Valuation and Hedging of Limited Price Indexed Liabilities

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British Actuarial Journal / Volume 10 / Issue 03 / August 2004, pp 627 - 663
DOI: 10.1017/S1357321700002713, Published online: 10 June 2011

B.A.J. **10**, III, 627-663 (2004)

VALUATION AND HEDGING OF LIMITED PRICE INDEXED LIABILITIES

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ABSTRACT

This paper considers the market or economic valuation and the hedging of Limited Price Indexed (LPI) liabilities. This involves finding optimal static and dynamic hedging strategies which minimise the riskiness of the investment portfolio relative to the liability.

In this paper we do not aim to find the perfect hedge in a perfect world. Instead, it is assumed that optimisation is restricted to three commonly used asset classes in pension funds: cash; long-term (or irredeemable) fixed-interest bonds; and long-dated index-linked bonds. The economic value of the liability is then defined as the value of the best matching portfolio using a mean/variance type of loss function. Specifically, we adopt the risk minimising approach of Föllmer & Sondermann (1986) and Schweizer & Föllmer (1988). Even with such a simple loss function, establishing the theoretically optimal solution can be difficult. We propose that a practical solution close to the theoretical optimum can be found using two approximations. First, we approximate the 'true' stochastic economic model by a vector autoregressive model of order one. Second, we use a sequence of linearisations to approximate non-linear by straightforward quadratic minimisation problems.

The proposed approach is illustrated with various numerical examples, and we compare the results of the approximately optimal hedging strategy with static strategies.

KEYWORDS

LPI Liability; Static Hedging; Regular Rebalancing; Dynamic Hedging; Risk Minimising Hedging

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1. Introduction

In this paper, we investigate some hedging methods for particular kinds of pension liability. Unlike short-term hedging problems (for example, derivative pricing, hedging and reserving), a pension fund normally requires a long-term view, because of the long-term nature of the liabilities. The levels of contribution income and benefit outgo are comparatively stable and predictable. Thus, pension fund investment strategies are less constrained by short-term considerations, allowing the actuary and fund managers to focus on long-term investment decisions. Pension fund income comes from the investment returns, and the employer's and the employees' contributions. A central feature of the decision making process is the choice of assets. These must strike the right balance between risk and return; for example, by maximising the expected return subject to an acceptable level of risk.

For a pension fund to meet a particular liability, it is important to choose the most appropriate investment strategy. We can investigate the effects of adopting a variety of investment strategies, and, in particular, establish the likelihood of meeting the fund's objectives. Nowadays, simulation can be used to determine the possible effects of various investment strategies. According to the results, actuaries can make well informed suggestions for future contribution rates and the expected surpluses from pension funds, and so choose a suitable investment strategy which provides the best balance between risk and return for a particular scheme. (See, for example, Huang, 2000; Cairns & Parker, 1997; Cairns, 2000; Dufresne, 1988, 1989, 1990; Haberman & Sung, 1994.)

Now, consider the valuation of liabilities and assets. Until recently, in the United Kingdom, values of both assets and liabilities were determined by discounting cash flows at an assumed rate, which is normally taken to represent the future long-term rate of return on investments. In recent years, ideas arising in financial economics have been applied to the valuation of insurance related liability when taken along side the market value of the assets. This is similar to estimating market prices for liabilities (the so-called *fair value* or *economic value*). This is discussed further by Head *et al.* (2000). They proposed various methods that take assets into the balance sheet at market value. The principle of fair value is discussed further by Cairns (2001).

From a practical point of view, pension funds are likely to review their investment strategies only periodically; for example, annually. This suggests the use of discrete time, stochastic investment models. In a discrete time model, the number of outcomes after each time step is usually infinite (for example, the Wilkie, 1995, model). This means that the market is *incomplete*; that is, few of a pension fund's liabilities can be precisely matched or *replicated*. (If a liability or financial derivative can be *replicated*, we mean the following. For the appropriate initial investment, in combination with a suitable dynamic hedging strategy using standard traded assets, we are able to reproduce *exactly* (that is, with certainty) the liability cash flow or the derivative payoff without the need for further injections of cash — positive or negative.) As a *benchmark*, it is desirable to find the hedging strategy which minimises the level of risk associated with a specific liability. (We stress, here, the word *benchmark*. This is intended as an objective point of reference. The objectives of the pension fund trustees may mean that a different investment mix from the benchmark is appropriate.) Where perfect hedging (matching) is possible (for example, Black & Scholes, 1973), the measure of risk is irrelevant. Under more realistic models, perfect hedging is not possible and the measure of risk is relevant. Here we use the concept of mean/variance hedging, well known to actuaries (see, for example, Wise, 1984a,b, 1987a,b, 1989; Wilkie, 1985; and Keel & Müller, 1995). However, mean/variance hedging is even more firmly established within the field of financial mathematics (see, for example, Musiela & Rutkowski, 1997, Chapter 4, and references therein). Alternative measures of risk include value-at-risk (in the present context this means *quantile hedging*; see Föllmer & Leukert, 1999, 2000) and semi-variance (Clarkson, 1995). In general, these different risk measures give rise to different hedging strategies and estimates of value. However, in many situations these differences are small, and not worth the considerable argument which rages around them. Cairns (2001) argues why, out of these, mean/variance hedging is, perhaps, the most appropriate choice, at least in a one-period setting.

In Section 2 we introduce the principles underlying mean/variance hedging: in particular, variance minimising hedging and local, risk minimising hedging. We propose that the value of the risk minimising portfolio be taken as the economic, or fair value, of a liability.

Section 3 defines the limited price indexation (LPI) liability. Valuation of this liability with reference to fixed and Retail Prices Index (RPI) linked liabilities is the main aim of the paper.

Section 4 describes the hedging strategies to be considered later on for pensions' liabilities. The valuation and hedging problem involves a combination of a potentially complex asset model and value functions which are non-linear in the state variables. This makes analytical solution of the optimal hedging problem virtually impossible. In order to circumvent this, we introduce two key forms of approximation: to the stochastic investment model (Section 5); and by linearisation of value functions (Sections 6 and 7).

Section 5 describes the vector-autoregressive model which we will use to determine liability values and hedging strategies. This model is proposed as an approximation to a more complex stochastic investment model (for example, Wilkie, 1995).

Sections 6 and 7 focus on valuation and hedging of fixed and RPI linked pensions. Here we introduce some linear approximations to the liabilities and hedging strategies (which are essential for us to make any progress), as a key step towards a practical solution for the optimal hedging problem. These provide overlays for the valuation of LPI liabilities. Consequently they play a key part in Section 8, in establishing values for the LPI liabilities. Finally, in Section 9 we use simulation to compare static hedging strategies with the (approximate) optimal hedging strategies.

2. Basic Principles

In order to develop our ideas further, we introduce some notation:

— $V(t)$ represents the economic value (also known as fair value or market value) of the liability at time t. How we calculate this will be discussed later.

— $F(t^-)$ represents the fund size just *before* any net injections of cash at time t.

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— $S_1(t)$ represents the value of a cash account at time t. In our discrete time model, $S_1(t+1)/S_1(t)$ equals the return over each year on the one-year, zero coupon bond (a quantity which is known at time t). It follows that $S_1(t+1)$ is known at time t. This gives rise to the commonly used notion that $S_1(t)$ represents the risk free investment; that is, risk free over each one-year time horizon.

— $S_k(t)$ for $k = 2, \ldots, m$ represents the value (with reinvestment of dividend or coupon income) of a unit investment at time 0 in risky asset k. (In this paper we will use $m = 3$.)

— We will assume that $S_i(0) = 1$ for all $i = 1, \ldots, m$.

— The vector $u(t) = (u_1(t), \ldots, u_m(t))'$ represents the numbers of units held in each of the assets $i = 1, \ldots, m$ from time t to $t+1$ for all $t = 0, \ldots, T-1$. Besides their dependence upon t, the $u_i(t)$ can depend upon the fund size at time t, current market conditions or on the history of the process up to and including time t.

Using this notation we have:

$$F(t^-) = \sum_{i=1}^{m} u_i(t-1)S_i(t)$$

just before rebalancing at time t, and

$$F(t) = \sum_{i=1}^{m} u_i(t)S_i(t)$$

just after rebalancing at time t.

An asset-allocation strategy is said to be *self-financing* if $F(t^-) = F(t)$ for $t = 1, 2, \ldots, T-1$; that is, at no time t is there a net inflow or outflow of cash.

There are two main approaches of mean/variance hedging (see, for example, Musiela & Rutkowski, 1997):

Variance Minimising Hedging

This type of hedging assumes that an investment strategy is *self-financing* (that is, there are no external injections or removals of cash except at the outset), and concentrates on minimising the *tracking error* at the terminal date only. In other words, we aim to minimise:

$$\mathrm{Var}\left(\frac{V(T) - F(T)}{S_1(T)}\right) \tag{2.1}$$

subject to:

$$\mathrm{E}\left(\frac{V(T) - F(T)}{S_1(T)}\right) = 0$$

by choosing an appropriate initial fund size $\hat{F}(0)$, and asset strategies $\hat{u}(t)$.

When we are following a variance minimising strategy, $\hat{F}(0)$ is regarded as the appropriate price or value of the liability being hedged. Between times 0 and T the fund size is not likely to be equal to the liability value at that time.

Risk Minimising Hedging

The second approach to establishing the economic value of a liability, by reference to an optimal hedging strategy, was proposed by Schweizer & Föllmer (1988) (see also Föllmer & Sondermann, 1986).

Valuation by reference to risk minimising hedging is more flexible, since it is not necessarily self-financing, and it gives us the value of the liability at all times $t = 0, \ldots, T-1$, rather than just at time 0. In this case, an optimality criterion is required at each date before the terminal date.

The values of the liabilities are determined backwards recursively. Let $V(t)$ be the economic value of the liability at time t. $F(t-1)$ represents the funds available at time $t-1$, made up of various numbers of units $u_i(t-1)$ in each of the assets.

We define $K(t)$ to be the shortfall (or tracking error) at time t; that is:

$$K(t) = V(t) - F(t^-) = V(t) - \sum_{i=1}^{m} u_i(t-1)S_i(t).$$

The local risk minimisation criterion requires that we minimise $\mathrm{Var}[K(t)|\mathcal{F}_{t-1}]$ over the $u_i(t-1)$ subject to $\mathrm{E}[K(t)|\mathcal{F}_{t-1}] = 0$ (where $\mathcal{F}_{t-1}$ represents the available information up to and including time $t-1$). (This is equivalent to the unconstrained minimisation of $\mathrm{E}[K(t)^2|\mathcal{F}_{t-1}]$, since we can always modify the amount invested in the cash account which is risk free over the period $t-1$ to t. Note also that, since $S_1(t)/S_1(t-1)$ is observable at time

$t-1$, the optimisation problem is the same as the minimisation of $\text{Var}[K(t)/S_1(t)|\mathcal{F}_{t-1}]$, subject to $\text{E}[K(t)/S_1(t)|\mathcal{F}_{t-1}] = 0$.) Schweizer & Föllmer (1988) refer to this as 'hedging by sequential regression', because optimisation of the $u_i(t-1)$ is equivalent to solving a linear regression problem.

The optimal $\hat{F}(t-1)$ (given $\mathcal{F}_{t-1}$) then provides us with our liability value at time $t-1$; that is, we choose to define $V(t-1) \equiv \hat{F}(t-1)$. (In a complete market, there will always exist suitable processes $V(t)$ and $u(t)$ such that the optimised $K(t)$ are all equal to zero with certainty. Under such circumstances the optimal strategy will be self financing and replicating, and the economic or market value of the liability is unambiguous and equal to the risk minimising price. In an incomplete market the definition of what we mean by economic value of the liability is ambiguous, and a number of different definitions exist, including the risk minimising price. However, in certain circumstances (see, for example, Cairns, 2001) the definition of the risk minimising price does coincide with the equilibrium price in a liquid market. This provides strong support for the use of the risk minimising price as a candidate for the economic value in the more general circumstances we find ourselves in here.)

