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Modified Logistic Model for Mortality Forecasting and the Application of Mortality-Linked Securities

Yawen Hwang and Hong-Chih Huang

Abstract

This article proposes the modified logistic mortality model as a potential moderation of current models. The fitting and forecasting effects of the proposed modified logistic mortality model are efficient for the mortality data of the United States, Japan, England and Wales, and the modified logistic model provides the best forecasting for persons older than age 30 years under MAPE criterion. In addition, this study considers how to use longevity bonds to manage longevity risk. We apply the proposed modified logistic mortality model to price the bond, and to improve the attractiveness of that bond, we design it to encompass more than one tranche, according to the concept of collateral debt obligation. This design offers investors more choices pertaining to their different risk preferences.

KEYWORDS: mortality model, logistic model, longevity bond, collateral debt obligation

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1. Introduction

Researchers keep trying to develop mortality models that will fit and accurately forecast human mortality rates. In the 1990s, so-called stochastic mortality models incorporated age and year effects, such as the famous Carter and Lee (1992) model (Lee-Carter model) that relies on the central death rate to fit and forecast mortality data for the United States. Other studies extend or apply their model, such as Wilmoth (1993), Lee (2000), Lee and Miller (2001), Booth et al. (2002), Renshaw and Haberman (2003a, 2006), Koissi et al. (2006), and Koissi and Shapiro (2006). In England, many researchers rely on the reduction factor model derived by the Continuous Mortality Investigation Bureau (CMI Committee, 1999) (e.g. Renshaw and Haberman, 2000, 2003b). The logistic model also appears to perform well for fitting mortality rates (Thatcher, 1999; Bongaarts, 2005; Jarner and Kryger, 2011); thus, Cairns et al. (2006) derive a logistic model, the two-factor model (CBD model), to map the mortality curve for United Kingdom citizens over age 60 years.

More recent research also notes the importance of cohort effects (e.g. Currie, 2006; Renshaw and Haberman, 2006). In Cairns et al.'s (2009) review of mortality models, they incorporate the cohort effect to test the parameter stability; through an examination of eight separate models, they show that the extended CBD model offers the best fit for data from a combined sample from England and Wales, whereas Renshaw and Haberman's (2006) model fits best for United States data. Both models incorporate cohort effects. However, the numbers of parameters are unavoidable concerns for mortality models that include the cohort effect. For example, if we were to extend the Lee-Carter model to consider the cohort effect and have access to mortality data for persons between the ages of 30 and 89 years during 1941–2000, we would need to fit 119 more parameters than appeared in the original model. In this study, we propose some methods to modify the logistic mortality model, instead. These revisions work as well as commonly used models, such that they provide insurers and researchers with another viable choice for this type of modeling.

Death rates and improvements in mortality remain highly uncertain – as Blake et al. (2006) assert, longevity risk is still among the greatest sources of risk for life insurance companies and pension funds. Accurate pricing to manage mortality risk is not sufficient for insurers. Therefore, Blake and Burrows (2001) derive the concept of a longevity bond (LB) and suggest that the government is the best issuer of these. Dowd (2003) echo this research and present situations in which the LB instrument works well for insurers. Lin and Cox (2005) also study two types of mortality-based securities, as well as mortality bonds and swaps, and they use a Wang transformation to price them. Securitization of mortality risks has a long duration, high capacity, and possibly a low cost. Cowley and Cummins (2005) further reveal that securitization may increase a firm's value by reducing transaction costs, agency costs, informational asymmetries, taxation and regulation. When Cairns et al. (2006) used the CBD model to price mortality-linked financial instruments, they discovered that the premium increases with both the term and the initial age of the reference cohort, although the reference cohort's initial age is more important for determining the premium than is the bond's maturity. In addition, Blake et al. (2006) introduced five types of LBs: longevity zeros, survivor bonds, principal-at-risk longevity bonds, inverse longevity bonds, and collateralized longevity obligations. Whereas Cox et al. (2006) used multivariate exponential tilting to price Swiss mortality bonds, Denuit et al. (2007) applied the Lee-Carter model and Wang transformation to price survivor bond. These investigations focus on how to apply mathematical methods to price mortality securities.

