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The simulation of a proposed nonlinear dynamic urban growth model

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Abstract. This paper develops a model which contains agglomeration economies, time variable and is based on both demand-side and supply side considerations and microeconomic foundations. The simulation result indicates that the proposed model can dynamically describe both monocentric and polycentric evolution of urban growth depending on different given conditions. The model also captures the features of decentralization of population and dispersal of economic activities from the central cities to the suburbs.

1. Introduction

The evolution of the urban area is open and dynamic since the urban structure and population distribution are not only determined by its history but also by the current fluctuations both externally and regionally. In addition, the distributions of residents and industries are the essential factors influencing the urban structure. Thus, the location choices of both industry and resident are critical in explaining the evolution of urban systems and the phenomenon of suburbanization.

The location decisions of residents and industries are influenced by each other and determined simultaneously. Meanwhile, land rent, which is one of the major factors affecting location decisions, is determined endogenously and simultaneously. These closely interacting relationships among the system variables make the dynamic urban system extremely complex. The complex interactions among the elements of the urban system can not be captured by a linear model. Furthermore, the dynamic growth process can hardly by presented by a static model. The purpose of this paper is to investigate a dynamic urban growth model to describe the evolution of the urban structure.

The location models of resident and industry are the foundations for investigating a complete urban growth model. Given the location of markets and of resource deposits, the standard location models assume monocentricity, perfect competition in goods markets, and uniform transport costs. Households choose their locations by trading off between transport and land costs. However, from both the viewpoints of theoretical completeness and practical usefulness, the assumption of monocentricity seems to be untenable. There are several pioneering non-monocentric urban land use models which do not assume an a priori location of either employment or households. Models by Beckmann (1976) and Borukhov and Hochman (1977) contain only one sector. Model by Capozza (1976) cannot generate multicentric city patterns at the equilibrium. The model by Amson (1976) assumes a fixed center. The non-monocentric urban land use model by Fujita and Ogawa (1982) is purely static and one-dimensional.

Most of the urban models in the economics literature did not have the dynamic feature; this began to change in the late '70s. Allen and Sanglier (1978, 1979) had introduced dynamic models with two-dimensional land surface into urban regional analysis. This model increased the interests of geographers and economists in applying them to practical situations. The appearance of dynamic model has made remarkable progress in urban economics, but the special structure of this model also makes it very complex and intractable.

Allen and Sanglier (1978, 1979, 1981) developed a spatial dynamic model that simultaneously analyzes the location determinants of households and employment, where employment and population interact and compete with each other via space constraints. This model, which will be referred as AS model in this paper, are free to exchange factors with the external world, and consists of changeable quantity and quality of elements based on the exogenous factors. The strength of the AS model is that it is able to present the evolution process of the urban system. Given the information about the fluctuations, the model is able to replicate both quantitatively and qualitatively changes.

There are two empirical studies (Pumain et al. 1987; Straussfogel 1991) investigating the application of the AS model. Pumain and her colleagues used the same version of the model as in Straussfogel's study to replicate the evolution of the 17 communes of Rouen in France for the years 1954 to 1975. Straussfogel used this model to simulate the evolution trends of the fourteen zones in Philadelphia. Their works support the utility of the model in an applied realm.

The AS model is a nonlinear dynamic model with time varying variables and parameters. In its empirical applications, the global dynamic growth has been rather accurately simulated. However, given the nonlinear structure of this model and the large number of fitting parameters, to calibrate the parameters is very costly in terms of time and computer usage, and it is impossible to define a unique set of parameters to fit the model. The lack of supply-side and microeconomic foundations are problems of this dynamic urban model (Haag 1989). All these problems have motivated me to investigate a proposed model which is based on both demand-side and supply-side foundations and is able to capture the evolution of the urban structure.

2. The nonlinear dynamic urban evolution model

2.1. *The proposed model: introduction*

The encouraging results from the empirical studies of the AS model make it quite promising to apply the idea in the AS model to explain the evolution of the urban system. A more complete dynamic urban model needs to be developed to overcome those problems and to reflect the sophisticated interacting relations in urban evolution. This proposed model, which is similar to the version of the AS model in Straussfogel or in Pumain, inherits the qualities in the AS model; i.e., employment and population will mutually interact and compete with each other via corresponding carrying capacities. It also includes demand-side considerations, micro-economic foundations, agglomeration effects and land prices. This model is expected to capture the existing fluctuations more specifically and to explain the transformation of the urban structure. To be more exact, the proposed model should be able to perform the shift of urban structures from monocentricity to polycentricity – shifts that have occurred in reality.