At each time t, additional finance of $K(t)$ is provided, to ensure that the fund size is at all times equal to the value of the liability; that is $F(t) = F(t^-) + K(t) = V(t)$. Thus, in contrast to variance minimising hedging, this strategy is not self financing.

3. The Definition of LPI

The valuation of the accrued liability is an important part of a pension scheme funding valuation. In recent years, Limited Price Indexation (LPI) has, by law, become a necessary feature of United Kingdom pension schemes. There are several forms of LPI, and we describe two of them:

— *Type 1* is the limited indexation of pensions in deferment introduced in the 1986 Pensions Act. Deferred pensions of early leavers from final salary pension schemes are increased at the lesser of some cap rate (U.K. 5% p.a.) compounded over the whole period, and the actual increase in the RPI in the U.K. (consumer prices index in a more international context), again measured over the full period. For an early leaver at time t retiring at a later time T, the pension payable at a time T will be:

$$pen(T) = \min\left\{\frac{RPI(T)}{RPI(t)}, 1.05^{T-t}\right\} \times pen(t)$$

where $pen(t)$ is the deferred pension calculated at the date of exit before statutory revaluation and $RPI(t)$ is the value of the RPI at time t.

— *Type 2* is the limited indexation of pensions once in payment, introduced under the 1990 Social Security Act. Under this type of indexation, a comparison is made, year-on-year, between the cap rate (U.K. 5% p.a.) and the annual increase in the RPI, and the lesser of the two increases is awarded. The pension at time t, $pen(t)$, is then:

$$pen(t) = pen(t-1) \times \min\left\{\frac{RPI(t)}{RPI(t-1)}, 1.05\right\}.$$

For some cases, we can additionally apply a floor (such as 0% p.a. increases) to Types 1 and 2 as well as a ceiling. We will not consider this possibility, because breaches of typical floors are not common in practice. In contrast to liabilities in respect of active employees, LPI liabilities are well defined (the former are subject to argument over the extent of salary risk and over the division between past and future service liabilities).

4. Hedging Strategies Considered

4.1 *Static Hedging*

In this section we investigate the use of static hedging to establish a first approximation to the value of an LPI liability. The aim is to achieve this using the standard asset classes used by pension funds, rather than achieve perfect matching of assets and liabilities using a more complex and detailed range of bonds. Here, static hedging means that we hold a fixed quantity of each asset over the full term n (that is, buy and hold). In investigating the LPI liability, we consider long-dated index-linked bonds, consols (that is, U.K., irredeemable, fixed-interest bonds) and cash in the portfolio. Compared to index-linked bonds, consols may be more volatile relative to inflation. However, since LPI includes a fixed cap, consols may be found to provide a useful asset for hedging outcomes where the cap locks in (see, for example, Cairns, 1999).

An LPI liability is closely correlated with two other liabilities; that is, an LPI liability is overlaid by both a liability with fixed percentage increases (FP) and a liability with increases in line with the RPI. We can conclude that it may be useful to take account of the pricing and hedging of these other liabilities first, before tackling the LPI problem.

For an FP liability, the pension in payment is subject to fixed percentage increases r each year; that is:

$$pen(t) = (1+r)^t \times pen(0).$$

For an RPI liability, the pension in payment increases in line with the retail prices index; that is:

$$pen(t) = \frac{RPI(t)}{RPI(0)} \times pen(0).$$

For an LPI liability, the pension in payment $pen(t)$ is governed by the equation:

$$pen(t) = pen(t-1) \times \min\left\{\frac{RPI(t)}{RPI(t-1)}, 1+r\right\}.$$

We will assume here that $r = 0.05$ or 5%, in line with U.K. regulations.

With an FP liability, we consider three cases of investment strategies in the portfolio:
— holding cash only in the portfolio;
— holding consols only in the portfolio; and
— holding cash, consols and index-linked bonds in the portfolio.

With an RPI liability, we consider two investment strategies in the portfolio:
— holding index-linked bonds only in the portfolio; and
— holding cash, consols and index-linked bonds in the portfolio.

With an LPI liability, we consider only one case in the portfolio:
— holding cash, consols and index-linked bonds in the portfolio.

Let $S_1(t), S_2(t), S_3(t)$ be the prices at time t of one unit of a cash account (invested in one-year zero-coupon bonds), an irredeemable fixed-interest bond (consols) and an irredeemable index-linked bond respectively, with $S_i(0) = 1$ for $i = 1, 2, 3$. (Strictly, $S_2(t), S_3(t)$ are just two risky assets. However, in the vector-autoregressive model which we will introduce, in Section 5, their dynamics are determined in a way which is consistent with their labels as fixed-interest and index-linked bonds.) We now define: $R_i(t) = S_i(t)/S_i(t-1) =$ total return on asset i from $t-1$ to t, with $i = 1$ (cash), $i = 2$ (consols) and $i = 3$ (index-linked bonds).

We also define $M^\alpha(t) =$ liability index of type α, where $\alpha = F, R, L$ represents the type of liability (fixed, RPI and LPI respectively). Thus, starting with $M^\alpha(0) = 1$, we have, for $t = 1, 2, \ldots$:

$$M^\alpha(t) = \begin{cases} (1+r)M^\alpha(t-1) = (1+r)^t & \text{for } \alpha = F \\ \frac{RPI(t)}{RPI(t-1)} M^\alpha(t-1) = \frac{RPI(t)}{RPI(0)} & \text{for } \alpha = R \\ \min\left(\frac{RPI(t)}{RPI(t-1)}, 1+r\right) M^\alpha(t-1) & \text{for } \alpha = L. \end{cases} \tag{4.2}$$

We now define the objective function for a liability of $M^{\alpha}(T)$ due at time T:

$$SB = \mathrm{Var}\left(\frac{\sum_{i=1}^{3} x_i^{\alpha} S_i(T) - M^{\alpha}(T)}{S_1(T)}\right) + \theta\left(\mathrm{E}\left[\frac{\sum_{i=1}^{3} x_i^{\alpha} S_i(T) - M^{\alpha}(T)}{S_1(T)}\right]\right)^2. \quad (4.3)$$

We minimise SB over the x_i^{α} (the amounts invested at time 0 in each asset) to obtain the optimal asset allocation strategy. This approach is similar to that of Wise (1984a,b, 1987a,b, 1989) and Wilkie (1985).

We can immediately note that x_1^{α} has no impact on the variance:

$$\mathrm{Var}\left(\left[\sum_{i=1}^{3} x_i^{\alpha} S_i(T) - M^{\alpha}(T)\right]/S_1(T)\right).$$

Thus, if $\theta > 0$, then the optimal solution is to minimise the variance over x_2^{α} and x_3^{α} first, before choosing x_1^{α} in a way which ensures that the expectation $\mathrm{E}\big(\big[\sum_{i=1}^{3} x_i^{\alpha} S_i(T) - M^{\alpha}(T)\big]/S_1(T)\big) = 0$. There is no unique solution when $\theta = 0$.

When $\theta > 0$, the optimal solutions are all the same, since, for any x_2^{α} and x_3^{α}, we are always able to find a value for x_1^{α} which results in the second component of SB being equal to zero. This means that we get the same optimal strategy for all $\theta > 0$. We will assume, therefore, that $\theta = 1$, without loss of generality.

If $\theta = 1$, the objective SB is to minimise the expected value of the square of the tracking error; that is:

$$\mathrm{E}\left[\left(\frac{\sum_{i=1}^{3} x_i^{\alpha} S_i(T) - M^{\alpha}(T)}{S_1(T)}\right)^2\right].$$

This optimisation problem will result in a variance minimising hedging strategy, subject to constraints on the investment strategy.

For notational convenience, we define, for $i, j = 1, 2, 3$ and $\alpha = F, R, L$:

$$\psi_i = \mathrm{E}\left[\frac{S_i(T)}{S_1(T)}\right]$$

$$\omega_{i,j} = \mathrm{Cov}\left(\frac{S_i(T)}{S_1(T)}, \frac{S_j(T)}{S_1(T)}\right)$$

$$\gamma_i^{\alpha} = \mathrm{Cov}\left(\frac{S_i(T)}{S_1(T)}, \frac{M^{\alpha}(T)}{S_1(T)}\right)$$

$$\psi_0^\alpha = \mathrm{E}\left[\frac{M^\alpha(T)}{S_1(T)}\right]$$

$$\omega_0^\alpha = \mathrm{Var}\left[\frac{M^\alpha(T)}{S_1(T)}\right]$$

$$X^\alpha = (x_1^\alpha, x_2^\alpha, x_3^\alpha)'$$

$$\Omega = \left(\omega_{i,j}\right)_{i,j=1}^3$$

$$\Gamma^\alpha = \left(\gamma_1^\alpha, \gamma_2^\alpha, \gamma_3^\alpha\right)'$$

$$\Psi = \left(\psi_1, \psi_2, \psi_3\right)'.$$

SB is quadratic in x_1, x_2 and x_3; that is:

$$SB = X^{\alpha\prime}\Omega X^\alpha - 2X^{\alpha\prime}\Gamma^\alpha + \omega_0^\alpha + \left(X^{\alpha\prime}\Psi - \psi_0^\alpha\right)^2. \tag{4.4}$$

Now $\omega_{ij} = 0$ and $\gamma_i^\alpha = 0$ whenever $i = 1$ or $j = 1$. It follows that this problem is solved by optimising first over x_2 and x_3, which gives us:

$$\begin{pmatrix} x_2^\alpha \\ x_3^\alpha \end{pmatrix} = \hat{\Omega}^{-1}\hat{\Gamma}^\alpha \tag{4.5}$$

where:

$$\hat{\Omega} = \begin{pmatrix} \omega_{22} & \omega_{23} \\ \omega_{32} & \omega_{33} \end{pmatrix}$$

and

$$\hat{\Gamma}^\alpha = \begin{pmatrix} \gamma_2^\alpha \\ \gamma_3^\alpha \end{pmatrix}.$$

We then optimise over x_1, and get:

$$x_1^\alpha = \psi_1^{-1}\left(\psi_0^\alpha - \sum_{i=2}^{3} x_i^\alpha \psi_i\right) \tag{4.6}$$

which ensures that $\left(X^{\alpha\prime}\Psi - \psi_0^\alpha\right) = 0$.

4.2 *Dynamic Hedging*

4.2.1 *Introduction*

We generally find that the optimal static hedge has relative proportions in each asset which vary with time to payment T and also with the prevailing economic conditions at time zero. This indicates that hedging strategies which vary over both time and with changing economic conditions will outperform static hedges.

The particular economic variables which we will make use of are as follows:

$$\begin{aligned}
y_1(t) &= \text{annualised inflation rate from time } t-1 \text{ up to time } t \\
y_2(t) &= \text{(historical) log dividend yield at time } t \\
y_3(t) &= \text{log consols yield at time } t \\
y_4(t) &= 1 \Big/ \text{price at } t \text{ for zero-coupon bond maturing at } t+1 \\
&= 1 + \text{the risk free rate of interest from } t \text{ to t+1} \\
y_5(t) &= \text{log real yield on index-linked bonds at } t \\
y(t) &= (y_1(t), \ldots, y_5(t))'.
\end{aligned} \tag{4.7}$$

We will assume, in this paper, that the model for $y(t)$ is Markov, and also that it is time homogeneous; that is, the distribution of $(y(t_1), \ldots, y(t_k))$, given $y(0) = y$ for any integers $0 < t_1 < \ldots, t_k$, is the same as the distribution of $(y(t_1 + s), \ldots, y(t_k + s))$, given $y(s) = y$ for all integers $s > 0$.

