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Intertemporal Stable Pension Funding

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Abstract

This paper proposes a discrete dynamic programming model to maintain pension contribution in a stable level. By assuming an intertemporal stable contribution rate, we derive an algorithm to calculate the optimal contribution that requires less exogenous information and produces more stable results. Our simulation results further confirm that our model helps pension fund managers to make more stable contributions and further reduce the contribution risk for the defined benefit pension fund than the traditional algorithms do.

Key Words: Pension fund, dynamic programming, intertemporal stable condition, optimal contribution

I. Introduction

To manage substantial uncertainties caused by stochastic economic and financial factors under defined-benefits pension funds, several researchers (O'Brien, 1986; Haberman, 1992, 1993a, 1993b, 1994; Cairn and Parker, 1997) have proposed a stochastic approach for pension fund managers to determine pension funding strategy efficiently. Haberman and Sung (1994) propose that pension funds need to cope with two main risks—a contribution rate risk and a solvency risk—and introduce a dynamic control theory to analyze the trade-off among the various risks. In addition, they also propose using dynamic programming with a backward induction algorithm to calculate the optimal contribution. Chang (1999) further modifies Haberman and Sung's methodology by adopting different measurements of the contribution rate risk and solvency risk. More recently, Chang, Tzeng, and Miao (2003, hereafter CTM) generalize previous dynamic models and further incorporate them into the downside risks for pension funding. Josa-Fombellida and Ricón-Zapatero (2001), on the other hand, extend this literature by simultaneously controlling both pension contributions and asset portfolio selection.

Although Haberman and Sung's (1994) methodology has provided many ingenious applications in both theory and real practice, their model relies on the backward induction algorithm and could limit the robustness in some cases. First, adopting a backward induction algorithm requires a boundary condition in the time horizon for pension funding. However, in real practice it is sometimes difficult to determine both the time horizon and the boundary conditions for pension funding. Furthermore, Chang (2000) finds that the time horizon of a pension fund could substantially influence the optimal contribution rate in pension budgeting. Thus, an incorrect decision in the time horizon for a pension fund could lead to a serious miscalculation in pension funding and thus induce a significant insolvency risk for the fund.

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To overcome previous problems, we propose in this paper a dynamic model with an intertemporal stable condition to calculate the optimal pension contribution. We extend the traditional methodology to the case where fund managers may not need to have clear idea of both the time horizon and the boundary conditions for pension funding. Following Josa-Fombellida and Ricón-Zapatero (2001) and Haberman and Sung (2005), we assume that pension fund managers would like to budget the funding for an infinite horizon. Since there is no end period for the pension plan in this case, the pension manager does not need to decide the bounded horizon in the continuous framework model. Although Haberman and Sung (2005) have proposed the optimal funding model over infinite horizon, their model need to assume that both investment rate of return and benefit outgo are stationary. The stationary assumption they used could limit their model only to some particular cases, for example, a mature pension scheme. To overcome this limitation, we assume an intertemporal stable condition to calculate the optimal contributions. The intertemporal stable condition assumption provides us two advantages. First, it improves the stableness of the contribution rate, since the intertemporal stable condition typically assumes that the contribution of pension funding would remain at a stationary level. Second, it ensures stable funding no matter a pension scheme can satisfy stationary valuation assumptions or not[1]. Furthermore, as proposed by Josa-Fombellida and Ricón-Zapatero (2001), we establish a discrete dynamic programming model to solve pension funding in the steady state.

Our paper differs from the previous literature in several ways. First, we use the intertemporal stable condition to calculate the optimal contribution, whereas Haberman and Sung (1994), Chang (1999), and CTM (2003) use backward induction to solve the optimal contribution rate. Second, Josa-Fombellida and Ricón-Zapatero (2001) have established a continuous dynamic model and solved the optimal solution in the steady state, whereas we set up a discrete dynamic model and derive the solution on the basis of an intertemporal stable condition. Third, the stationary properties of investment return and benefit outgo are required in advance in Haberman and Sung's (2005) model, whereas our model can calculate the optimal contribution without those assumptions.

In summary, the dynamic model with an intertemporal stable condition proposed in this paper requires less exogenous information than the traditional backward induction model and provides a new alternative method of forward dynamic programming in the pension funding literature. Thus, our paper complements the methodologies in existing literature for analyzing several cases where it may be difficult to cope with traditional models. More importantly, the simulation results in this paper confirm that our proposed model produces a more stable contribution rate because of the intertemporal stable assumption. Thus, our proposed model could further reduce the contribution risk for the defined benefit pension fund.

In Section 2, we introduce our model and compare it with previous traditional models. In Section 3, we employ an actual case to demonstrate the advantages of our model and analyze the financial impacts on pension funding and contribution strategies. The final section concludes this paper.

II. The Model

Following CTM (2003)[2], we assume that pension fund managers choose the optimal contribution rates to minimize four main risks—contribution risk, solvency risk, and two types

[1] Without intertemporal stable condition assumption, Haberman and Sung (2005) must assume there is no economic and demographic growth over time. That is, in their model the normal cost, accrual liability and benefit outgo are constant over time.

