic function is available in the next section.

3 Inversion formulas

First, we give a version of the inversion formula of the Stieltjes transformation given by Widder (1946).

Theorem 4 (Widder (1946), p. 340) *If $f(x) \in L[0, r]$ for every positive r, and is such that the integral*

$$F(s) = \int_0^\infty \frac{f(x)}{x + s}\, dx$$

converges, then

$$\lim_{\epsilon\to 0^+} \frac{F(-x - i\epsilon) - F(-x + i\epsilon)}{2\pi i} = \frac{f(x^+) + f(x^-)}{2}$$

for any positive x at which $f(x^+)$ and $f(x^-)$ exist.

We give another version of inverse formula for the Stieltjes transform in the following corollary.

Corollary 5 *If $f(x) \in L[-a, r]$ for positive a and every positive r, and is such that the integral*

$$F(s) = \int_{-a}^\infty \frac{f(x)}{x + s}\, dx$$

converges, then

$$\lim_{\epsilon\to 0^+} \frac{F(-x - i\epsilon) - F(-x + i\epsilon)}{2\pi i} = \frac{f(x^+) + f(x^-)}{2}$$

for any $x \geq -a$ at which $f(x^+)$ and $f(x^-)$ exist.

Proof. To distinguish the notations between Theorem 4 and Corollary 6, we shall temporarily use subscript "1" on x, f, s, and F in this corollary. For example, notation $f(x)$ is replaced by $f_1(x_1)$ for now. By letting $x_1 = x - a$, $f_1(x_1) = f(x)$, $s_1 = a + s$, and $F_1(s_1) = F(s)$ in Theorem 4 and using the change of variables, we have

$$\begin{aligned}
&f_1(x_1) = f(x_1 + a) \in L[-a, r],\\
&F_1(s) = F(s_1 - a) = \int_0^\infty \frac{f(x)}{x + s_1 - a}\, dx = \int_{-a}^\infty \frac{f_1(x_1)}{x_1 + a + s_1 - a}\, dx_1,\\
&F_1(-x_1 - i\epsilon) = F(-x_1 - i\epsilon - a) = F(-x - i\epsilon),\\
&F_1(-x_1 + i\epsilon) = F(-x_1 + i\epsilon - a) = F(-x + i\epsilon),\\
&f_1(x_1^+) = f(x^+), \text{ and}\\
&f_1(x_1^-) = f(x^-).
\end{aligned}$$

The proof is then completed if we drop all the subscript "1" in the above notations. ■

We then have the following inversion formula for our c-characteristic function when $c = 1$.

Theorem 6 *If* $g(t; X, c) = \int_{-a}^a \frac{1}{(1-itx)^c} f(x)\, dx$ *and converges, then*

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left[\frac{g\left(\frac{i}{-x-i\epsilon}; X, 1\right)}{-x - i\epsilon} - \frac{g\left(\frac{i}{-x+i\epsilon}; X, 1\right)}{-x + i\epsilon} \right] = \frac{f(x^+) + f(x^-)}{2},$$

for any $x \in [-a, a]$ *at which* $f(x^+)$ *and* $f(x^-)$ *exist.*

Proof. Since

$$g\left(\frac{i}{-x - i\epsilon}; X, 1\right) = -(x + i\epsilon) \int_{-a}^a \frac{1}{-x - i\epsilon + y} f(y)\, dy$$

and

$$g\left(\frac{i}{-x + i\epsilon}; X, 1\right) = -(x - i\epsilon) \int_{-a}^a \frac{1}{-x + i\epsilon + y} f(y)\, dy,$$

then, by Corollary 5, we have

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \left[\frac{g\left(\frac{i}{-x-i\epsilon}; X, 1\right)}{-x - i\epsilon} - \frac{g\left(\frac{i}{-x+i\epsilon}; X, 1\right)}{-x + i\epsilon} \right] = \frac{f(x^+) + f(x^-)}{2},$$

for any $x \in [-a, a]$ at which $f(x^+)$ and $f(x^-)$ exist. ■

Let X be a beta distribution with parameters (b_1, b_2), denoted by $X \sim Beta(b_1, b_2)$, then p.d.f of X can be expressed as $f(x) = \frac{1}{B(b_1, b_2)} x^{b_1 - 1}(1 - x)^{b_2 - 1}$, where $0 < x < 1$ and $B(b_1, b_2) = \Gamma(b_1)\Gamma(b_2)/\Gamma(b_1 + b_2)$. In the next example, we shall restrict the beta distribution to the case that $c = b_1 + b_2 = 1$ and use it to illustrate the above inversion formula.