This choice of economic factors reflects the later use in this paper of the Wilkie (1995) model. Specifically, the equity dividend yield is included here, because of its importance within the cascade structure of the Wilkie model. For other models, other factors may be appropriate.

Now, recall that $M^\alpha(t)$ equals the amount of a type-α liability payable at time t (see equation 4.2). These expressions can now be defined in terms of the $y(t)$:

$$M^\alpha(t) = \begin{cases} (1+r)^t & \text{for } \alpha = F \\ \prod_{s=1}^{t}(1 + y_1(s)) & \text{for } \alpha = R \\ \prod_{s=1}^{t} \min\{1 + y_1(s), 1 + r\} & \text{for } \alpha = L. \end{cases} \tag{4.8}$$

4.2.2 *True optimisation for dynamic hedging*

For an LPI liability, we consider cash, consols and irredeemable index-linked bonds as the available assets for hedging. Index-linked bonds and consols are risky assets in the portfolio; that is, unlike the cash account (one-year zero-coupon bonds), the values of these investments are not known one year in advance. We suppose that the economic model governing future liabilities and asset returns is Markov and time homogeneous. We then let the vector $y(t)$ represent relevant market conditions at time t; that is, knowledge of $y(t)$ is sufficient for probability forecasts of the future.

Recall that $R_i(t)$ is the total return on asset i from $t-1$ to t. Define the vector $R(t) = (R_1(t), R_2(t), R_3(t))'$.

Let $\tilde{x}_j^\alpha(t, y(t), T-t)$ be the optimal amount of asset j for a true optimisation (based on risk minimising hedging) at time t for a type-α liability due in $T-t$ years, given market conditions $y(t)$ at time t.

Also, let $\tilde{V}^\alpha(t, y(t), T-t)$ be the economic value at t for the liability $M^\alpha(T)$ payable at time T, given $y(t)$. By definition (using the Schweizer & Föllmer, 1988, definition of fair value, discussed in Section 2), $\tilde{V}^\alpha(t, y(t), T-t) = \sum_{j=1}^{3} \tilde{x}_j^\alpha(t, y(t), T-t)$.

Now, the economic model $y(t)$ is Markov and time homogeneous. This means that, for each $s = 1, 2, \ldots,$ the distribution of the vector $y(t+s)$, given $y(t) = y$, is the same as the distribution of the vector $y(s)$, given $y(0) = y$. This implies that, for each $s = 1, 2, \ldots,$ $M^\alpha(t+s)/M^\alpha(t)$, given $y(t) = y$, has the same distribution as $M^\alpha(s)/M^\alpha(0) = M^\alpha(s)$, given $y(0) = y$.

Consequently, if we take as given $y(t) = y$ and $M^\alpha(t) = m$, we have the result:

$$\tilde{V}^\alpha(t, y(t), T-t) = m \times \tilde{V}^\alpha(0, y(t), T-t) \tag{4.9}$$

meaning that it is sufficient for us to establish the form of $\tilde{V}^\alpha(0, y, T-t)$ only. We also note that, given $y(t) = y$, $M^\alpha(t) = M^\alpha$, we can write $M^\alpha(t+1) \stackrel{\mathcal{D}}{=} M^\alpha \times M^\alpha(1)$, given $y(0) = y$.

The $\tilde{x}_i^\alpha(t, y(t), T-t)$ and $\tilde{V}^\alpha(t, y(t), T-t)$ are established by means of a backwards recursion starting at $t = T-1$, and stepping backwards a year at a time to $t = 0$. Thus, for a general t, we let the $\tilde{x}_i^\alpha(t, y(t), T-t)$ be the values of x_1, x_2, x_3 that minimise:

$$\mathrm{E}\left[\left(\sum_{i=1}^{3} x_i R_i(t+1) - \tilde{V}^\alpha(t+1, y(t+1), T-t-1)\right)^2 \middle| y(t) = y, M^\alpha(t) = m\right].$$

Using the Markov property with $M^\alpha(t) = m$, this is equivalent to the minimisation of:

$$\mathrm{E}\left[\left(\sum_{i=1}^{3} m x_i R_i(1) - m M^\alpha(1) \tilde{V}^\alpha(0, y(1), T-t-1)\right)^2 \middle| y(0) = y\right].$$

The advantage of this second optimisation problem is that we have reduced the dimension of the problem by one.

To develop formulae for the optimal asset allocation for the LPI liability, we need to specify some model for the economic variables $y(t)$ and $R(t)$.

5. Model Structure and Assumptions for Hedging

In this section we use multivariate regression to build a vector autoregressive model of order one for the economic variables and a linear estimated liability model.

5.1 *The Vector Autoregressive Model*

Let us suppose that there is some underlying stochastic economic model, such as the Wilkie (1995) model or the TY model (Yakoubov *et al.*, 1999). Often such models are sufficiently complex to render optimisation infeasible. Here, we propose the use of a simple vector autoregressive model (VAR(1)) as an approximation to these more complex models. The VAR model is fitted by the use of multivariate regression on simulated data generated by the more complex model. For the three assets under consideration, the Wilkie (1995) model requires the five drivers $(y_1(t), \ldots, y_5(t))$ defined in equation (4.7).

Consider a long simulation run using the underlying stochastic economic model running from time 0 to time N. This gives us values for the market indicators $y_i(t)$, for $i = 1, \ldots, 5$ and $t = 0, 1, \ldots, N$, and for the total returns $R_i(t)$, for $i = 1, 2, 3$ and $t = 1, \ldots, N$. Let:

$$X(t) = (X_1(t), \ldots, X_5(t))'$$
$$Y(t) = (Y_1(t), \ldots, Y_8(t))'$$

where:

$$X_i(t) = y_i(t-1) \quad \text{for } i = 1, \ldots, 5$$

and

$$Y_i(t) = \begin{cases} y_i(t) & \text{for } i = 1, \ldots, 5 \\ R_{i-5}(t) & \text{for } i = 6, 7, 8. \end{cases}$$

Let μ_{yi} be the unconditional mean of $X_i(t)$ and μ_{Yi} be the unconditional mean of $Y_i(t)$. For $i = 1, \ldots, 5$, we have $\mu_{Yi} = \mu_{yi} = \mathrm{E}[y_i(t)]$ (the unconditional expectation), and, for $i = 6, 7, 8$, we have $\mu_{Yi} = \mu_{R,i-5} = \mathrm{E}[R_{i-5}(t)]$.

This leads us to the following mutivariate regression model:

$$Y(t) - \mu_Y = A(X(t) - \mu_y) + \epsilon(t) \quad \text{for } t = 1, \ldots, N$$

where $A = (A_{ij})$ is an 8×5 matrix, and the random innovations denoted by $\epsilon(t) = (\epsilon_1(t), \ldots, \epsilon_8(t))'$ are i.i.d. multivariate normal vectors with mean $(0, \ldots, 0)'$ and 8×8 covariance matrix $C_\epsilon = (C_{ij})$.

For convenience later, we will write:

$$A = \begin{pmatrix} A_y \\ A_R \end{pmatrix} \qquad \mu_Y = \begin{pmatrix} \mu_y \\ \mu_R \end{pmatrix} \qquad \epsilon(t) = \begin{pmatrix} \epsilon^{(y)}(t) \\ \epsilon^{(R)}(t) \end{pmatrix}$$

where the 5×5 matrix A_y is the first five rows of A, the 3×5 matrix A_R is the last three rows of A, the vector $\epsilon^{(y)}(t)$ is the first five elements of $\epsilon(t)$, and the vector $\epsilon^{(R)}(t)$ the last three elements of $\epsilon(t)$. We can then write:

$$y(t) = \mu_y + A_y[y(t-1) - \mu_y] + \epsilon^{(y)}(t)$$

and

$$R(t) = \mu_R + A_R[y(t-1) - \mu_y] + \epsilon^{(R)}(t). \tag{5.10}$$

For the process to be stationary, we require that the eigenvalues of A_y should all have magnitude less than one (see, for example, Wei, 1990).

Now, let $U = (U_{ij})$ be the 5×5 simulation covariance matrix for $X(t)$ and $W = (W_{ij})$ be the 8×5 simulation covariance matrix for $Y(t)$ and $X(t)$; that is:

$$U_{ij} = \frac{1}{N}\sum_{k=1}^{N}(X_i(k) - \hat{\mu}_{yi})(X_j(k) - \hat{\mu}_{yj}) \quad \text{for } i, j = 1, \ldots, 5$$
$$W_{ij} = \frac{1}{N}\sum_{k=1}^{N}(Y_i(k) - \hat{\mu}_{Yi})(X_j(k) - \hat{\mu}_{yj}) \quad \text{for } i = 1, \ldots, 8 \text{ and } j = 1, \ldots, 5$$

where:

$$\hat{\mu}_{yi} = \frac{1}{N}\sum_{k=1}^{N} X_i(k) \quad \text{for } i = 1, \ldots, 5$$

and

$$\hat{\mu}_{Yi} = \begin{cases} \hat{\mu}_{yi} & \text{for } i = 1, \ldots, 5 \\ \frac{1}{N}\sum_{k=1}^{N} Y_i(k) & \text{for } i = 6, 7, 8. \end{cases}$$

Then (for example, see Srivastava & Carter, 1983), the 8×5 matrix A has the estimate:

$$\hat{A} = WU^{-1}.$$

Now, let $\hat{\epsilon}(k) = Y(k) - \hat{\mu}_Y - \hat{A}(X(k) - \hat{\mu}_y)$. Then the 8×8 matrix C_ϵ has the estimate:

$$\hat{C}_\epsilon = (\hat{C}_{ij}) = \frac{1}{N}\sum_{k=1}^{N} \hat{\epsilon}(k)\hat{\epsilon}(k)'$$

or

$$\hat{C}_{ij} = \frac{1}{N}\sum_{k=1}^{N} \hat{\epsilon}_i(k)\hat{\epsilon}_j(k) \quad \text{for } i, j = 1, \ldots, 8.$$

6. Finding the Optimal Asset Allocation

In this section we consider the fixed and RPI liabilities ($\alpha = F$ and $\alpha = R$ only). Our aim is to establish the optimal value function $\tilde{V}(t, y, T-t)$. From the practical point of view, this is almost an impossible task. In particular, suppose that we have a relatively complex model, where the state variable y has several dimensions. Typically (as we have in the present context), no analytical formula exists for $\tilde{V}(t, y, T-t)$, meaning that any numerical method must evaluate $\tilde{V}$ at every single value of y before we can move one step backwards. In order to avoid this problem, we will approximate the true, optimal solution by a function which is linear in y, and which is much easier to evaluate. A second linearisation is then employed for each t, to help to derive the optimal hedging strategy for the previous time step. Thus, we have a sequence of linear approximations as we step backwards down the time line. We will now describe this procedure in more detail. (Readers who are more interested in the numerical results may choose to skip forward to Section 7 and return to this theoretical section later.)

Previously, we defined the theoretical optima $\tilde{x}_i^\alpha(t, y(t), T-t)$ and $\tilde{V}^\alpha(t, y(t), T-t)$. We will begin now to develop some approximations, which will result in a set of allocations $\bar{x}_i^\alpha(t, y(t), T-t)$, which will optimise a linearised version of the one-step valuation problem. These then give us an approximation to the true economic value by defining:

$$\bar{V}^\alpha(t, y(t), T-t) = \sum_{i=1}^{3} \bar{x}_i^\alpha(t, y(t), T-t).$$

As before, the Markov, time homogeneous structure of the model implies that, given $y(t) = y$ and $M^\alpha(t) = m$:

$$\bar{x}_i^\alpha(t, y, T-t) = m\bar{x}_i^\alpha(0, y, T-t) \quad \text{for } i = 1, 2, 3$$

and

$$\bar{V}^{\alpha}(t, y, T-t) = m\bar{V}^{\alpha}(0, y, T-t).$$

We derive these functions by backwards recursion. Suppose, then, that the form of $\bar{V}^{\alpha}(0, y, T-t)$ is known for all y and for all $0 \leq s \leq T-t$.