[2] We set up our model on the basis of Chang, Tzeng, and Miao (2003) because it is a generalization model and includes many models from the earlier literature as special cases.

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of downside risks. As the same setting in Haberman and Sung (1994), Chang (2000) and CTM (2003), we assume that B_t, NC_t, F_t, and AL_t are, respectively, the benefit outgo, normal cost, fund asset and accrued liability of a pension fund at time t. We also assume that the contribution rate and total wage are c_t and W_t at time t, and the expected wage growth rate is g_W. Thus, total contribution C_t is equal to $c_t \times W_t$.

Let $\alpha_{1,t}(C_t/NC_t - 1)^2$ and $\alpha_{3,t}(C_t/NC_t - 1)$ denote the contribution rate risks; $\alpha_{2,t+1}(1 - F_{t+1}/\eta AL_{t+1})^2$ and $\alpha_{4,t+1}(1 - F_{t+1}/\eta AL_{t+1})$ denote the solvency risks where η is the target fund ratio and $\alpha_{1,t}$, $\alpha_{2,t}$, $\alpha_{3,t}$, and $\alpha_{4,t}$ are weighted ratios for the contribution rate risks and solvency risks at time t. Let v_t and β_t denote the discount factor and relative weight between the contribution rate risk and solvency risk at time t. The performance criterion function in Haberman and Sung (1994) is the case with $\alpha_{1,t} = v_t NC_t^2$, $\alpha_{2,t+1} = v_{t+1}\beta_{t+1}\eta^2 AL_{t+1}^2$, $\alpha_{3,t} = 0$ and $\alpha_{4,t+1} = 0$, while that in Chang (1999) is the case with $\alpha_{1,t} = v_t$, $\alpha_{2,t+1} = v_{t+1}\beta_{t+1}$, $\alpha_{3,t} = 0$, and $\alpha_{4,t+1} = 0$. Further assume that r_t denotes the investment rate of return for a pension fund at time t and H_t denotes the state variables at time t.

Since we assume that the planning horizon of the pension fund is infinite, the optimalization model for calculating the pension contribution can be expressed as Equation (1) as follows:

$$\min_{c_t,\dots} E\left[\sum_{t=s}^{\infty}\alpha_{1,t}\left(\frac{c_tW_t}{NC_t}-1\right)^2+\alpha_{2,t+1}\left(1-\frac{F_{t+1}}{\eta AL_{t+1}}\right)^2+\alpha_{3,t}\left(\frac{c_tW_t}{NC_t}-1\right)+\alpha_{4,t+1}\left(1-\frac{F_{t+1}}{\eta AL_{t+1}}\right)\middle|H_t\right] \tag{1}$$

subject to $F_{t+1} = (F_t + c_tW_t - B_t)(1 + r_{t+1})$,

where the return r_{t+1} between time t and $t+1$ is assumed to be a sequence of independent and identically distributed normal variables with mean μ and variance σ^2. In Equation (1), $H_t = \{F_t\}$ and the control variable is c_t. Thus, the Bellman equation can be written as Equation (2):

$$\begin{aligned} V(F_t) = \min_{c_t} E\{&\alpha_{1,t}\left(\frac{c_tW_t}{NC_t}-1\right)^2+\alpha_{2,t+1}\left(1-\frac{F_{t+1}}{\eta AL_{t+1}}\right)^2+\alpha_{3,t}\left(\frac{c_tW_t}{NC_t}-1\right)+\alpha_{4,t+1}\left(1-\frac{F_{t+1}}{\eta AL_{t+1}}\right) \\ &+V(F_{t+1})|F_t\} \end{aligned} \tag{2}$$

Differentiating Equation (2) with c_t and setting it equal to zero, we can obtain Equation (3) as follows:

$$\begin{aligned} &-\frac{2\alpha_{1,t}W_t}{NC_t}+\frac{2\alpha_{1,t}c_tW_t^2}{NC_t^2}-\frac{2\alpha_{2,t+1}W_tH}{\eta AL_{t+1}}+\frac{2\alpha_{2,t+1}(F_t+c_tW_t-B_t)W_tK}{\eta^2 AL_{t+1}^2}+\frac{\alpha_{3,t}W_t}{NC_t}-\frac{\alpha_{4,t+1}W_tH}{\eta AL_{t+1}} \\ &+EV'(F_{t+1})W_t(1+r_{t+1})=0 \end{aligned} \tag{3}$$

where $H = 1 + \mu$ and $K = H^2 + \sigma^2$.