A LB with a maturity of 25 years was first issued in November 2004 by the European Investment Bank (EIB) and BNP Paribas. This EIB/BNP LB was designed to help pension plans hedge their exposure to longevity risk, although it was withdrawn later due to its unsuccessful design (Blake et al., 2006). Despite a design flaw, a LB suggests a potentially effective way to hedge longevity risk, and a more attractive design might induce greater popularity. Therefore, we attempt to design a more attractive LB and to investigate its appropriate pricing and cash flow. We employ collateral debt obligation (CDO) structures to make the LB more attractive to investors, and then we investigate the risk of a special purpose vehicle (SPV).

In Section 2, we discuss the modification of the logistic mortality model and compare the fitting and forecasting quality of our proposed model with other commonly used mortality models. In Section 3, we introduce our new LB and price it. Next, we undertake brief net present value (NPV) tests of the SPV, including some analyses of its risks. We conclude in Section 4 with some implications and discussion of the results.

2. Modification of Logistic Mortality Model

The basic form of the logistic mortality model, as shown in Equation (1), includes two parts: the senescent death rate, a fraction that increases with age x; and the background death rate, $\gamma(t)$, which represents the influence of year t:

$$q_{x,t} = \frac{\alpha(t)e^{\beta(t)x}}{1+\alpha(t)e^{\beta(t)x}} + \gamma(t)\,. \qquad (1)$$

2.1. Proposed Modified Logistic Mortality Model

According to mortality data, we find that improvements of mortality rates occur at all ages, but the improvement ratios differ for different ages. However, the logistic mortality model proposed by Bongaarts (2005) is not able to show the effects of different mortality improvement for different ages. Therefore, we propose the segment approach to extend the original logistic model to improve this effect. According to our numerical results, the slope parameter $\beta(t)$ is nearly constant, which is consistent with the results of Bongaarts (2005). Therefore, we follow his work to set $\beta(t)$ as a fixed constant parameter. In addition, we set the background death rate as the parameter with age as it might be related more reasonably to age. Thus, we propose the following model and call it the modified logistic (beta) mortality model:

$$q_{x,t} = \frac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta x}} + \gamma(x)\,, \qquad (2)$$

where $\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } x < seg1 \\ \alpha_2(t) & \text{if } x \geq seg1 \end{cases}$, and $\beta = \begin{cases} \beta_1 & \text{if } x < seg2 \\ \beta_2 & \text{if } x \geq seg2 \end{cases}$.

2.2. Numerical Analysis

In this section, we analyze the goodness-of-fit data for the modified logistic (beta) model and compare the six mortality models in Table 1. The M1 model is the Lee-Carter model, and it is the most commonly used. The M2 model is the reduction factor model proposed by CMI Committee. The M3 model is the CBD model introduced by Cairns et al. (2006), who propose that this model is suitable for older ages. The above three models are dynamic mortality models related to age and year. Moreover, we also consider the M7 model (Cairns et al., 2009), which is an age-year-cohort mortality model and we call it the M4 model. The M5 model is the original logistic model in Bongaarts (2005), and the M6 model is our proposed modified logistic (beta) model.

2.2.1. Measurements of Fitting and Forecasting Results

In this paper, we apply the mean absolute percentage error (MAPE), the root mean square error (RMSE) and the Bayesian Information Criterion (BIC) as our criteria to measure the quality of the fit and forecast. First, we use the following definition of MAPE:

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\frac{|\varepsilon_t|}{X_t}\times 100 = \frac{1}{n}\sum_{t=1}^{n}\frac{|\hat{X}_t - X_t|}{X_t}\times 100, \qquad (3)$$

where ε_t is the difference between observation X_t and estimation $\hat{X}_t$, and n is the number of samples.