Step 1. Find $\bar{x}_i^{\alpha}(t-1, y(t-1), T-t+1)$

For a general t, we first aim to find the $\bar{x}_i^{\alpha}(t-1, y(t-1), T-t+1)$ for $i = 1, 2, 3$, which minimise:

$$\mathrm{E}\left[\left(\sum_{i=1}^{3} \bar{x}_i^{\alpha}(t-1, y, T-t+1)R_i(t) - \bar{V}_i^{\alpha}(t, y(t), T-t)\right)^2 \middle| y(t-1) = y\right].$$

Since $\bar{x}_i^{\alpha}(t-1, y(t-1), T-t+1) = M^{\alpha}(t-1)\bar{x}_i^{\alpha}(0, y(t-1), T-t+1)$ and $\bar{V}^{\alpha}(t, y(t), T-t) = M^{\alpha}(t)\bar{V}^{\alpha}(0, y(t), T-t)$, this is equivalent to minimisation over the $\bar{x}_i^{\alpha}$ of:

$$\mathrm{E}\left[\left(\sum_{i=1}^{3} \bar{x}_i^{\alpha}(0, y, T-t+1)R_i(1) - M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t)\right)^2 \middle| y(0) = y\right]$$

where $M^{\alpha}(1)$ equals either $1+r$ (FP), or $1+y_1(1)$ (RPI), or $\min\{1+r, 1+y_1(1)\}$ (LPI).

For the purpose of constructing formulae for the optimal asset allocation, we need to linearise $M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t)$; that is:

$$M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t) \approx \hat{a}^{\alpha}_{T-t}(y(1)-\mu_y) + \hat{b}^{\alpha}_{T-t}$$

where $\hat{a}^{\alpha}_{T-t}$ is a row vector, where the jth component is:

$$\hat{a}^{\alpha}_{j,T-t} = \frac{\partial}{\partial y_j(1)}\Big(M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t)\Big)\Big|_{y(1)=\mu_y} \tag{6.11}$$

and $\hat{b}^{\alpha}_{T-t}$ is the scalar:

$$\hat{b}^{\alpha}_{T-t} = M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t)\Big|_{y(1)=\mu_y}. \tag{6.12}$$

It would be useful to be able to quantify the accuracy of this approximation. However, the complexity of the problem, even with a relatively simple model as we have here, has left us unable to achieve this aim even over just two time steps. Despite this, we believe that the approximation is reasonable, and we comment further on this in Section 9.

Since $R_1(1)$ (the return on cash) is known at time 0, for any $\bar{x}_2^\alpha \equiv \bar{x}_2^\alpha(0, y, T-t+1)$ and $\bar{x}_3^\alpha \equiv \bar{x}_3^\alpha(0, y, T-t+1)$ we can find an $\bar{x}_1^\alpha \equiv \bar{x}_1^\alpha(0, y, T-t+1)$, for which:

$$\mathrm{E}\left[\sum_{i=1}^{3} \bar{x}_i^\alpha R_i(1) - \left\{\hat{a}_{T-t}^\alpha(y(1)-\mu_y) - \hat{b}_{T-t}^\alpha\right\}\middle| y(0) = y\right] = 0.$$

Thus, step 1 is equivalent to minimisation over $\bar{x}_2^\alpha$ and $\bar{x}_3^\alpha$ of:

$$\mathrm{Var}\left[\sum_{i=2}^{3} \bar{x}_i^\alpha R_i(1) - \left\{\hat{a}_{T-t}^\alpha(y(1)-\mu_y) - \hat{b}_{T-t}^\alpha\right\}\middle| y(0) = y\right] \tag{6.13}$$

with $\bar{x}_1^\alpha \equiv \bar{x}_1^\alpha(0, y, T-t+1)$ chosen subsequently to satisfy:

$$\mathrm{E}\left[\sum_{i=1}^{3} \bar{x}_i^\alpha R_i(1) - \left\{\hat{a}_{T-t}^\alpha(y(1)-\mu_y) - \hat{b}_{T-t}^\alpha\right\}\middle| y(0) = y\right] = 0.$$

Then, given $y(t-1) = y$ and $M^\alpha(t-1) = m$, we have:

$$\bar{x}_2^\alpha(t-1, y, T-t+1) = m\bar{x}_2^\alpha(0, y, T-t+1).$$

Recall (equation 5.10) that the multivariate regression model is given by:

$$y(t) = \mu_y + A_y[y(t-1) - \mu_y] + \epsilon^{(y)}(t)$$

and

$$R(t) = \mu_R + A_R[y(t-1) - \mu_y] + \epsilon^{(R)}(t)$$

where:

$$\epsilon(t) = \begin{pmatrix} \epsilon^{(y)}(t) \\ \epsilon^{(R)}(t) \end{pmatrix} \sim MVN(0, C_\epsilon).$$

Equation (6.13) can be rewritten as:

$$\begin{aligned}
&\mathrm{Var}\left(\bar{x}_2^\alpha\epsilon_7(1) + \bar{x}_3^\alpha\epsilon_8(1) - \hat{a}_{T-t}^\alpha\epsilon^{(y)}(1)\right) \\
&= (-1, \bar{x}_2^\alpha, \bar{x}_3^\alpha)\begin{pmatrix} \tilde{C}_{11}(T-t+1) & \tilde{C}_{17}(T-t+1) & \tilde{C}_{18}(T-t+1) \\ \tilde{C}_{17}(T-t+1) & C_{77} & C_{78} \\ \tilde{C}_{18}(T-t+1) & C_{78} & C_{88} \end{pmatrix}\begin{pmatrix} -1 \\ \bar{x}_2^\alpha \\ \bar{x}_3^\alpha \end{pmatrix}
\end{aligned}$$

where the optimal values for $\bar{x}_2^\alpha$ and $\bar{x}_3^\alpha$ give us $\bar{x}_i^\alpha(0, y, T-t+1)$ for $i = 2, 3$.

For convenience, we write $X = (\bar{x}_2^\alpha, \bar{x}_3^\alpha)'$:

$$\tilde{C}_{11}(T-t+1) = \hat{a}_{T-t}^\alpha \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{pmatrix} \hat{a}_{T-t}^{\alpha\,\prime}$$

$$\tilde{C}_{17}(T-t+1) = \hat{a}_{T-t}^\alpha \begin{pmatrix} C_{17} \\ C_{27} \\ C_{37} \\ C_{47} \\ C_{57} \end{pmatrix} \qquad \text{and} \qquad \tilde{C}_{18}(T-t+1) = \hat{a}_{T-t}^\alpha \begin{pmatrix} C_{18} \\ C_{28} \\ C_{38} \\ C_{48} \\ C_{58} \end{pmatrix}.$$

In order to develop the formula for the asset allocation $\bar{x}_i^\alpha(0, y, T-t+1)$, we define:

$$C_X = \begin{pmatrix} C_{77} & C_{78} \\ C_{78} & C_{88} \end{pmatrix} \qquad \text{and} \qquad h_X(T-t+1) = -\begin{pmatrix} \tilde{C}_{17}(T-t+1) \\ \tilde{C}_{18}(T-t+1) \end{pmatrix}.$$

Then, the objective function becomes:

$$\begin{aligned} f(X) &= \mathrm{Var}\left(\sum_{i=2}^{3} \bar{x}_i^\alpha R_i(1) - \hat{a}_{T-t}^\alpha (y(1) - \mu_y) - \hat{b}_{T-t}^\alpha \,\middle|\, y(0) = y \right) \\ &= X' C_X X + 2 h_X(T-t+1)' X + \tilde{C}_{11}(T-t+1). \end{aligned}$$

To establish the optimal values $\bar{x}_2^\alpha(0, y, T-t+1)$ and $\bar{x}_3^\alpha(0, y, T-t+1)$, we minimise the objective function $f(X)$ over $\bar{x}_2^\alpha$ and $\bar{x}_3^\alpha$; that is:

$$\frac{\partial f(X)}{\partial X} = 2 C_X X + 2 h_X(T-t+1) = 0.$$

We then obtain the optimal asset allocation of consols and index-linked bonds at time zero as follows:

$$\hat{X} = \begin{pmatrix} \bar{x}_2^\alpha(0, y, T-t+1) \\ \bar{x}_3^\alpha(0, y, T-t+1) \end{pmatrix} = -C_X^{-1} h_X(T-t+1).$$

We note that C_X is constant and that $h_X(T-t+1)$ depends upon $T-t+1$ and μ_y, but not on y. It follows that $\bar{x}_2^{\alpha}(0, y, T-t+1)$ and $\bar{x}_3^{\alpha}(0, y, T-t+1)$ depend upon $T-t+1$ and μ_y only, and not on y. This lack of dependence on y happens because of the linear approximation in the one-year ahead estimate of the liability. This means that the variance of the hedging error at the end of the present year, given x_2 and x_3, is independent of the current value of y. Thus, the values of x_2 and x_3, which minimise the variance of the hedging error, do not depend on the current value of y.

With the values of $\bar{x}_2^{\alpha}(0, y, T-t+1)$ and $\bar{x}_3^{\alpha}(0, y, T-t+1)$ established, we can find $\bar{x}_1^{\alpha}(0, y, T-t+1)$ to satisfy the identity:

$$\mathrm{E}\left[\sum_{i=1}^{3} \bar{x}_i^{\alpha}(0, y, T-t+1)R_i(1) - \left\{\hat{a}_{T-t}^{\alpha}\big(y(1)-\mu_y\big) + \hat{b}_{T-t}^{\alpha}\right\}\middle| y(0) = y\right] = 0.$$

Hence, the optimal asset allocation of cash at time $t-1$ is:

$$\begin{aligned}
\bar{x}_1^{\alpha}(t-1, y, T-t+1) &= M^{\alpha}(t-1)\bar{x}_1^{\alpha}(0, y, T-t+1) \\
&= \frac{M^{\alpha}(t-1)}{\mathrm{E}[R_1(1)|y(0)=y]}\Big[\big(\hat{a}_{T-t}^{\alpha}\mathrm{E}[y(1)-\mu_y|y(0)=y] + \hat{b}_{T-t}^{\alpha}\big) \\
&\quad - (\bar{x}_2^{\alpha}(0, y, T-t+1)\mathrm{E}[R_2(1)|y(0)=y]) \\
&\quad - (\bar{x}_3^{\alpha}(0, y, T-t+1)\mathrm{E}[R_3(1)|y(0)=y])\Big]
\end{aligned}$$

where:

$$\begin{aligned}
\mathrm{E}[R_1(1)|y(0)=y] &= [y]_4 \\
\mathrm{E}[R_2(1)|y(0)=y] &= \big[\mu_R + A_R(y-\mu_y)\big]_2 \\
\mathrm{E}[R_3(1)|y(0)=y] &= \big[\mu_R + A_R(y-\mu_y)\big]_3 \\
\mathrm{E}[y(1)-\mu_y|y(0)=y] &= A_y(y-\mu_y)
\end{aligned}$$

where $[\ldots]_i$ represents the value of the ith component of the vector $[\ldots]$ and A_y and A_R are the regression matrices in equation (5.10).