On the basis of the Benveniste-Scheinkman equation, we can differentiate Bellman equation with respect to F_t and obtain the following Equation:[3]

$$V'(F_t) = \frac{-2\alpha_{2,t+1}H}{\eta AL_{t+1}} + \frac{2\alpha_{2,t+1}(F_t + c_t W_t - B_t)K}{\eta^2 AL_{t+1}^2} - \frac{\alpha_{4,t+1}H}{\eta AL_{t+1}}. \quad (4)$$

Equation (4) can be rewritten as:

$$V'(F_{t+1}) = \frac{-2\alpha_{2,t+2}H}{\eta AL_{t+2}} + \frac{2\alpha_{2,t+2}(F_{t+1} + c_{t+1} W_{t+1} - B_{t+1})K}{\eta^2 AL_{t+2}^2} - \frac{\alpha_{4,t+2}H}{\eta AL_{t+2}}. \quad (5)$$

Rearranging Equation (3) in the form of Equation (5), we obtain the Euler Equation:

$$\begin{aligned} &-\frac{2\alpha_{1,t}W_t}{NC_t} + \frac{2\alpha_{1,t}c_t W_t^2}{NC_t^2} - \frac{2\alpha_{2,t+1}W_t H}{\eta AL_{t+1}} + \frac{2\alpha_{2,t+1}(F_t + c_t W_t - B_t)W_t K}{\eta^2 AL_{t+1}^2} + \frac{\alpha_{3,t}W_t}{NC_t} - \frac{\alpha_{4,t+1}W_t H}{\eta AL_{t+1}} \\ &+ E\left\{\left[\frac{-2\alpha_{2,t+2}H}{\eta AL_{t+2}} + \frac{2\alpha_{2,t+2}(F_{t+1} + c_{t+1}W_{t+1} - B_{t+1})K}{\eta^2 AL_{t+2}^2} - \frac{\alpha_{4,t+2}H}{\eta AL_{t+2}}\right] W_t(1 + r_{t+1})\right\} = 0 \end{aligned} \quad (6)$$

Note that $F_{t+1} = (F_t + c_t W_t - B_t)(1 + r_{t+1})$. Thus, Equation (6) can be expressed as:

$$\begin{aligned} &\frac{2\alpha_{1,t}c_t W_t^2}{NC_t^2} + \frac{2\alpha_{2,t+1}c_t W_t^2 K}{\eta^2 AL_{t+1}^2} + \frac{2\alpha_{2,t+2}c_t W_t^2 K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2}E[c_{t+1}W_t W_{t+1}(1 + r_{t+1})]K}{\eta^2 AL_{t+2}^2} \\ &= \frac{2\alpha_{1,t}W_t}{NC_t} + \frac{2\alpha_{2,t+1}W_t H}{\eta AL_{t+1}} - \frac{2\alpha_{2,t+1}(F_t - B_t)W_t K}{\eta^2 AL_{t+1}^2} + \frac{2\alpha_{2,t+2}W_t H^2}{\eta AL_{t+2}} - \frac{2\alpha_{2,t+2}(F_t - B_t)W_t K^2}{\eta^2 AL_{t+2}^2} \\ &+ \frac{2\alpha_{2,t+2}B_{t+1}W_t HK}{\eta^2 AL_{t+2}^2} - \frac{\alpha_{3,t}W_t}{NC_t} + \frac{\alpha_{4,t+1}W_t H}{\eta AL_{t+1}} + \frac{\alpha_{4,t+2}W_t H^2}{\eta AL_{t+2}} \end{aligned} \quad (7)$$

In our model, we further assume an intertemporal stable condition to keep contribution rates at a stationary level. The intertemporal stable condition is as follows:

$$E(c_{t+1}W_{t+1}(1 + r_{t+1})) = c_t E(W_{t+1}(1 + r_{t+1})). \quad (8a)$$

In another form, we can express the intertemporal stable condition as:

$$c_t = E(c_{t+1}) \text{ and } \operatorname{cov}(c_{t+1}, W_{t+1}(1 + r_{t+1})) = 0. \quad (8b)$$

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[3] Let $L(F_{t+1}, c_t) = \alpha_{1,t}(\frac{c_t W_t}{NC_t} - 1)^2 + \alpha_{2,t+1}(1 - \frac{F_{t+1}}{\eta AL_{t+1}})^2 + \alpha_{3,t}(\frac{c_t W_t}{NC_t} - 1) + \alpha_{4,t+1}(1 - \frac{F_{t+1}}{\eta AL_{t+1}})$ and $F_{t+1} = (F_t + c_t W_t - B_t)(1 + r_{t+1}) = g(F_t, c_t)$. In our setting, the value function is differentiable with respect to F_t when (1) $L(F_{t+1}, c_t)$ is convex and differentiable; (2) the constraint set generated by g is convex. Since $\frac{\partial^2 L}{\partial F_{t+1}^2} > 0$ and g is a linear function, the value function we proposed is differentiable with respect to F_t. Thus, equation (2) satisfies these conditions.

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Obviously, Equation (8b) implies Equation (8a). Thus, we adopt a weaker condition (8a) in this paper. Although the model can be processed with condition (8a), condition (8b) provides a more intuitive rationale for the intertemporal stable condition. Typically, Equation (8b) indicates two decision mechanisms for pension managers. First, they may consider that the contribution rate in the next period is not correlated to the wage and investment rate of return in the next period. Second, pension managers would try to maintain the expected contribution rate in the next period as equal to the current contribution rate. One rationale for explaining why pension fund managers would behave in this way is the so-called "habit formation". It is well known that the consumption of the current period will influence an individual's utility in the next period. That is, an individual's consumption behavior is generally like a habit. By the same token, we assume that pension fund managers may behave as a result of "habit formation". Thus, they like to set up a stationary contribution level as characterized in Equation (8a) or (8b).