Table 1: Mortality models' formulae

Model	Formula
M1 (Lee-Carter model)	$\ln(q_{x,t}) = \beta_x^{(1)} + \beta_x^{(2)} \cdot k_t + \psi_{x,t}$
M2 (RF model[1])	$\frac{q_{x,t}}{q_{x,0}} \equiv RF(x,t) = \alpha(x) + [1-\alpha(x)][1-f(x)]^{\frac{t}{20}}$
M3 (CBD model[2])	$\log it\ \ q_{x,t} = \beta_x^{(1)} k_t^{(1)} + \beta_x^{(2)} k_t^{(2)}$
M4 (M7 model; Cairns et al., 2009)	$\log it\ \ q_{x,t} = k_t^{(1)} + k_t^{(2)}(x-\bar{x}) + k_t^{(3)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^4$
M5 (logistic model)	$q_{x,t} = \frac{\alpha(t)e^{\beta\cdot x}}{1+\alpha(t)e^{\beta\cdot x}} + \gamma(t)$
M6 [modified logistic (beta) model]	$q_{x,t} = \frac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta x}} + \gamma(x)$ $\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } x < seg1 \\ \alpha_2(t) & \text{if } x \geq seg1 \end{cases}, \quad \beta = \begin{cases} \beta_1 & \text{if } x < seg2 \\ \beta_2 & \text{if } x \geq seg2 \end{cases}$

[1] We use the modified reduction factor model proposed by Huang and Hsu (2006). Their idea is that the segment points and corresponding parameters differ for different countries.
[2] We use the revised form of the CBD model (Cairns et al., 2009).

Second, the formula of RMSE is as follows:

$$RMSE = \frac{1}{n}\sum_{t=1}^{n}\left(\hat{X}_t - X_t\right)^2, \tag{4}$$

where X_t and $\hat{X}_t$ are the observation and estimation, respectively, and *n* is the numbers of sample.

The above criteria are based on the estimation errors; however, we have to consider the abuse of the number of parameters. Hence, we apply the BIC to measure the fitting quality of mortality models. The BIC is defined as follows:

$$BIC = l(\hat{\phi}) - \frac{1}{2} v \log N \tag{5}$$

where $l(\hat{\phi})$ is the maximum-likelihood estimate and v and N are the numbers of parameters and observations, respectively.

2.2.2 Fitting Results

We gather mortality data for the United States, Japan, and England and Wales from the Human Mortality Database.[3] Although our study data are somewhat constrained, the logistic model is an increasing function, which is not suitable for persons younger than 1 year of age. Therefore, we fit the mortality rates of single ages between the ages of 30 and 89 years during 1960–1995.

If we have the mortality data of x_1-year-old to x_2-year-old from the year y_1 to y_2, then the optimization criterion under the M6 model is to minimize the total MAPE of fitting on the whole mortality data. The optimization equation is as follows:

$$\underset{\alpha_1(t),\alpha_2(t),\beta_1,\beta_2,\gamma(x),seg1,seg2}{\text{Min}} \quad \frac{1}{(y_2-y_1)(x_2-x_1)}\sum_{t=y_1}^{y_2}\sum_{x=x_1}^{x_2}\left(\frac{\left|\hat{q}_{x,t}-q_{x,t}\right|}{q_{x,t}}\right)\times 100 \tag{6}$$

In this paper, we apply the MATLAB program to search the optimal solutions of the parameters of the M6 model, and the objective function of Equation (6) is to minimize the total MAPE of fitting.[4]

[3] The website is www.mortality.org.

[4] The segment ages for these three countries are in Appendix 1.

The MAPE of fitting results are listed in Table 2, and we rank efficiency according to the numbers in parentheses, with (1) representing the most efficient model of fitting results. Model M4 is the most efficient across the three countries under MAPE, which is a reasonable finding, because the M4 model (Cairns et al., 2009) is incorporated with the cohort effect and contains the most parameters. The fitting results of the modified logistic (beta) model (M6) and Lee-Carter model (M1) are also efficient. Note that the M6 model performs better than the original logistic model (M5), which supports the quality of the modified logistic (beta) model we propose. Although the M6 model does not provide the best fit, its MAPEs are close to that of the model that achieves this best ranking. In Table 3, we represent the RMSE of the fitting results, and we find that the best mortality model is the M1 model under RMSE criterion. However, the RMSE of our proposed M6 model is similar to the M1 model.