In contrast to $\bar{x}_2$ and $\bar{x}_3$, it is clear that $\bar{x}_1^{\alpha}(t-1, y(t-1), T-t+1)$ does depend on $y(t-1)$.

Step 2. Set up the estimated economic value of the liability

Once we have determined the optimal asset allocation at a given time, the estimated economic value of the liability is then defined as the sum of the values of the holdings in the three assets. Thus:

$$\hat{V}^{\alpha}(t-1, y(t-1), T-t+1) = \sum_{i=1}^{3} \bar{x}_i^{\alpha}(t-1, y(t-1), T-t+1).$$

This will be non-linear in y. It represents the best estimate of the liability at time $t-1$, given the linear approximations at times $t, t+1, \ldots, T$. Thus, the estimated economic value of the liability is:

$$\begin{aligned}
&\hat{V}^{\alpha}(t-1, y(t-1), T-t+1) \\
&\quad = \bar{x}_2^{\alpha}(t-1, y(t-1), T-t+1) + \bar{x}_3^{\alpha}(t-1, y(t-1), T-t+1) \\
&\qquad + \frac{1}{y_4(t-1)} \Big\{ M^{\alpha}(t-1)\big(\hat{a}^{\alpha}_{T-t} A_y (y(t-1)-\mu_y) + \hat{b}^{\alpha}_{T-t}\big) \\
&\qquad - \bar{x}_2^{\alpha}(t-1, y(t-1), T-t+1)\big[\mu_R + A_R(y(t-1)-\mu_y)\big]_2 \\
&\qquad - \bar{x}_3^{\alpha}(t-1, y(t-1), T-t+1)\big[\mu_R + A_R(y(t-1)-\mu_y)\big]_3 \Big\}.
\end{aligned}$$

Step 3. Linearise the estimated economic value of the liability

Let:

$$\bar{V}^{\alpha}(t-1, y(t-1), T-t+1) = M^{\alpha}(t-1)\big[a^{\alpha}_{T-t+1}(y(t-1)-\mu) + b^{\alpha}_{T-t+1}\big]$$

where a^{α}_{T-t+1} is a row vector, and

$$a^{\alpha}_{j,T-t+1} = \left.\frac{\partial \hat{V}^{\alpha}(0, y, T-t+1)}{\partial y_j}\right|_{y=\mu_y}$$

$$b^{\alpha}_{T-t+1} = \hat{V}^{\alpha}(0, \mu_y, T-t+1).$$

Note that the linearisations involving the a^{α}_{T-t+1} and b^{α}_{T-t+1} are different from the linearisations, introduced earlier in this section, involving the $\hat{a}^{\alpha}_{T-t+1}$ and $\hat{b}^{\alpha}_{T-t+1}$. However, the two linearisations and the a and b functions are very closely linked, as we will see below.

The parameters of the linearised liability are as follows.

For $j = 1, 2, 3, 5$:

$$\begin{aligned}
&a^{\alpha}_{j,T-t+1} \\
&\quad = \frac{1}{\mu_4}\left\{\sum_{i=1}^{5} \hat{a}^{\alpha}_{i,T-t}[A_y]_{i,j} - \bar{x}_2^{\alpha}(0, \mu_y, T-t+1)[A_R]_{2,j} - \bar{x}_3^{\alpha}(0, \mu_y, T-t+1)[A_R]_{3,j}\right\}.
\end{aligned}$$

For $j = 4$:

$$\begin{aligned}
&a^{\alpha}_{4,T-t+1} \\
&\quad = \frac{1}{\mu_4}\left\{\sum_{i=1}^{5} \hat{a}^{\alpha}_{i,T-t}[A_y]_{i,4} - \bar{x}_2^{\alpha}(0, \mu_y, T-t+1)[A_R]_{2,4} - \bar{x}_3^{\alpha}(0, \mu_y, T-t+1)[A_R]_{3,4}\right\} \\
&\qquad - \frac{1}{\mu_4^2}\big(\hat{b}^{\alpha}_{T-t} - \bar{x}_2^{\alpha}(0, \mu_y, T-t+1)\mu_7 - \bar{x}_3^{\alpha}(0, \mu_y, T-t+1)\mu_8\big)
\end{aligned}$$

and the average liability is:

$$b^{\alpha}_{T-t+1} = \hat{b}^{\alpha}_{T-t}\mu_4^{-1} + \bar{x}^{\alpha}_2(0, \mu_y, T-t+1)(1-\mu_7\mu_4^{-1}) + \bar{x}^{\alpha}_3(0, \mu_y, T-t+1)(1-\mu_8\mu_4^{-1})).$$

Now, for the purpose of constructing approximate formulae for the optimal asset allocation one time-step further back, we aim to linearise:

$$M^{\alpha}(1)\bar{V}^{\alpha}(0, y(1), T-t+1) \approx \hat{a}^{\alpha}_{T-t+1}(y(1)-\mu_y) + \hat{b}^{\alpha}_{T-t+1}.$$

For the FP liability, the parameters of the linear liability with pension increase f are:

$$\hat{a}^F_{j,T-t+1} = (1+f)a^F_{j,T-t+1}$$
$$\hat{b}^F_{T-t+1} = (1+f)b^F_{T-t+1}$$

and for the RPI liability, the parameters of the linear liability with pension increase $1 + y_1(1)$ are:

$$\hat{a}^R_{1,T-t+1} = b^R_{T-t+1} + (1+\mu_1)a^R_{1,T-t+1}$$
$$\hat{a}^R_{j,T-t+1} = (1+\mu_1)a^R_{j,T-t+1} \quad j = 2, 3, 4, 5$$

and

$$\hat{b}^R_{T-t+1} = (1+\mu_1)b^R_{T-t+1}.$$

Step 4. Reduce t *by 1 and go to step 1*

Using the backward method, we can derive the asset allocation step by step from the last year back to the first year.

7. Some Numerical Results

In Sections 5 and 6 we constructed formulae for approximately optimal asset allocations for the FP and RPI liabilities using dynamic hedging. In this section we provide some numerical results obtained using those formulae.

Based on a single 10,000-year simulation of the Wilkie (1995) model (using the parameter values quoted in that paper), we obtained values of A and C_{ϵ} as follows:

$$A = \begin{pmatrix} A_y \\ \\ \cdots \\ A_R \end{pmatrix} = \begin{pmatrix} 0.58202 & 0.00167 & 0.00037 & -0.00558 & -0.00086 \\ 0.07273 & 0.56505 & 0.00310 & -0.11103 & -0.00805 \\ 0.35627 & -0.00500 & 0.93178 & 0.00303 & -0.05921 \\ 0.02054 & 0.00077 & 0.01200 & 0.73368 & -0.00252 \\ -0.00262 & 0.00058 & -0.00489 & 0.04145 & 0.54888 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 \\ -0.37395 & 0.00689 & 0.14036 & 0.04523 & 0.07199 \\ 0.60820 & 0.00131 & 0.00546 & -0.04873 & 0.51492 \end{pmatrix}.$$

(Recall that $y_1(t) =$ inflation rate from time $t-1$ to t, $y_2(t) =$ log(dividend yield) at t, $y_3(t) =$ log(consols yield) at t, $y_4(t) = 1+$risk free interest rate from t to $t+1$, $y_5(t) =$ log(real yield on index-linked bonds) at t, while $R_1(t)$, $R_2(t)$ and $R_3(t)$ equal the total returns on cash, consols, and index-linked bonds respectively.)

$$C_\epsilon = \begin{pmatrix} 0.00199 & 0.00332 & 0.00112 & 0.00006 & 0.00000 & 0 & -0.00113 & 0.00207 \\ 0.00332 & 0.02961 & 0.00525 & 0.00039 & -0.00010 & 0 & -0.00528 & 0.00356 \\ 0.00112 & 0.00525 & 0.00782 & 0.00059 & 0.00308 & 0 & -0.00789 & -0.00207 \\ 0.00006 & 0.00039 & 0.00059 & 0.00019 & 0.00020 & 0 & -0.00050 & -0.00014 \\ 0.00000 & -0.00010 & 0.00308 & 0.00020 & 0.00407 & 0 & -0.00309 & -0.00428 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.00113 & -0.00528 & -0.00789 & -0.00050 & -0.00309 & 0 & 0.00804 & 0.00207 \\ 0.00207 & 0.00356 & -0.00207 & -0.00014 & -0.00428 & 0 & 0.00207 & 0.00669 \end{pmatrix}.$$

Consols and index-linked bonds are risky assets in the portfolio, whereas cash is the riskless asset. The matrix of $\hat{C}_\epsilon$ shows that the return on consols has the largest conditional variance of 0.00804 or 0.090^2.

The unconditional means of $(y_1(t), y_2(t), y_3(t), y_4(t), y_5(t))$ and $(R_1(t), R_2(t), R_3(t))'$ are (to four significant figures):

$$\mu_y' = (0.04827, -3.197, -2.570, 1.065, -3.218)$$

and

$$\mu_R' = (1.065, 1.084, 1.093).$$

From the values of μ_R, we see that index-linked bonds offer the highest mean return at 1.093. Cash provides the lowest mean return at 1.065.

The Wilkie model is used here for illustration only. In principle, the vector autoregressive model could be used as an approximation to any stochastic asset model or estimated directly from historical data.

Table 7.1. Asset allocation for FP liabilities (5% increase) with terms up to 10 years: the $\bar{x}_1^F(0, y(0), T-t)$ are those for $y(0) = \mu_y$; the $\bar{x}_i^F(0, y(0), T-t)$ for $i = 2, 3$ do not depend upon $y(0)$; for a ten-year liability the hedge quantities, $\bar{x}_i^F(t, y(0), T-t)$, are found by multiplying the $\bar{x}_i^F(0, y(0), T-t)$ by the relevant liability multiplier $M^F(t)$

t	Term to maturity $T-t$	$\bar{x}_1^F(0, \mu_y, T-t)$	$\bar{x}_2^F(0, y(0), T-t)$	$\bar{x}_3^F(0, y(0), T-t)$	Liability multiplier
0	10	0.2999	0.4972	0.0221	1
1	9	0.3722	0.4479	0.0204	1.05
2	8	0.4458	0.3969	0.0185	1.05^2
3	7	0.5206	0.3443	0.0162	1.05^3
4	6	0.5966	0.2902	0.0137	1.05^4
5	5	0.6736	0.2345	0.0110	1.05^5
6	4	0.7515	0.1773	0.0081	1.05^6
7	3	0.8300	0.1189	0.0052	1.05^7
8	2	0.9083	0.0595	0.0024	1.05^8
9	1	0.9855	0	0	1.05^9

Now we consider liabilities with terms of up to ten years for both FP and RPI pensions.

In Table 7.1 we show the optimal asset allocation of cash, consols and index-linked bonds for FP liabilities, with ten years down to one year to maturity, given $y(0) = \mu_y$.

For example, consider the row $T - t = 8$ years. The table tells us that, on average, we require 0.4458 in cash, 0.3969 in consols and 0.0185 in index-linked to hedge, from time zero to time one, a payment of 1.05^8 at time eight. For a ten-year dynamic hedge, we multiply row s by 1.05^s to account for those pension increases which have already vested.

For the FP liability, we see from Table 7.1 that we should invest most funds (about 61% of the total assets) in consols at the beginning of the term. We also notice that, for FP liability, we should hold very few assets in index-linked bonds in the portfolio, as we might expect. This is because, for the FP liability, the pension liability increases by a fixed percentage (here 5%) every year, and so we know the exactly liability at the end of the term. Thus, consols are the more appropriate choice for this type of liability. We also see that the optimal investments shift gradually from consols early on into cash (100% in the final year). Again, this shift is intuitively reasonable; year-on-year reductions in the duration of the liability $(T - t)$ are being matched by reductions in the mean duration of the assets. However, it is interesting to note that the vector autoregressive model does not explicitly take the duration of the consols into account, so it does not obviously follow that it would be optimal to shift over time from one asset to the other.