Under the condition of Equation (8a), we further assume that wages are not correlated with investment return. Thus, the optimal stationary contribution rate is:

$$c_t^* = \frac{D_t}{G_t}, \tag{9}$$

where:

$$G_t = \frac{2\alpha_{1,t} W_t}{NC_t^2} + \frac{2\alpha_{2,t+1} W_t\, K}{\eta^2 AL_{t+1}^2} + \frac{2\alpha_{2,t+2} W_t\, K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2} HKW_t\,(1+g_W)}{\eta^2 AL_{t+2}^2},$$

$$\begin{aligned} D_t = {} & \frac{2\alpha_{1,t}}{NC_t} + \frac{2\alpha_{2,t+1} H}{\eta AL_{t+1}} - \frac{2\alpha_{2,t+1}(F_t - B_t)K}{\eta^2 AL_{t+1}^2} + \frac{2\alpha_{2,t+2} H^2}{\eta AL_{t+2}} - \frac{2\alpha_{2,t+2}(F_t - B_t)K^2}{\eta^2 AL_{t+2}^2} \\ & + \frac{2\alpha_{2,t+2} B_{t+1} HK}{\eta^2 AL_{t+2}^2} - \frac{\alpha_{3,t}}{NC_t} + \frac{\alpha_{4,t+1} H}{\eta AL_{t+1}} + \frac{\alpha_{4,t+2} H^2}{\eta AL_{t+2}} \end{aligned}.$$

To further analyze the contribution of this paper, we compare our proposed model with earlier models in the literature. From CTM (2003), the optimal contribution is:

$$C_t^* = \frac{D_t'}{G_t'}, \tag{10}$$

where:

$$\begin{aligned} D_t' &= \left\{ \frac{2\alpha_{1,t}}{NC_t} + \frac{2\alpha_{2,t+1} H}{\eta AL_{t+1}} - \frac{2\alpha_{2,t+1}(F_t - B_t)K}{\eta^2 AL_{t+1}^2} - \frac{\alpha_{3,t}}{NC_t} + \frac{\alpha_{4,t+1} H}{\eta AL_{t+1}} \right\} + \\ & \quad \left\{ 2a_{1,t+1} B_t K - a_{2,t+1} H - a_{1,t+1} K F_t \right\} \\ &= \{Non-recursive \quad terms\} + \{recursive \quad adjustments(adjD_{bw})\} \end{aligned}, \tag{11}$$

$$\begin{aligned} G_t' &= \left\{ \frac{2\alpha_{1,t}}{NC_t^2} + \frac{2\alpha_{2,t+1} K}{\eta^2 AL_{t+1}^2} \right\} + \left\{ 2a_{1,t+1} K \right\} \\ &= \{Non-recursive \quad terms\} + \{recursive \quad adjustments(adjG_{bw})\} \end{aligned}. \tag{12}$$

In CTM (2003) model, $a_{1,t}$ and $a_{2,t}$ are recursive terms[4] that represent an adjustment to the contribution for the periods from time *t* to the terminal period. Both D_t' and G_t' can be separated further into two parts—one is no-recursive and the other is calculated by a backward recursive adjustment.

To compare our model with CTM (2003), we multiply Equation (9) by W_t to obtain the optimal contribution in our model.

$$D_t = \left\{ \frac{2\alpha_{1,t}}{NC_t} + \frac{2\alpha_{2,t+1}H}{\eta AL_{t+1}} - \frac{2\alpha_{2,t+1}(F_t - B_t)K}{\eta^2 AL_{t+1}^2} - \frac{\alpha_{3,t}}{NC_t} + \frac{\alpha_{4,t+1}H}{\eta AL_{t+1}} \right\} + \left\{ \frac{2\alpha_{2,t+2}H^2}{\eta AL_{t+2}} - \frac{2\alpha_{2,t+2}(F_t - B_t)K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2}B_{t+1}HK}{\eta^2 AL_{t+2}^2} + \frac{\alpha_{4,t+2}H^2}{\eta AL_{t+2}} \right\} \tag{13}$$

$$\frac{G_t}{W_t} = \left\{ \frac{2\alpha_{1,t}}{NC_t^2} + \frac{2\alpha_{2,t+1}K}{\eta^2 AL_{t+1}^2} \right\} + \left\{ \frac{2\alpha_{2,t+2}K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2}HK(1+g_W)}{\eta^2 AL_{t+2}^2} \right\} \tag{14}$$

We find that the terms in the first blanket of both Equation (13) and Equation (14) are the same as the non-recursive terms in Equation (11) and Equation (12), respectively. We further define the terms in the second blanket in Equation (13) and Equation (14) as follows:

$$adjD_{fw} = \frac{2\alpha_{2,t+2}H^2}{\eta AL_{t+2}} - \frac{2\alpha_{2,t+2}(F_t - B_t)K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2}B_{t+1}HK}{\eta^2 AL_{t+2}^2} + \frac{\alpha_{4,t+2}H^2}{\eta AL_{t+2}} \tag{15}$$

$$adjG_{fw} = \frac{2\alpha_{2,t+2}K^2}{\eta^2 AL_{t+2}^2} + \frac{2\alpha_{2,t+2}HK(1+g_W)}{\eta^2 AL_{t+2}^2} \tag{16}$$