Table 2: MAPE for various models fits (%)

	United States		**England and Wales**		**Japan**	
	Male	**Female**	**Male**	**Female**	**Male**	**Female**
M1	3.0640 (3)	2.8619 (3)	3.5859 (3)	3.5627 (2)	3.4199 (2)	3.7021 (3)
M2	6.7149 (6)	5.9076 (6)	6.7860 (6)	5.7468 (5)	6.2193 (5)	5.8733 (4)
M3	5.0780 (5)	5.8576 (5)	6.4519 (5)	6.3624 (6)	7.3414 (6)	13.736 (6)
M4	**2.5029 (1)**	**2.1833 (1)**	**2.9945 (1)**	**2.5620 (1)**	**1.7147 (1)**	**2.5907 (1)**
M5	4.5085 (4)	4.9477 (4)	5.6100 (4)	5.5581 (4)	4.3718 (4)	7.1608 (5)
M6	2.6363 (2)	2.7345 (2)	3.2855 (2)	3.7478 (3)	3.4622 (3)	3.4583 (2)

Note: the numbers in bold font represent the most efficient model of fitting results.

Table 3: RMSE for various models fits

	United States		England and Wales		Japan	
	Male	**Female**	**Male**	**Female**	**Male**	**Female**
M1	**0.0015**	**0.0007**	**0.0026**	**0.0012**	0.0020	**0.0021**
M2	0.0028	0.0021	0.0046	0.0029	0.0067	0.0060
M3	0.0022	0.0073	0.0056	0.0070	0.0053	0.0156
M4	0.0017	0.0012	0.0026	0.0013	**0.0013**	0.0024
M5	0.0037	0.0044	0.0035	0.0050	0.0035	0.0060
M6	0.0018	0.0009	0.0029	0.0015	0.0020	0.0021

Note: the numbers in bold font represent the most efficient model of fitting results.

Table 4: BIC for various models fits

	United States		England and Wales		Japan	
	Male	**Female**	**Male**	**Female**	**Male**	**Female**
M1	–4,234 (3)	–3,896 (3)	–4,408 (2)	–4,372 (2)	–4,175 (2)	–4,420 (3)
M2	–5,048 (5)	–4,727 (5)	–4,989 (4)	–4,738 (4)	–4,737 (5)	–4,689 (4)
M3	–5,969 (6)	–5,677 (6)	–5,820 (6)	–5,579 (6)	–6,285 (6)	–7,634 (6)
M4	–4,089 (2)	**–3,457 (1)**	–4,624 (3)	**–4,054 (1)**	**–3,181 (1)**	**–3,841 (1)**
M5	–4,901 (4)	–4,577 (4)	–5,194 (5)	–4,830 (5)	–4,452 (4)	–5,494 (5)
M6	**–4,023 (1)**	–3,737 (2)	**–4,180 (1)**	–4,380 (3)	–4,337 (3)	–4,365 (2)

Note: the numbers in bold font represent the most efficient model of fitting results.

Then, we apply BIC of fitting results to measure the quality of these six mortality models. In Table 4, we find that the best model under the BIC is not the same for different country's mortality data. Overall, the M4 model still performs well for female data and the M6 models also perform well for male data of the United States and England and Wales under the BIC. Moreover, our proposed M6 model outperforms the original logistic model (M5).

2.2.3. Forecasting Results

We forecast the mortality rates for single ages from 30 to 89 years of age. The data from Japan and England and Wales pertain to the years between 1996 and

2006. For the United States, the data reflect the years between 1996 and 2005. We also rank efficiency according to the numbers in parentheses in Table 5. Although the M4 model is the most efficient for fit, it is not the best for forecasting. Instead, the M6 model, the modified logistic (beta) model, achieves the best forecast across the three countries under the MAPE criterion. Moreover, we measure the efficiency of forecasting by RMSE in Table 6. We find that our proposed M6 model still dominates the male mortality data of these three countries but not for the entire data. These results show that our proposed modified logistic (beta) model is useful for life insurers, pension funds, and government because it is efficient for mortality forecasting.