Example 1 Let $X \sim Beta(b_1, b_2)$, where $b_1 + b_2 = 1$. Consider a new random variable $Y = w_1 X + w_2(1 - X)$, where $w_1 < w_2$, then by Corollary 3.3 of Jiang (1988) the c-characteristic function of Y is $g(t; Y, 1) = (1 - itw_1)^{-b_1}(1 - itw_2)^{-b_2}$, where the range of Y is

(w_1, w_2). First, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[\frac{g(\frac{i}{-y-i\epsilon};Y,1)}{-y-i\epsilon} - \frac{g(\frac{i}{-y+i\epsilon};Y,1)}{-y+i\epsilon}\right] \\
=& \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[\frac{(1-i\frac{i}{-y-i\epsilon}w_1)^{-b_1}(1-i\frac{i}{-y-i\epsilon}w_2)^{-b_2}}{-y-i\epsilon} - \frac{(1-i\frac{i}{-y+i\epsilon}w_1)^{-b_1}(1-i\frac{i}{-y+i\epsilon}w_2)^{-b_2}}{-y+i\epsilon}\right] \\
=& \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[\frac{1}{(-y+w_1-i\epsilon)^{b_1}(-y+w_2-i\epsilon)^{b_2}} - \frac{1}{(-y+w_1+i\epsilon)^{b_1}(-y+w_2+i\epsilon)^{b_2}}\right] \quad (3.1)
\end{aligned}$$

Since $w_1 < y < w_2$, hence $-y+w_1 < 0$ and $-y+w_2 > 0$. Next, we have

$$\begin{aligned}
(-y+w_1-i\epsilon)^{-b_1} &= e^{-b_1 ln(-y+w_1-i\epsilon)} \\
&= e^{-b_1[ln|-y+w_1-i\epsilon| + i\arg(-y+w_1-i\epsilon)]} \\
&= |-y+w_1-i\epsilon|^{-b_1} e^{-ib_1\arg(-y+w_1-i\epsilon)}.
\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to 0^+}(-y+w_1-i\epsilon)^{-b_1} = (y-w_1)^{-b_1}e^{ib_1\pi}. \quad (3.2)$$

Similarly,

$$\begin{aligned}
\lim_{\epsilon\to 0^+}(-y+w_2-i\epsilon)^{-b_2} &= (w_2-y)^{-b_2} \quad (3.3)\\
\lim_{\epsilon\to 0^+}(-y+w_1+i\epsilon)^{-b_1} &= (y-w_1)^{-b_1}e^{-ib_1\pi} \quad (3.4)
\end{aligned}$$

and

$$\lim_{\epsilon\to 0^+}(-y+w_2+i\epsilon)^{-b_2} = (w_2-y)^{-b_2}. \quad (3.5)$$

By Theorem 6 and equations (3.1)-(3.5), the p.d.f. of Y can be expressed as

$$\begin{aligned}
g(y) &= \frac{1}{2\pi i}\left[(y-w_1)^{-b_1}e^{ib_1\pi}(w_2-y)^{-b_2} - (y-w_1)^{-b_1}e^{-ib_1\pi}(w_2-y)^{-b_2}\right] \\
&= \frac{1}{2\pi i}(y-w_1)^{-b_1}(w_2-y)^{-b_2}\left[e^{ib_1\pi} - e^{-ib_1\pi}\right] \\
&= \frac{1}{2\pi i}(y-w_1)^{-b_1}(w_2-y)^{-b_2}\left[2i\sin(b_1\pi)\right] \\
&= \frac{1}{\pi}(y-w_1)^{-b_1}(w_2-y)^{-b_2}\left[2i\frac{\pi}{\Gamma(b_1)\Gamma(b_2)}\right] \\
&= \frac{1}{\Gamma(b_1)\Gamma(b_2)}(y-w_1)^{-b_1}(w_2-y)^{-b_2}
\end{aligned}$$

where $w_1 < y < w_2$. Note that $\sin(b_1\pi) = \frac{\pi}{\Gamma(b_1)\Gamma(b_2)}$ can be obtain from theorem 3.9-1 of Carlson (1977, P48).

4 Conclusions

The c-characteristic function is an important alternative to the traditional characteristic function. However, its utility has been somewhat limited due to the lack of its inverse formula. In this paper, we give an important inverse formula. In addition, we give an example to illustrate this formula. In the future research, we shall use this inverse formula to study random moments of the Ferguson-Dirichlet process that are not currently available.

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