Recall that the linear approximation to the liability is:

Table 7.2. Liability valuation for 5% fixed pension liabilities: the b^F_{T-t} column gives the liability value for a $(T-t)$ year liability starting from $y(0) = \mu_y$; the $a^F_{i,T-t}$ show how sensitive the values are to deviations in $y_i(0)$ from μ_i; the final column shows the implied valuation interest rate when $y(t) = \mu_y$, $i_{T-t} = 1.05/(b^F_{T-t})^{1/(T-t)} - 1$

t	$T-t$	b^F_{T-t}	$a^F_{1,T-t}$	$a^F_{2,T-t}$	$a^F_{3,T-t}$	$a^F_{4,T-t}$	$a^F_{5,T-t}$	Implied i_{T-t} (%)
0	10	0.8192	0.0002	−0.0064	−0.4169	−1.5682	−0.0374	7.12
1	9	0.8405	0.0010	−0.0064	−0.3601	−1.7493	−0.0342	7.05
2	8	0.8611	0.0017	−0.0063	−0.3034	−1.9091	−0.0305	6.98
3	7	0.8812	0.0022	−0.0060	−0.2472	−2.0378	−0.0265	6.91
4	6	0.9005	0.0026	−0.0055	−0.1925	−2.1221	−0.0220	6.85
5	5	0.9191	0.0026	−0.0047	−0.1403	−2.1434	−0.0172	6.79
6	4	0.9370	0.0024	−0.0037	−0.0925	−2.0763	−0.0122	6.72
7	3	0.9540	0.0017	−0.0025	−0.0509	−1.8857	−0.0072	6.66
8	2	0.9702	0.0008	−0.0011	−0.0188	−1.5239	−0.0029	6.60
9	1	0.9855	0	0	0	−0.9250	0	6.54

$$\bar{V}^F(t, y(t), T-t) = (1+f)^t\left(a^F_{T-t}(y-\mu_y) + b^F_{T-t}\right)$$

or

$$\bar{V}^F(0, y, T-t) = a^F_{T-t}(y-\mu_y) + b^F_{T-t}.$$

Values for the b^F_{T-t} and $a^F_{i,T-t}$ are given in Table 7.2. The b^F_{T-t} give the basic liability (economic value) when $y(t) = \mu_y$, while the $a^F_{i,T-t}$ indicate how sensitive the liability is to deviations from μ_y. Also shown is the implied valuation interest rate when $y(t) = \mu_y$:

$$i_{T-t} = \left(\frac{1.05^{T-t}}{b^F_{T-t}}\right)^{1/(T-t)} - 1.$$

When $T-t=1$ this is equal to the cash rate. As $(T-t)$ increases, so does i_{T-t} towards the consols rate, as the balance between the two assets shifts. With ten years to maturity, $i_{10} = 7.12\%$. This reflects both the predominance of consols early on, shifting to heavier investments in cash later on.

We see from Table 7.2 that, for the FP liability, the linearised liability is strongly related to the risk free return on one-year zero-coupon bonds (that is, the column headed $a^F_{4,T-t}$). Also, the liability is related to the consol real yield, becoming larger as the term to maturity increases.

Consider next RPI liabilities with terms of up to ten years. Table 7.3 shows how the optimal asset allocations of cash, consols and index-linked bonds for RPI liability vary with term to maturity. From Table 7.3, we see that we should hold a very high proportion of assets in index-linked bonds,

Table 7.3. Asset allocation for RPI liabilities with terms up to ten years; the $\bar{x}_1^R(0, y(0), T-t)$ are for $y(0) = \mu_y$; the $\bar{x}_i^R(0, y(0), T-t)$ for $i = 2, 3$ do not depend upon $y(0)$; for a ten-year liability the $\bar{x}_i^R(0, y(0), T-t)$ should be multiplied by the liability multiplier; in the present context the liability multiplier $M^R(t)$ equals $RPI(t)/RPI(0)$

t	$T-t$	$\bar{x}_1^R(0, \mu_y, T-t)$	$\bar{x}_2^R(0, y(0), T-t)$	$\bar{x}_3^R(0, y(0), T-t)$	Liability multiplier
0	10	−0.1047	0.0420	0.7532	1
1	9	−0.0803	0.0305	0.7722	$M^R(1)$
2	8	−0.0456	0.0143	0.7866	$M^R(2)$
3	7	0.0024	−0.0081	0.7943	$M^R(3)$
4	6	0.0679	−0.0383	0.7928	$M^R(4)$
5	5	0.1556	−0.0778	0.7783	$M^R(5)$
6	4	0.2714	−0.1275	0.7454	$M^R(6)$
7	3	0.4216	−0.1849	0.6847	$M^R(7)$
8	2	0.6104	−0.2369	0.5779	$M^R(8)$
9	1	0.8337	−0.2393	0.3836	$M^R(9)$

especially in the early years of the term. This makes sense, since, for an RPI liability, IL bonds provide a reasonable match in the long run for the RPI-linked pension increases. Like the FP liability, with the RPI liability we shift gradually towards cash in the later years. For example, in the RPI liability we should hold 0.7532 in index-linked bonds, 0.0420 in consols and short by an average of 0.1047 in cash to minimise risk in the first year of the term. In the last year of the term, we require 0.3836 in index-linked bonds, −0.2393 in consols and an average holding of 0.8337 in cash. This presumably means that cash is a better hedge for an RPI over one year than index-linked bonds. It reflects the relative certainty of the liability one year ahead, the certainty (in nominal terms) of cash versus the relative riskiness of the long-dated index-linked bond (because of the variable real yield) in the short term.

The investments for the one-year liability can be interpreted in more detail, as follows. The long investment of 0.3836 in index-linked bonds will give a partial hedge against the inflation increase in the pension. The short position of −0.2393 in consols is a result of correlation between changes in nominal and real yields in the model. Finally, the long position in cash acts over the one year to remove any bias in the hedge. Similar arguments can be applied to longer-dated liabilities. Values for the b^R_{T-t} and $a^R_{i,T-t}$ are given in Table 7.4. The b^R_{T-t} are lower than the corresponding values of b^F_{T-t} in Table 7.2. This is mainly due to the differences in mean returns on index-linked bonds and consols in the Wilkie model, rather than the mean inflation assumption, which is close to the fixed increases of 5%.

From Table 7.4, we notice that, for the RPI liability, the linearised liability is affected by the price of one-year zero-coupon bonds (especially for the later years of the term), by the real yield on index-linked bonds

Table 7.4. Liability valuation for RPI pension liabilities: the b^R_{T-t} column gives the liability value for a $(T-t)$ year liability starting from $y(0)=\mu_y$; the $a^R_{i,T-t}$ show how sensitive the values are to deviations in $y_i(0)$ from μ_i

t	$T-t$	b^R_{T-t}	$a^R_{1,T-t}$	$a^R_{2,T-t}$	$a^R_{3,T-t}$	$a^R_{4,T-t}$	$a^R_{5,T-t}$	Implied real yield (%)
0	10	0.6905	−0.0302	−0.0015	−0.0028	−0.0296	−0.8073	3.77
1	9	0.7225	−0.0173	−0.0014	0.0049	−0.1714	−0.8149	3.68
2	8	0.7552	0.0004	−0.0012	0.0138	−0.3379	−0.8128	3.57
3	7	0.7886	0.0243	−0.0008	0.0238	−0.5256	−0.7980	3.45
4	6	0.8224	0.0559	−0.0001	0.0342	−0.7263	−0.7661	3.31
5	5	0.8561	0.0969	0.0008	0.0441	−0.9233	−0.7119	3.16
6	4	0.8894	0.1481	0.0018	0.0515	−1.0874	−0.6292	2.97
7	3	0.9214	0.2065	0.0029	0.0540	−1.1708	−0.5118	2.77
8	2	0.9513	0.2563	0.0034	0.0480	−1.0987	−0.3560	2.53
9	1	0.9780	0.2434	0.0026	0.0299	−0.7601	−0.1700	2.25

(especially for the earlier years of the term; that is for large $T-t$), and by the inflation rate (especially for when $T-t$ is small). The fact that, for example, $a^R_{1,10} = -0.030 < 0$ indicates that, even if the rate of inflation is currently high (suggesting a higher liability), returns on the matching assets must be correspondingly higher in the long run (that is, the liability is actually lowered).

8. Optimal Asset Allocation for LPI Liability

For an LPI liability, the pension increase rate is the lower of the fixed rate (5%) and RPI; that is:

$$M^L(t) = M^L(t-1)\min\left\{\frac{RPI(t)}{RPI(t-1)}, 1.05\right\} = M^L(t-1)\min\{1+y_1(t), 1.05\}.$$

It follows that an LPI liability has strong links with both the FP and RPI liabilities (for example, we can immediately note that the liability will be lower than the corresponding fixed and RPI liabilities). (Note that, if X and Y are random variables, then $\mathrm{E}[\min\{X, Y\}] \le \min\{\mathrm{E}[X], \mathrm{E}[Y]\}$ with strict inequality if $Pr[X > Y] > 0$ and $Pr[X < Y] > 0$.) We will use the RPI and FP liabilities, developed earlier in this section, to propose approximate formulae for an LPI liability, by choosing an appropriate function which satisfactorily distributes the proportions of these two types of liability.

8.1 *Model Setting for the LPI Liability*

In this section we aim to build a suitable model to connect the FP and RPI liabilities with the LPI liability. Since the main factor affecting the relationship among these three liabilities is the inflation rate, we assume, for simplicity, that this connection is a function only of the inflation rate.

We now assume that there exist two functions of $(T-t)$ and the inflation rate, $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$, which allow approximation of the asset allocation for the LPI liability as follows:

$$\hat{x}_j^L(t, y(t), T-t) = p(T-t, y_1(t))\bar{x}_j^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{x}_j^F(t, y(t), T-t)$$

where $j = 1, 2, 3$.

Note that it will not be possible, in general, to find a p and q such that the linear combination of $\bar{x}_j^R$ and $\bar{x}_j^F$ is precisely equal to the true $\bar{x}_j^L$. Instead, we aim to find a p and q which make $\hat{x}_j^L = p\bar{x}_j^R + q\bar{x}_j^F$ the best approximation (in some sense) to $\bar{x}_j^L$. Note also that $\bar{x}_1^R$ and $\bar{x}_1^F$ depend upon all of the $y_i(t)$. To this extent, $\hat{x}_1^L$ will also depend upon $y(t)$. $\hat{x}_2^L$ and $\hat{x}_3^L$ will depend on $y_1(t)$ and μ_y only, since $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ are functions only of $y_1(t)$ and $\bar{x}_2^R$, $\bar{x}_2^F$, $\bar{x}_3^R$ and $\bar{x}_3^F$ are dependent on μ_y.