Table 5: MAPE for various models forecasts (%)

	United States		**England and Wales**		**Japan**	
	Male	**Female**	**Male**	**Female**	**Male**	**Female**
M1	14.2972 (6)	6.2737 (3)	15.6493 (4)	14.8519 (6)	12.3802 (5)	15.6201 (5)
M2	12.6528 (5)	8.2910 (6)	22.0467 (6)	12.7179 (4)	13.8168 (6)	10.7441 (2)
M3	11.8382 (4)	6.6915 (5)	19.3693 (5)	13.6675 (5)	10.9077 (4)	16.1992 (6)
M4	10.7716 (2)	5.9161 (2)	12.9405 (2)	10.8415 (2)	9.1543 (2)	13.5546 (3)
M5	11.1938 (3)	6.3701 (4)	15.2468 (3)	12.2479 (3)	9.6164 (3)	14.0022 (4)
M6	**7.6255 (1)**	**5.8027 (1)**	**10.0496 (1)**	**6.7918 (1)**	**7.8988 (1)**	**10.5600 (1)**

Note: the numbers in bold font represent the most efficient model of forecasting results.

Table 6: RMSE for various models forecasts

	United States		**England and Wales**		**Japan**	
	Male	**Female**	**Male**	**Female**	**Male**	**Female**
M1	0.0054 (6)	0.0014 (2)	0.0105 (4)	0.0045 (5)	0.0097 (6)	0.0097 (6)
M2	0.0044 (5)	0.0041 (5)	0.0108 (5)	**0.0037 (1)**	0.0063 (5)	**0.0027 (1)**
M3	0.0039 (4)	0.0049 (6)	0.0104 (3)	0.0044 (3)	0.0041 (4)	0.0088 (5)
M4	0.0031 (3)	0.0023 (3)	0.0060 (2)	0.0109 (6)	**0.0031 (1)**	0.0037 (3)
M5	0.0023 (2)	0.0028 (4)	0.0119 (6)	0.0044 (3)	0.0037 (3)	0.0033 (2)
M6	**0.0018 (1)**	**0.0012 (1)**	**0.0038 (1)**	0.0042 (2)	**0.0031 (1)**	0.0062 (4)

Note: the numbers in bold font represent the most efficient model of forecasting results.

3. Securitization of Longevity Risk

The EIB bond issued in November 2004 was the first bond for longevity risk, but it failed due to its simplistic design. Therefore, we follow Lin and Cox (2005) and use the concept of CDOs to create a new LB with two tranches, which should allow different investors to choose their preferred investments. Moreover, we apply our proposed modified logistic (beta) mortality model to price the bond.

3.1. Longevity Bond

The cash flow between the insurer and the SPV appears in Figure 1. At time 0, the insurer pays a premium P to buy insurance with the SPV, which covers the longevity risk of the original annuity product. Then the SPV issues a LB, designed according to the longevity risk of the annuity product. We assume that the amount of the insured annuity product is A and the maturity is T years. Furthermore, the insurer and SPV must determine the survivor index S_t upfront. At time *t*, if the real survivor $\widehat{S}_t$ is greater than S_t, the SPV pays claim B_t to the insurer.

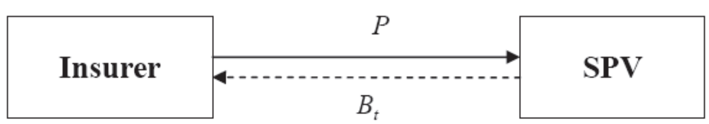

Figure 1: Structure between the insurer and SPV

The uncertainty due to possible improvements in the mortality rate is high. Therefore, we also design a deductible and claim limit, as in Equation (7), where k_1 refers to a deductible (or trigger) that the SPV must pay the insurer when $\hat{S}_t > k_1 S_t$. Moreover, k_2 indicates the claim limit, such that the greatest responsibility of the SPV to the insurer is $(k_2 - k_1)S_t$.

$$B_t = \begin{cases} 0 & \hat{S}_t \le k_1 S_t \\ (\hat{S}_t - k_1 S_t)A & k_1 S_t < \hat{S} \le k_2 S_t \\ (k_2 - k_1) S_t A & \hat{S} > k_2 S_t \end{cases}. \tag{7}$$