$adjD_{fw}$ and $adjG_{fw}$ are the adjustments to the contribution in our model, which, respectively, play the same roles as the recursive adjustments of D_t' and G_t' in CTM (2003). The main difference in the contribution adjustment between CTM (2003) model and

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[4] In their model, when the return is identical and is an independent normal distribution,

$$a_{1,t} = \frac{\alpha_{1,t}E_t^2}{G_t^2 NC_t^2} + \frac{\alpha_{2,t+1}(G_t + E_t)^2 K}{G_t^2 \eta^2 AL_{t+1}^2} + \frac{a_{1,t+1}(G_t + E_t)^2 K}{G_t^2}$$, and

$$a_{2,t}(r_t) = \frac{-2\alpha_{1,t}E_t}{G_t NC_t}\left(1 - \frac{D_t}{G_t NC_t}\right) - \frac{2\alpha_{2,t+1}H(G_t + E_t)}{\eta AL_{t+1}G_t} + \frac{2\alpha_{2,t+1}K(G_t + E_t)(D_t - B_t G_t)}{\eta^2 AL_{t+1}^2 G_t^2} + \frac{\alpha_{3,t}E_t}{G_t NC_t} - \frac{\alpha_{4,t+1}H(G_t + E_t)}{\eta AL_{t+1}G_t} + \frac{2a_{1,t+1}K(G_t + E_t)(D_t - B_t G_t)}{G_t^2} + \frac{a_{2,t+1}H(G_t + E_t)}{G_t}$$

where: $a_{1,T} = 0$, $a_{2,T} = 0$, $D_t = \frac{2\alpha_{1,t}}{NC_t} + \frac{2\alpha_{2,t+1}H}{\eta AL_{t+1}} + \frac{2\alpha_{2,t+1}B_t K}{\eta^2 AL_{t+1}^2} - \frac{\alpha_{3,t}}{NC_t} + \frac{\alpha_{4,t+1}H}{\eta AL_{t+1}} + 2a_{1,t+1}B_t K - a_{2,t+1}H$,

$E_t = \frac{-2\alpha_{2,t+1}K}{\eta^2 AL_{t+1}^2} - a_{1,t+1}K$,and $G_t = \frac{2\alpha_{1,t}}{NC_t^2} + \frac{2\alpha_{2,t+1}K}{\eta^2 AL_{t+1}^2} + 2a_{1,t+1}K$.

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ours is the required information. $adjD_{bw}$ and $adjG_{bw}$ are composed by recursive calculation. To obtain $adjD_{bw}$ and $adjG_{bw}$, pension managers need all the information for the normal cost, benefit outgo, accrued liability, and investment return from time *t* to the terminal period. This may cause an unrealistic bias when the terminal period is far from time *t*. It is important to note that $adjD_{fw}$ and $adjG_{fw}$ in our model provide several advantages in this respect. The information used to calculate these two adjustments in our model comes only from time *t* to *t*+2. This implies that we can calculate the contributions by means of more realistic assumptions instead of simple conjecture.

III. Simulation Result and Analysis

To demonstrate the advantages of using a dynamic model with an intertemporal stable condition, we use the actual data of a pension fund in real practice as an example for capturing the difference between our proposed model and traditional models. The company's profile and the underlining assumptions used to estimate the actuarial accrued liability, normal costs and benefit payment are as follows:

- Population: company service table based on 2001-2002; 1989 TSO for the retiree's annuity table
- Number of employees in the sample: 591
- Employees' average age: 34
- Average years of service: 8.7
- Actuarial cost method: individual entry age normal cost method
- Salary scale and inflation rate: 5% for the annual salary increase and 3% for the annual inflation rate
- Discount rate: 4%

Under this actual case, we calculate the contribution ratios, contribution rates, funding ratios, and adjustments to contribution for both our proposed model and CTM (2003) model. We further draw a comparison between these two models. In order to compare the results, we follow CTM (2003) setting and assume that $\alpha_{1,t} = v^{-t}$, $\alpha_{2,t+1} = v^{-t}\beta_{2,t+1}$, $\alpha_{3,t} = v^{-t}\beta_{3,t}$, and $\alpha_{4,t+1} = v^{-t}\beta_{4,t+1}$, where v is equal to 1.08. $\beta_{s,t}$ is the relative importance among risks to a fund manager. We make the following assumptions and simulate 1,000 times under two different scenarios of the parameter setting.

- Target fund ratio: η = 100% for every year.
- Risk measurement weight: we consider two scenarios of the parameter setting to be relative importance. In Scenario 1, we assume $\beta_{2,t}$ = 1, $\beta_{3,t}$ =0.5, and $\beta_{4,t}$ =2. In Scenario 2[5], we assume $\beta_{2,t}$ = 1, $\beta_{3,t}$ =0.5, and $\beta_{4,t}$ =10.
- Annual investment return: $r_t \sim N(\mu, \sigma^2)$. We assume u = 8%, σ = 4%. We assume that the fund manager sets up the investment plan each year.