Then, the estimated LPI liability is taken to be:

$$\begin{aligned}&\hat{V}^L(t, y(t), T-t)\\ &\quad = p(T-t, y_1(t))\bar{V}^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{V}^F(t, y(t), T-t).\end{aligned}$$

As indicated earlier, $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$ are functions only of $(T-t)$ and $y_1(t)$, the rate of inflation from $(t-1)$ to t. If current inflation rates are low (that is, much lower than 5%), then we anticipate that the LPI liability will be closer to the RPI liability than if current inflation rates are high, especially when $(T-t)$ is small. In this case, $p(T-t, y_1(t))$ should be near to one and $q(T-t, y_1(t))$ should be near to zero, and vice versa if recent inflation has been high (much higher than 5%). Following this observation, we propose these two functions to be:

$$p(T-t, y_1(t)) = \frac{\gamma_{T-t}^p \exp(-\alpha_{T-t}(y_1(t) - \beta_{T-t}^p))}{1 + \exp(-\alpha_{T-t}(y_1(t) - \beta_{T-t}^p))} \tag{8.14}$$

and

$$q(T-t, y_1(t)) = \frac{\gamma_{T-t}^q \exp(\alpha_{T-t}(y_1(t) - \beta_{T-t}^q))}{1 + \exp(\alpha_{T-t}(y_1(t) - \beta_{T-t}^q))}. \tag{8.15}$$

To establish $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$, we need to minimise the $S(p, q)$ function, below, over $p = p(T-t, y_1(t))$ and $q = q(T-t, y_1(t))$, where:

$$S(p, q) = \mathrm{E}\Bigg[\Big\{\big(p\bar{x}^R(t, y(t), T-t) + q\bar{x}^F(t, y(t), T-t)'R(t+1) \\ - \tilde{V}^L(t+1, y(t+1), T-t-1)\Big\}^2 \Big| y(t), LPI(t)\Bigg].$$

This function is minimised for each $y_1(t)$, giving different values of p and q for each $y_1(t)$. (This step is implemented before we parametrise p and q according to equations (8.14) and (8.15).)

To obtain estimates for $p(T-t, y_1(t))$ and $q(T-t, y_1(t))$, we differentiate S with respect to p and q and equate to zero, in combination with 40,000 simulations of the vector autoregressive model as an approximation to exact expectation. This is implemented for 21 values of $y_1(t)$ ($y_1^{(i)}(t) = \mu_{y1} + 0.02(i-11)$ for $i = 1, \ldots, 21$). The estimated p and q values for $T-t=1$ are plotted in Figure 8.1.

With the estimated optimal values of $\hat{p}(T-t, y_1(t))$ and $\hat{q}(T-t, y_1(t))$, the next step is to estimate the values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t}, introduced in equations (8.14) and (8.15). These are determined by minimising:

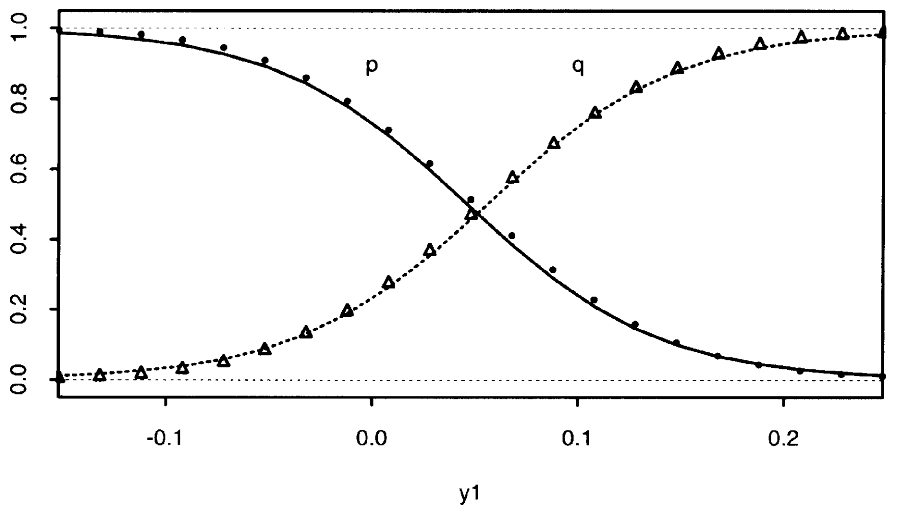

Figure 8.1. Estimated $\hat{p}$ (dots) and $\hat{q}$ (triangles) against different inflation rates for $T-t=1$; fitted curves $p(1, y_1)$ (solid curve) and $q(1, y_1)$ (dotted curve); for $j = 2, 3, 4, 5$ we have $y_j = \mu_{y,j}$

$$S(\alpha, \beta^p, \beta^q, \gamma^p, \gamma^q) = \sum_{i=1}^{21}\Big[p(T-t, y_1^{(i)}(t))(\alpha, \beta^p, \gamma^p) - \hat{p}(T-t, y_1^{(i)}(t))\Big]^2 + \sum_{i=1}^{21}\Big[q(T-t, y_1^{(i)}(t))(\alpha, \beta^q, \gamma^q) - \hat{q}(T-t, y_1^{(i)}(t))\Big]^2. \quad (8.16)$$

We can see, from Figure 8.1, that the quality of fit for $\hat{p}(1, y_1(t))$ and $\hat{q}(1, y_1(t))$ is good. We then can obtain graphs similar to Figure 8.1 for $t = 0, \ldots, T-2$. In Figure 8.2 we plot the liability values for $T-t=1$. As expected, we find that the LPI value is close to the RPI liability when y_1 is very low, and close to the 5% fixed liability when y_1 is very high. We can also see that the LPI liability is very much lower than both the RPI and fixed liabilities when y_1 is close to the LPI threshold of 5%, indicating that the effect of the stochastic minimum ($\min\{1.05, RPI(1)/RPI(0)\}$) is significant.

The backward method is used to calculate, in sequence, the values for α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $T-t = 1, \ldots, T$. At each time $T-t+1$, we calculate $\hat{p}(T-t+1, y_1)$ and $\hat{q}(T-t+1, y_1)$ on the assumption that:

$$\begin{aligned}&\bar{V}^L(t, y(t), T-t)\\ &\quad = p(T-t, y_1(t))\bar{V}^R(t, y(t), T-t) + q(T-t, y_1(t))\bar{V}^F(t, y(t), T-t)\end{aligned}$$

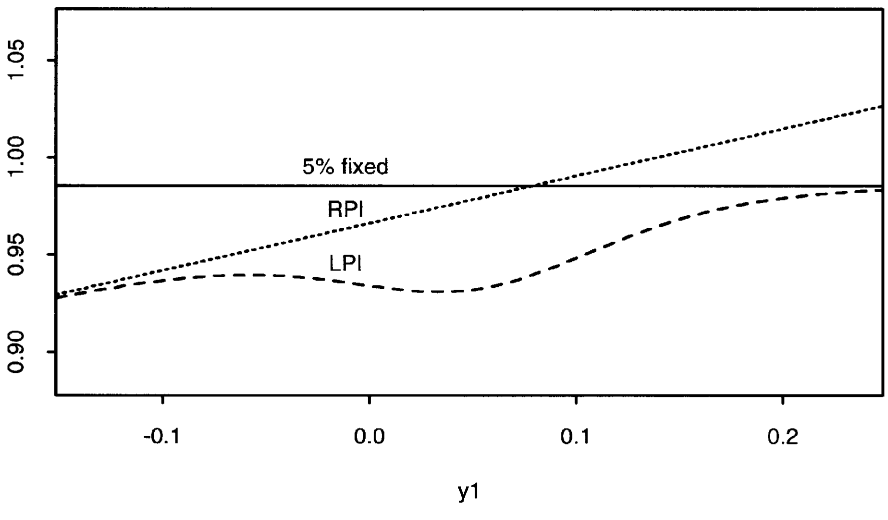

Figure 8.2. Estimated liability values for different values of y_1; the liability is due for payment in $T-t=1$ year; for $j = 2, 3, 4, 5$ we have $y_j = \mu_{y,j}$

Table 8.5. Values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $T = 10$ and $t = 0, \ldots, 9$

t	$T-t$	α_{T-t}	β^p_{T-t}	β^q_{T-t}	γ^p_{T-t}	γ^q_{T-t}
0	10	8.910	0.06997	0.08294	0.8726	0.7617
1	9	8.917	0.07028	0.08317	0.8824	0.7867
2	8	8.960	0.07032	0.08313	0.8925	0.8125
3	7	9.062	0.07014	0.08286	0.9028	0.8391
4	6	9.257	0.06898	0.08157	0.9132	0.8669
5	5	9.583	0.06538	0.07782	0.9242	0.8950
6	4	10.100	0.06032	0.07252	0.9380	0.9211
7	3	9.824	0.06213	0.07422	0.9531	0.9437
8	2	7.237	0.05264	0.07998	0.9775	0.9998
9	1	21.607	0.04649	0.05591	1.0000	1.0000

where the parametric forms for p and q (equations 8.14 and 8.15) are used with the already estimated values for the function parameters.

The estimated values of α_{T-t}, β^p_{T-t}, β^q_{T-t}, γ^p_{T-t} and γ^q_{T-t} for $t = T-1$ to 0 are presented in Table 8.5. These estimates for $t = 0, \ldots, T-1$ give us a means of connecting the FP and RPI liabilities with the LPI liability. From this, we are able to deduce the approximately optimal asset allocations for the LPI liability.

In Figures 8.3 and 8.4 we plot the functions p and q and the liability

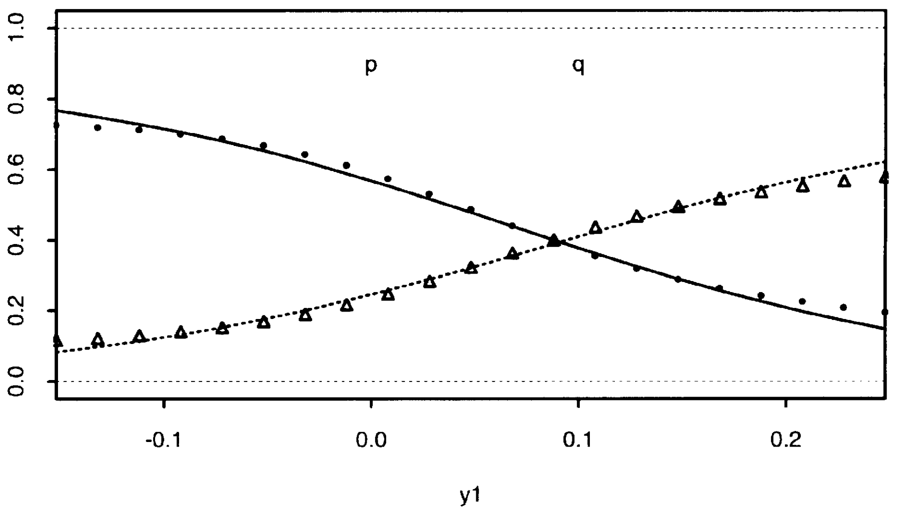

Figure 8.3. Estimated $\hat{p}$ (dots) and $\hat{q}$ (triangles) against different inflation rates for $T - t = 10$; fitted curves $p(10, y_1)$ (solid curve) and $q(10, y_1)$ (dotted curve)

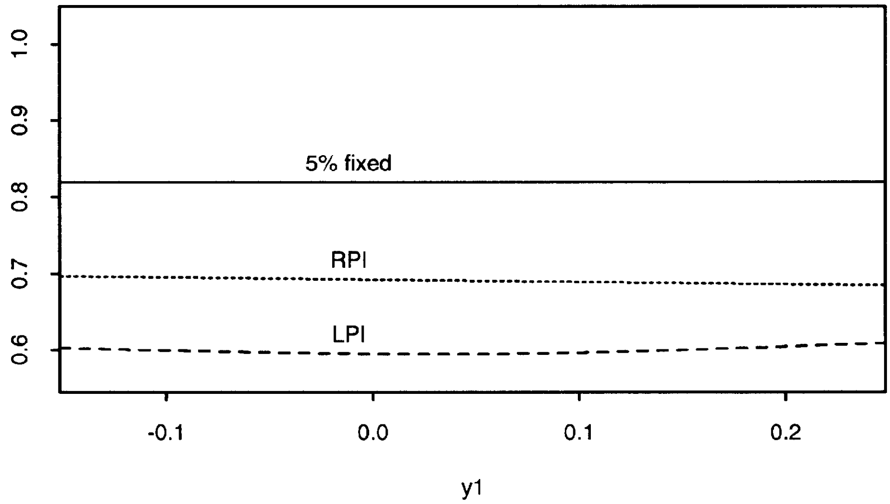

Figure 8.4. Estimated liability values for different values of y_1; the liability is due for payment in $T - t = 10$ years

estimates for $T - t = 10$. In contrast to Figure 8.1, the p and q functions are flatter and have upper limits which are below one (that is, $\gamma^p_{10} = 0.8726$ and $\gamma^q_{10} = 0.7617$, see Table 8.5). Even more striking is the comparison with Figure 8.2. The three liability curves are now almost independent of y_1. The difference between LPI and RPI amounts to a difference in the assumed rate of increase in the pension of 1.5% p.a. The size of this difference reflects the magnitude of the volatility in price inflation. It is also appropriate here to compare RPI with 5% fixed increases. Average price inflation is just below 5%, whereas the liability values suggest something rather larger. The bigger difference is a result of the hedging portfolio for each liability. In the case of RPI, the liability is hedged with a much larger proportion invested in index-linked bonds, which have a higher expected rate of return, resulting in a lower liability.