Next, we depict the structure between the SPV and investor in Figure 2, which integrates the CDO concept. Several tranches represent different risk levels to attract various investors. Investors who choose Tranche A pay V to buy LBs whose face value is $\Pr_0^A$ at time 0. Investors in Tranche B instead pay V to buy LBs whose face value is $\Pr_0^B$ at time 0. The coupon rates of Tranche A and Tranche B are fixed at c_A and c_B, respectively. Thus, an investor in Tranche A receives a coupon $c_A \cdot \Pr_t^A$ from the SPV at time *t*, whereas the investor in Tranche B receives the coupon $c_B \cdot \Pr_t^B$ at time *t*. If the SPV pays a claim to the insurer, the principal of Tranche B, $\Pr_t^B$, is decreasing at time *t*, whereas the principal of Tranche A will be deducting when $\Pr_t^B$ is zero. Therefore, Tranche B is more risky than Tranche A, such that $c_B > c_A$. Because the coupon rate of Tranche B is bigger than that of Tranche A, it should attract riskier investors who seek high return rates. D_t^A and D_t^B are the amount of interest at time t for investors A and B, respectively.

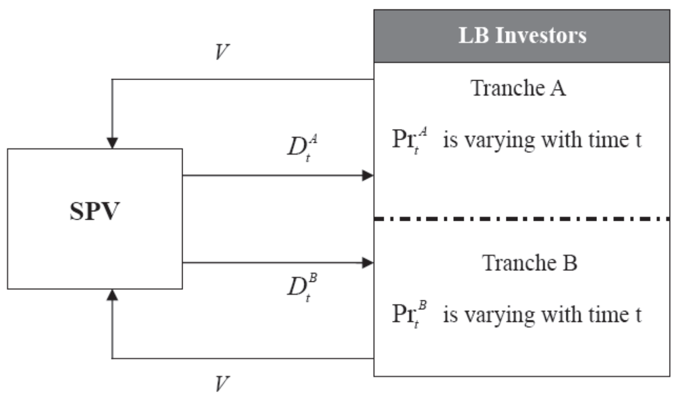

Figure 2: Structure between the SPV and investors

3.2. Numerical Analysis

We assume that the insurer issues N policies of annuity insurance to males in the United States of age x. The insurer pays *Ann.* to a policyholder who is still alive at time $t < T$. The SPV issues the longevity bond in line with this annuity insurance.

The insurer also uses the mortality data from the annuity table to price the annuity insurance premium, in consideration of improvements to the mortality rates. We further assume that the insurer uses improvement trends from Lin and Cox (2005) to price the annuity products. In Table 7, we detail the improvement trends for the United States.

Therefore, the insurer faces the longevity risk arising from an incorrect estimation of mortality rates. To hedge this longevity risk, the insurer pays a premium to the SPV and receives a payment when the real survivor rate is greater than the survivor index from Table 7.

Table 7: Three improvement levels

Age range, years	Change in mortality
65–74	–0.0070
75–84	–0.0093
85–94	–0.0103

The SPV issues these LBs to transform the longevity risk and uses the modified logistic (beta) mortality model to forecast future mortality. Moreover, in line with the Lee-Carter model, we add a stochastic term to the end of the equation, e^{ε}, because some error exists in the model. Thus,

$$q_{x,t} = \left[\frac{\alpha(t)e^{\beta x}}{1+\alpha(t)e^{\beta x}} + \gamma(x) \right] \cdot e^{\varepsilon}, \tag{8}$$

where $\varepsilon \sim N(0,\sigma^2)$,[5] $\alpha(t) = \begin{cases} \alpha_1(t) & \text{if } x < seg1 \\ \alpha_2(t) & \text{if } x \geq seg1 \end{cases}$, and $\beta = \begin{cases} \beta_1 & \text{if } x < seg2 \\ \beta_2 & \text{if } x \geq seg2 \end{cases}$.

Moreover, we assume that the interest rate follows Vasicek (1977), such that

$$dr(t) = a(b - r(t))dt + \sigma_r dW^r(t), \; r(0) = r_0. \tag{9}$$

We list the parameters in Table 8.