Table 1 and Table 2 illustrate results of the estimated contribution ratios and contribution rate. In Table 1, we find that the standard deviations of the contribution ratios in our model are less than those in CTM's model, except in the last period for Scenario 1. The pattern of means is also more stable in our model than in CTM's model. The same results can also be found in Table 2. Under Scenario 1, the standard deviations of the contribution rates in our

[5] Scenario 2 is an arbitrary parameter setting that makes the fund ratios in our model at the end of the time similar to those in Chang, Tzeng, and Miao's (2003) model.

model are less than or equal to those in CTM's model in each period. That is, given the same parameter setting, the pattern of means of the contribution rates in our model is more stable than in CTM's model.

This analysis confirms that our model provides more stable contributions than the traditional algorithm does. It is worth noting that the levels of the contribution ratios as well as the contribution rates in our model are also lower than in CTM's model in each period. Consequently, lower contributions make the fund ratios in our model lower than in CTM's model.

Table 3 reports the results of the estimated fund ratio. In Table 3, the differences in fund ratios between CTM's model and our model fall between –0.041 in time periods 12 and 13 and 0.008 in time period 28. Given the pattern of fund ratios, we can compare the contributions of both models from the second and fourth columns in Table 1 and Table 2. The contribution ratios in our model are lower than those in CTM's model for the first eleven periods, whereas the contribution ratios in our model are higher than those in CTM's model for the remaining periods. We note that the values of the contribution ratios on our model fall between 1.566 to 1.089 and those in CTM's fall between 1.618 to 0.775. This result implies that the pattern of the contribution ratios is flatter in our model than in CTM's model. Similar results are also found in the contribution rates. This analysis confirms that the intertemporal stable condition makes the contributions more stable than the traditional algorithm does when the same objective funding goal is required. Thus, the optimal funding schedule derived by this paper could be useful for a pension's sponsors and managers who prefer a more stationary contribution strategy.

However, it is important to note that there is a trade-off or even a conflict between contribution rate risk and solvency risk. To fortify the contribution stability we assume the intertemporal stable condition in our model by adopting equation (8). From the result of Table 3, we can clearly observe that the fund ratios of CTM's model are slightly higher than those of our model in most of the cases. It implies that adopting intertemporal stable condition may somewhat weaken the solvency condition of pension fund. Thus, pension fund managers need to consider this trade-off and try to balance it appropriately.

Table 4 illustrates the results for adjustment terms. Under Scenario 1, we find that $adjD_{bw}$ and $adjG_{bw}$ are greater than $adjD_{fw}$ and $adjG_{fw}$ in each period (except the last period). This implies that adjustments to contributions in our model are smaller than in CTM's model. Moreover, the differences in the adjustment terms become smaller in each succeeding period. This pattern may result from the information involved in the models. As mentioned in the above section, the adjustment terms in CTM's model require all the actuarial information from time *t* to the terminal period. Thus, it is possible to have more prediction error in their model. On the other hand, the adjustment terms in our model depend only on the information from time *t* to *t*+2. Thus, our model not only requires less information but also uses more reliable and realistic information for pension fund managers. Thus, our model has a smaller adjustment to the contribution than CTM's model has.

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Table 1: Contribution Ratios
Contribution Ratio = Contribution/Normal Cost

Time	CTM's model in Scenario 1	Our model in Scenario 1	Our model in Scenario 2
1	1.618(0.051)	0.859(0.027)	1.566(0.050)
2	1.573(0.053)	0.845(0.027)	1.517(0.048)
3	1.531(0.054)	0.833(0.027)	1.473(0.047)
4	1.498(0.056)	0.824(0.027)	1.436(0.046)
5	1.466(0.057)	0.815(0.027)	1.402(0.045)
6	1.438(0.058)	0.808(0.027)	1.376(0.045)
7	1.414(0.059)	0.801(0.027)	1.351(0.044)
8	1.389(0.060)	0.797(0.027)	1.328(0.044)
9	1.360(0.061)	0.795(0.027)	1.304(0.043)
10	1.324(0.060)	0.794(0.027)	1.284(0.043)
11	1.277(0.059)	0.793(0.027)	1.264(0.042)
12	1.232(0.057)	0.792(0.027)	1.248(0.042)
13	1.183(0.054)	0.790(0.027)	1.227(0.041)
14	1.140(0.052)	0.788(0.027)	1.207(0.040)
15	1.104(0.050)	0.787(0.027)	1.190(0.039)
16	1.064(0.047)	0.786(0.027)	1.173(0.039)
17	1.041(0.046)	0.788(0.027)	1.170(0.039)
18	1.014(0.044)	0.788(0.027)	1.162(0.039)
19	0.985(0.042)	0.789(0.027)	1.153(0.038)
20	0.957(0.040)	0.790(0.027)	1.146(0.038)
21	0.932(0.038)	0.790(0.026)	1.140(0.038)
22	0.903(0.035)	0.789(0.026)	1.129(0.037)
23	0.874(0.033)	0.788(0.026)	1.117(0.037)
24	0.847(0.031)	0.785(0.026)	1.105(0.036)
25	0.828(0.029)	0.785(0.026)	1.102(0.036)
26	0.812(0.028)	0.785(0.026)	1.101(0.036)
27	0.792(0.027)	0.783(0.026)	1.092(0.036)
28	0.775(0.025)	0.782(0.026)	1.089(0.036)
29	0.758(0.024)	NA NA	NA NA
30	NA NA	NA NA	NA NA

The numbers without blanket are means of contribution ratios.