Figure 8.5 displays illustrations of asset allocations in the three different types of liabilities when $y(t) = \mu_y$. We can see more clearly how the various allocations change over time. For example, in all cases cash becomes more important as term to payment $(T - t)$ decreases. Also, we can see that longer-term LPI liabilities make use of a mixture of consols and IL bonds, as we might have expected.

8.2 *The Efficiency of Dynamic Hedging*

From the preceding sections we have derived the formula for the optimal

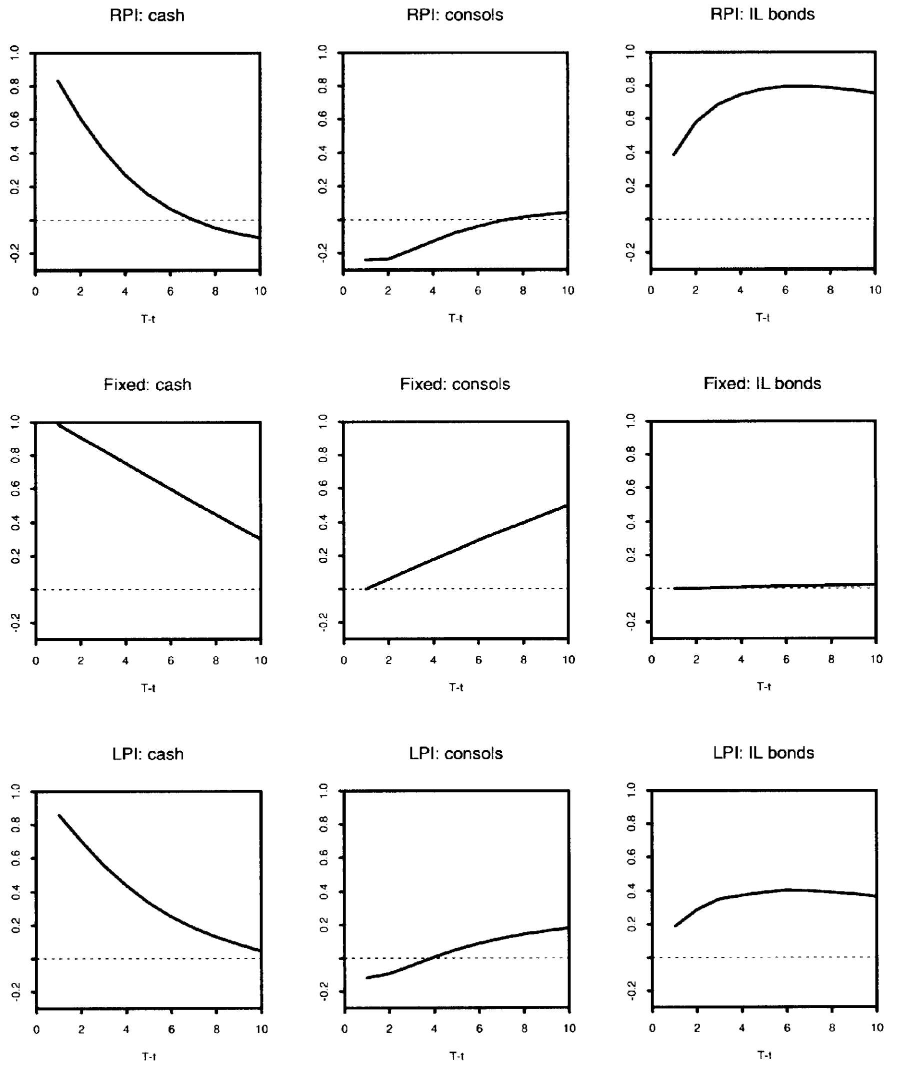

Figure 8.5. Asset allocations for a ten-year liability for the cases of RPI, FP and LPI liabilities; values plotted are representative values for $y(t) = \mu_y$

asset allocation $\hat{x}_j^{\alpha}(t, y(t), T-t)$, where $\alpha = F, R, L$ and $j = 1, 2, 3$. In this section we can now use these formulae to examine the effectiveness of the proposed (approximate) dynamic hedging strategy.

In dynamic strategies, the extra cash required at time t is:

$$C_t^{\alpha} = \bar{V}^{\alpha}(t, y(t), T-t) - \sum_{j=1}^{3} \bar{x}_j^{\alpha}(t-1, y(t-1), T-t+1) R_j(t).$$

Thus, C_t^{α} is the difference between the new liability at t, $\bar{V}^{\alpha}(t, y(t), T-t)$, and the value at t of the available assets held from $(t-1)$ to t.

Then, the present value at time zero of the total extra cash required up to time T is:

$$TC^{\alpha} = \sum_{t=1}^{n} C_t^{\alpha} \frac{S_1(0)}{S_1(t)}$$

where $S_1(u)$ is the unit value of the cash account at time u (that is, $S_1(0) = 1$ and $S_1(u+1) = S_1(u)R_1(u+1) = S_1(u)y_4(u)$). This measure is consistent with those commonly used in financial mathematics (see, for example, Musiela & Rutkowski, 1997, Chapter 4). It is, in particular, consistent with the risk minimising approach taken in earlier sections, where we minimise variances over each time step.

To test the efficiency of dynamic hedging, we need to calculate the values of $\mathrm{E}[TC^{\alpha}]$ and $\mathrm{Var}(TC^{\alpha})$ by making numerous simulations. Numerical results of comparisons with other hedging strategies are shown in Section 9.

9. Comparison of Hedging Strategies

In this section we will consider numerical results to allow comparison of the static and dynamic hedging strategies.

To assess the effect of static strategies, as in dynamic hedging, we denote TC^{α} as the present value at time zero of total extra cash (total cost) for the static strategies at time T for pensions $\alpha = F, R, L$. We have:

$$TC^{\alpha} = \frac{1}{S_1(T)} \left[M^{\alpha}(T) - \sum_{i=1}^{3} \hat{x}_i^{\alpha} \frac{S_i(T)}{S_i(0)} \right]$$

where $\alpha = F, R$ or L. Our objective with static hedging was to minimise the function SB (equation (4.3)), which is:

$$SB = \mathrm{E}\left[TC^{\alpha 2}\right].$$

We will now consider how much of an improvement in SB is provided by the switch to dynamic hedging (based on the linear approximations for 5% fixed and RPI liabilities and the non-linear approximation for LPI liabilities). We will assume that we are starting from neutral conditions at time zero (that is, $y(0) = \mu_y$) and $T = 10$.

Results for a ten-year liability are presented in Table 9.6. The value of TC^{α} was calculated for each of 40,000 simulations.

First, we note that (with the exception of the RPI liability) static hedging with all three assets proves to give a substantial improvement in performance relative to investment in a single asset class (for example, consols only for the fixed pension liability).

Second, we can observe from this that the proposed form of dynamic hedging does reduce the primary objective function $E[TC^{\alpha 2}]$. However, the improvement is not substantial, suggesting that dynamic hedging does not help greatly over static hedging.

An important observation to note is that all of the dynamic hedging strategies have $E[TC^{\alpha}]$ significantly different from zero (LPI especially so). This is not the case (explicitly by construction) for static hedging. For the 5% fixed and RPI liabilities, this will be the result of the linear approximation, but, in any event, the error is relatively insignificant. We remarked on this point in Section 6; that is, we believe that the linear approximation is reasonable. If the linear approximation was poor, then we would expect to see a greater bias in $E[TC^{\alpha}]$ away from zero.

Table 9.6. Comparison of static and dynamic hedging strategies for a ten-year liability; statistics are based on 40,000 simulations; starred (*) values should theoretically be zero, but differ from zero because of simulation error

	$E[TC^{\alpha}]$	$Var[TC^{\alpha}]$	$E[TC^{\alpha 2}]$
5% Fixed			
Static (cash only)	−0.000873*	0.012321	0.012322
Static (consols only)	−0.024918	0.020034	0.020655
Static (three assets)	0.000278*	0.005150	0.005150
Dynamic	0.007496	0.004585	0.004641
RPI			
Static (IL bonds only)	−0.006211	0.004409	0.004448
Static (three assets)	−0.000728*	0.003825	0.003825
Dynamic	0.005707	0.003527	0.003559
LPI			
Static (cash only)	0.001417*	0.015153	0.015154
Static (consols only)	−0.029089	0.036549	0.037394
Static (IL bonds only)	−0.030400	0.014991	0.015915
Static (three assets)	0.000656*	0.004866	0.004866
Dynamic	0.039129	0.002924	0.004455

The larger bias in the LPI liability is a significant factor, contributing to the size of the objective function E[$TC^{\alpha 2}$]. In contrast, we can see that, of all the liability types, dynamic hedging works best for LPI when we consider its effect on the variance Var[TC^{α}].

Further investigation suggests that much of the bias arises close to the liability payment date. This indicates that we should focus our attention in the future on improving both the dynamic hedging and our assessment of the liability value for shorter-term liabilities. Conversely, longer-dated liabilities (for example, 20 years) benefit more from dynamic hedging relative to static hedging.

An alternative line of investigation is to replace consols and long-dated index-linked bonds with zero-coupon fixed-interest and IL bonds maturing on the same date as the pension liability.

10. Conclusions

In this paper we have proposed some methods for pricing and hedging pensions' liabilities using cash, consols and index-linked bonds. We use the methods of static and dynamic hedging to hedge LPI liabilities. For static hedging, we find that investing solely in cash, index-linked bonds or long-dated bonds creates higher errors than when holding a suitable mixture of the three assets in the portfolio.

With dynamic hedging, we develop formulae for finding an approximation to the optimal asset allocation for hedging FP, RPI and LPI liabilities. For the FP liability, it is shown that most of the portfolio should be invested in consols at the beginning of the term, switching into cash later on, with very few assets in index-linked bonds. This switch to cash reflects the decreasing duration of the liability.

For an RPI liability, a high proportion of index-linked bonds should be held, especially in the early years of the term. Like the FP liability, more risky assets should be held in the early years of a term and more cash in later years. When making comparisons between FP, RPI and LPI liabilities, all the lines of asset allocation curves for the three types of liabilities are similar, and especially for both RPI and LPI liabilities. When the current inflation rate is very high (significantly above 5%), then the optimal asset allocation of the LPI liability is closer to that of the FP liability. Also, if the inflation rate is very low (always lower than 5%), then the optimal asset allocation of the LPI liability will be closer to that of the RPI liability.

Acknowledgements

This research was carried out while the authors were in receipt of a research grant from the Faculty and Institute of Actuaries.

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