[5] $\hat{\sigma}^2$ is 0.0017 for the mortality data between 1982 and 2000 for United States males.

Table 8: The values of the parameters

Interest rate	a	0.2	σ_r	0.02
	b	0.05	r_0	0.03
LBs	k_1	1.02	k_2	1.22
	N	10,000	$Ann.$	1,000
	x	60	T	30
	Tranche A		$\Pr_0^A$	6,500,000
	Tranche B		$\Pr_0^B$	6,500,000

If the SPV issues LBs with a 20% price premium and a face value of \$6,500,000, we can compute the premium, or coupon A and coupon B, as in Table 9. Note that the modified logistic (beta) mortality model is stochastic; therefore, we also apply VaR(95) to calculate the premium.

Table 9: SPV premium and coupons

Pricing mortality model	Premium	VaR(95)	Coupon A	Coupon B
Modified logistic (beta) model	1,212,100	2,467,000	5.7167%	6.9531%

Because the interest rate risk of SPV is massive, we cannot calculate the premium accurately without considering the stochastic property of the interest rate. Subsequently, we assume that the interest rate and death rate are stochastic. We run 50,000 simulations to calculate the premium and coupon rates.

We begin by calculating the premium for every trial and summarize the characteristic values in Table 10. Comparing Table 10 to Table 9, the values of the mean are similar. However, the VaR(95) in Table 10 is larger than that in Table 9, which confirms that the interest rate risk of the SPV is significant.

Table 10: Results of premiums

Median	VaR(70)	VaR(75)	VaR(80)	VaR(85)
1,233,800	1,658,200	1,779,900	1,923,700	2,093,500
VaR(90)	VaR(95)	Max	Mean	Std.
2,314,500	2,645,900	8,639,600	1,288,100	761,410

We next calculate the coupons for Tranche A and Tranche B for every simulation. The characteristic values of the coupons appear in Table 11; we depict the density functions of coupons A and B in Figure 3.

Table 11: Results of coupon rates (%)

	Min	Median	VaR(45)	VaR(40)	VaR(35)	VaR(30)
Coupon A	3.7316	5.6858	5.5972	5.5076	5.4165	5.3152
Coupon B	3.9114	6.9052	6.7758	6.6481	6.5173	6.3836
	VaR(25)	VaR(20)	VaR(15)	Max	Mean	Std.
Coupon A	5.2066	5.0735	4.9122	7.7441	5.6323	0.0062
Coupon B	6.2346	6.0734	5.8904	10.5678	6.9329	0.0099

Comparing Table 11 to Table 9, the mean of coupon A and coupon B becomes 5.6323% and 6.9329%, respectively, when we consider the stochastic interest rate and death rate. However, the worst rates for coupon A and coupon B reach 3.7316% and 3.9114%. In other words, if the future interest rates are very low and the future death rates fall, we cannot offer high coupon rates to the bond investor. Moreover, according to Figure 3, we find that the distributions of coupon A and coupon B differ. Thus, the spread between coupon A and coupon B exists, and this could attract the various risk attitude investors.