The numbers with blanket are standard deviations of contribution ratios.

Table 2: Contribution Ratios
Contribution Rate = Contribution/Wage

Time	CTM's model in Scenario 1	Our model in Scenario 1	Our model in Scenario 2
1	0.142(0.005)	0.076(0.002)	0.138(0.004)
2	0.130(0.004)	0.070(0.002)	0.125(0.004)
3	0.120(0.004)	0.065(0.002)	0.116(0.004)
4	0.112(0.004)	0.061(0.002)	0.107(0.003)
5	0.103(0.004)	0.058(0.002)	0.099(0.003)
6	0.097(0.004)	0.054(0.002)	0.093(0.003)
7	0.092(0.004)	0.052(0.002)	0.088(0.003)
8	0.086(0.004)	0.049(0.002)	0.082(0.003)
9	0.080(0.004)	0.047(0.002)	0.077(0.003)
10	0.073(0.003)	0.044(0.002)	0.071(0.002)
11	0.066(0.003)	0.041(0.001)	0.066(0.002)
12	0.061(0.003)	0.039(0.001)	0.062(0.002)
13	0.054(0.002)	0.036(0.001)	0.056(0.002)
14	0.049(0.002)	0.034(0.001)	0.052(0.002)
15	0.044(0.002)	0.032(0.001)	0.048(0.002)
16	0.039(0.002)	0.029(0.001)	0.043(0.001)
17	0.038(0.002)	0.029(0.001)	0.043(0.001)
18	0.035(0.002)	0.028(0.001)	0.041(0.001)
19	0.033(0.001)	0.026(0.001)	0.039(0.001)
20	0.031(0.001)	0.025(0.001)	0.037(0.001)
21	0.029(0.001)	0.024(0.001)	0.035(0.001)
22	0.026(0.001)	0.023(0.001)	0.032(0.001)
23	0.023(0.001)	0.021(0.001)	0.029(0.001)
24	0.020(0.001)	0.019(0.001)	0.026(0.001)
25	0.019(0.001)	0.018(0.001)	0.025(0.001)
26	0.019(0.001)	0.018(0.001)	0.025(0.001)
27	0.017(0.001)	0.017(0.001)	0.024(0.001)
28	0.016(0.001)	0.016(0.001)	0.023(0.001)
29	0.015(0.000)	NA NA	NA NA
30	NA NA	NA NA	NA NA

The numbers without blanket are means of contribution rates.

The numbers with blanket are standard deviations of contribution rates.

Table 3: Fund Ratios
Fund Ratio = Fund/Accrued Liability

Time	CTM's model in Scenario 1	Our model in Scenario 1	Our model in Scenario 2
1	1.000(0.000)	1.000(0.000)	1.000(0.000)
2	1.124(0.054)	1.073(0.052)	1.122(0.054)
3	1.244(0.074)	1.147(0.069)	1.240(0.074)
4	1.332(0.090)	1.195(0.083)	1.330(0.090)
5	1.433(0.106)	1.254(0.097)	1.425(0.104)
6	1.494(0.119)	1.283(0.107)	1.483(0.119)
7	1.599(0.135)	1.350(0.120)	1.585(0.135)
8	1.675(0.149)	1.389(0.129)	1.654(0.150)
9	1.745(0.167)	1.425(0.140)	1.716(0.167)
10	1.770(0.177)	1.420(0.148)	1.737(0.179)
11	1.788(0.192)	1.411(0.156)	1.752(0.188)
12	1.793(0.203)	1.385(0.169)	1.752(0.204)
13	1.787(0.213)	1.356(0.178)	1.746(0.216)
14	1.776(0.226)	1.326(0.186)	1.739(0.227)
15	1.803(0.239)	1.330(0.198)	1.771(0.242)
16	1.774(0.247)	1.290(0.204)	1.740(0.249)
17	1.729(0.259)	1.232(0.212)	1.695(0.259)
18	1.696(0.268)	1.184(0.220)	1.664(0.273)
19	1.640(0.278)	1.122(0.225)	1.614(0.282)
20	1.598(0.289)	1.065(0.235)	1.577(0.296)
21	1.514(0.292)	0.979(0.237)	1.501(0.303)
22	1.458(0.303)	0.912(0.242)	1.451(0.315)
23	1.387(0.308)	0.836(0.245)	1.380(0.320)
24	1.345(0.313)	0.786(0.250)	1.341(0.330)
25	1.311(0.323)	0.739(0.256)	1.305(0.338)
26	1.308(0.334)	0.719(0.265)	1.306(0.352)
27	1.303(0.348)	0.698(0.274)	1.308(0.368)
28	1.300(0.356)	0.685(0.282)	1.308(0.376)
29	1.308(0.365)	NA NA	NA NA
30	1.323(0.377)	NA NA	NA NA

The numbers without blanket are means of fund ratios.

The numbers with blanket are standard deviations of fund ratios.

Table 4: Adjustment Terms

$adjD_{bw}$ in eq. (11), $adjG_{bw}$ in eq. (12), $adjD_{fw}$ in eq. (15), and $adjG_{fw}$ in eq.(16)

Time	CTM's model in Scenario 1		Our model in Scenario 1		Our model in Scenario 2	
	$adjD_{bw}$	$adjG_{bw}$	$AdjD_{fw}$	$adjG_{fw}$	$adjD_{fw}$	$adjG_{fw}$
1	6.71E-05	1.61E-10	4.31E-06	2.49E-11	2.22E-05	2.49E-11
2	5.92E-05	1.32E-10	3.50E-06	2.01E-11	1.88E-05	2.01E-11
3	5.21E-05	1.09E-10	2.84E-06	1.66E-11	1.60E-05	1.66E-11
4	4.59E-05	8.87E-11	2.33E-06	1.32E-11	1.35E-05	1.32E-11
5	4.04E-05	7.23E-11	1.88E-06	1.11E-11	1.16E-05	1.11E-11
6	3.56E-05	5.87E-11	1.53E-06	9.01E-12	9.94E-06	9.01E-12
7	3.14E-05	4.75E-11	1.25E-06	7.47E-12	8.53E-06	7.47E-12
8	2.76E-05	3.82E-11	1.07E-06	5.92E-12	7.27E-06	5.92E-12
9	2.42E-05	3.07E-11	9.35E-07	4.79E-12	6.26E-06	4.79E-12
10	2.10E-05	2.47E-11	8.57E-07	4.00E-12	5.51E-06	4.00E-12
11	1.80E-05	1.96E-11	7.75E-07	3.30E-12	4.81E-06	3.30E-12
12	1.52E-05	1.56E-11	6.93E-07	2.68E-12	4.16E-06	2.68E-12
13	1.28E-05	1.23E-11	6.04E-07	2.24E-12	3.64E-06	2.24E-12
14	1.09E-05	9.63E-12	5.46E-07	1.78E-12	3.14E-06	1.78E-12
15	9.20E-06	7.52E-12	5.07E-07	1.44E-12	2.74E-06	1.44E-12
16	7.71E-06	5.85E-12	4.67E-07	1.19E-12	2.40E-06	1.19E-12
17	6.36E-06	4.52E-12	4.32E-07	9.63E-13	2.10E-06	9.63E-13
18	5.22E-06	3.47E-12	4.01E-07	7.90E-13	1.85E-06	7.90E-13
19	4.24E-06	2.63E-12	3.73E-07	6.30E-13	1.61E-06	6.30E-13
20	3.40E-06	1.99E-12	3.47E-07	5.19E-13	1.42E-06	5.19E-13
21	2.68E-06	1.48E-12	3.17E-07	4.12E-13	1.24E-06	4.12E-13
22	2.07E-06	1.09E-12	2.87E-07	3.36E-13	1.08E-06	3.36E-13
23	1.57E-06	7.92E-13	2.60E-07	2.78E-13	9.53E-07	2.78E-13
24	1.16E-06	5.59E-13	2.32E-07	2.33E-13	8.40E-07	2.33E-13
25	8.22E-07	3.78E-13	2.08E-07	1.95E-13	7.42E-07	1.95E-13
26	5.49E-07	2.39E-13	1.83E-07	1.60E-13	6.47E-07	1.60E-13
27	3.26E-07	1.35E-13	1.62E-07	1.35E-13	5.69E-07	1.35E-13
28	1.46E-07	5.74E-14	1.43E-07	1.13E-13	4.99E-07	1.13E-13
29	0	0	NA	NA	NA	NA
30	NA	NA	NA	NA	NA	NA

The numbers are means of adjustment terms.

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V. Conclusions

This paper proposes a discrete dynamic programming model to solve pension funding in a stable level. By assuming an intertemporal stable contribution rate, we derive an algorithm to calculate the optimal contribution rate that requires less exogenous information and produces a more stable funding. Compared to traditional backward induction approaches, our model has the following advantages. First, it does not require a boundary condition or a time horizon for pension funding.

Second, it uses more reliable and realistic information and thus reduces possible prediction error that could result in a miscalculation in pension funding. Our simulation results further confirm that our model helps pension fund managers to make more stable contributions than the traditional algorithm does. Thus, our proposed model could further reduce the contribution risk for the defined benefit pension fund. However, there is a trade-off between contribution rate risk and solvency risk.

To fortify the contribution stability our model adopts the intertemporal stable condition which may somewhat weaken the solvency condition of pension fund. It is very important for pension fund managers to realize this trade-off and try to balance it appropriately in order to ensure the long-term security of pension funding.

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