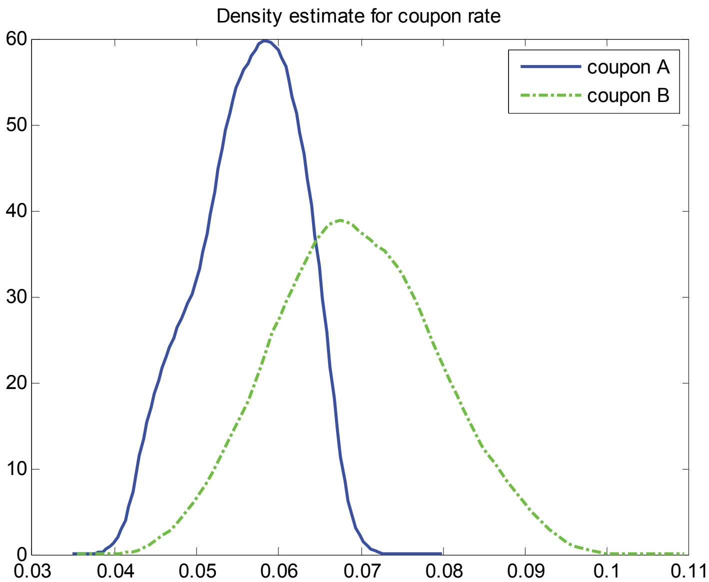

Figure 3. Distributions of coupon rates

3.3 NPV Analyses of SPV

For this analysis, we assume that the SPV charges a premium of $1,288,100, which is the mean of simulated results. For coupons A and B, we use the median of the simulated results, or 5.6858% and 6.9052%, respectively. In Table 12, we find that the risk of the SPV is large because the NPV's standard deviation is large and the probability that NPV < 0 is 49.33%.

Compared to Table 12, we charge the premium according to VaR(90) and VaR(95), or 2,314,500 and 2,645,900, and use VaR(40) for the coupon rates – 5.5076% for coupon A and 6.6481% for coupon B – we attain the results in Table 13. With VaR(40) as the coupon rate, we can reduce over 29% of the probability of a negative NPV. That is, this approach decreases the SPV's interest rate risk and mortality risk. If the SPV's risk attitude is more risk averse, it might even calculate the coupon rate using VaR(25), which would decrease the probability of a negative NPV to 7.71%.

These NPV analyses show that the mortality and interest rate risks of SPV are significant. Thus, SPV should price the LB carefully and apply the accurate mortality model to price.

Table 12: NPV with mean as the premium

	Min	Mean	VaR(90)	VaR(95)	Max	Std.	NPV<0
Insurer	–4,089,800	44.8145	976,340	1,102,200	1,288,100	761,410	47.26%
Investor	–6,615,800	–214,750	1,682,100	2,082,600	4,526,200	1,577,600	50.98%
All	–5,947,900	–214,700	1,569,000	1,839,000	3,035,900	1,514,500	49.33%

Table 13. NPV with VaR as the premium

PREMIUM: VaR(90)							
Coupon	MIN	MEAN	VaR(90)	VaR(95)	MAX	STD.	NPV<0
Median	-4,921,500	811,700	2,595,400	2,865,400	4,062,300	1,514,500	27.47%
VaR(40)	-4,315,200	1,271,900	3,012,200	3,278,500	4,434,100	1,477,900	20.28%
PREMIUM: VaR(95)							
Coupon	MIN	MEAN	VaR(90)	VaR(95)	MAX	STD.	NPV<0
Median	-4,590,100	1,143,100	2,926,800	3,196,800	4,393,700	1,514,500	22.44%
VaR(40)	-3,983,800	1,603,300	3,343,600	3,609,900	4,765,500	1,477,900	16.17%
VaR(25)	-2,988,200	2,359,300	4,030,400	4,288,400	5,378,800	1,418,000	7.71%

4. Conclusion

Investigations of mortality models have increased recently; we extend this research by modifying a logistic model by two lines, which are segment method and the modified background death rate. Numerical results show that our proposed model (M6) achieves better quality than the Lee-Carter and CBD models for fitting. It is important to note that our proposed modified logistic (beta) model provides the best forecasting rates for persons aged 30–80 years across three countries under the MAPE criterion. This is useful for life insurers, to forecast the mortality rates and to price the life insurance products.

The new LB design we propose to manage longevity risk can be priced according to our modified logistic (beta) mortality model. In an attempt to improve the attractiveness of the bond, we also adopt the concept of CDO and design the LB with more than one tranche, which allows investors to make choices according to their risk preferences. Our numerical results reveal that the coupon rate for Tranche B is 1.2% greater than that for Tranche A, which implies it can attract riskier investors who seek high return rates. Moreover, we find that both interest and mortality rates have important effects on the SPV's NPV.

Therefore, SPVs should carefully evaluate premium and coupon rates to control their risk.

Appendix 1

The optimal segment ages of the modified logistic (beta) model are listed in Table A14. We find that the optimal segment ages are different for different data sets.

Table A14: The optimal segment ages of the M6 model

	UNITED STATES		ENGLAND AND WALES		JAPAN	
	Male	Female	Male	Female	Male	Female
seg1	40	54	43	48	68	63
seg2	47	70	80	76	60	68

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