The model is derived from the following processes and methods. First, I postulate a resident's utility function and a firm's profit function based on microeconomic theory. These functions presumably represent a certain locality's attractiveness to residents and firms; consequently, each zone's relative attractiveness can be derived. The attractiveness of a certain zone is defined as the extent of this zone's attraction to households or to firms. A ratio of a particular zone's attractiveness to the sum of all zones' attractiveness is defined to be the relative attractiveness. This relative attractiveness is the major factor in determining the location choices of resident and industry; moreover, these location decisions will influence the potential occupation, i.e., the long-run maximum employment or population that can be sustained at the locality.

Second, a land rent function is developed. The land rent of a certain zone is based on the land demand from residents and industries, and is constrained by the amount of land available. Correspondingly, the location choices of resident and industry will be influenced by the distributions of current land rents. Finally, urban growth equations are established, where the growth of population and employment are proportional to the difference between their potential occupations and their existing ones. In this model, the commodity market, labor market and land market are all taken into account.

2.2. *The model*

There are three types of actors in the model: residents, industries and land developers. Industries are separated into export and nonexport (service) industries. The demand for the export (nonexport) goods is from outside (inside) the region. The inputs of the production function are land and labor. Besides these two factors, the agglomeration effect, which is the external economies by firms' gathering together, will also influence firms' final productions. Residents, who are suppliers of labor and demanders of commodities, make location decisions to maximize their utilities subject to the budget constraint. Industries, which produce commodities and create employment opportunities, choose plant sites to maximize their profits given the producing costs, land rent, net agglomeration effect and the demand for their product. Land developers make their development decisions based on the rising rate of land price in the long run, and in the short run, a change in the amount of developed land will lead to a change in the land rent. Firms, residents and developers make their location and development decisions simultaneously and interactively.

2.2.1. Net agglomeration effect. Firms tend to gather to benefit from each other. Gathering promotes technology transition, saving of communication costs, and improvement of productivity. This attraction will continue until the development of this area reach the saturation point. After that point, the increase in the number of firms will not promote the productivity of this area. Rather, it will cause a series of negative influences which hinder urban growth. The net effect of these positive and negative externalities is called the net agglomeration effect. This process of agglomeration can express the basic acceleration-deceleration-stagnation cycle of an urban area.

There are two kinds of agglomeration effects: one is localization economies, where activities are more productive when similar activities gather together. Another is urbanization economies, where activities are more productive when located in large cities. The net agglomeration economies, A_i^k, include both positive and negative effects. The positive effect is economies of scale, and the negative effect includes air pollution, crowding and congestion.[1] The net agglomeration effect is measured by

$$A_i^k = 1 + \rho^k \bar{J}_i (1 - \Psi^k \bar{J}_i) \ , \tag{1}$$

where

$$\bar{J}_i = \sum_j J_i^k \exp^{-\omega_i d_{ij}} \ ; \qquad k = E, S \ . \tag{2}$$

$A_i^E (A_i^S)$: the net agglomeration effect of export (service) activities,
$J_j^E (J_j^S)$: the numbers of export (service) employment in zone j,
ρ^k: a parameter representing cooperativity or economies of scale,
ψ^k: a parameter representing negative externalities and saturation effects,
$\exp(-\omega_i d_{ij})$: distance decay function, where ω measures the sensitivity of the firms at i to commuting distances (the friction of distance).

The variable $\bar{J}_i$ expresses the regional employment accessibility of zone i based on commuting distance and communting costs. It indicates the locational advantage of zone i for having access to other idustries. The economies of scale in zone i will be accelerated not only by the employment in zone i but also by the employment of nearby zones. The degree of influences from other zones' employ-

[1] Fujita and Ogawa (1982) P. 165 – 164, Pumain, Saint-Julien and Sanders (1987) P. 154, Straussfogel (1991) P. 4.

ment on the agglomeration effect of zone i depends on zone i's accessibility to other zones. In this way, the model can explain centralization.

2.2.2. Population. Residential attractiveness, which depends on resident's utility function, is the major consideration in the residential decision. Residential indirect utility function is composed of commodity price, land price, wage rate of working zone, commuting cost and distance for workers, and the level of service activities of residential zone. The residential attractiveness to the resident locating in zone i but working in zone m is measured by

$$\exp^{\mu V_{im}} = \left[\frac{(W_m - \eta d_{im})^{\alpha+\beta}}{P_i^{\alpha} \cdot R_i^{\beta}} \right]^{\mu} . \tag{3}$$

V_i: the observed utility function of residents located at zone i,
R_i: land rent for a unit of land at i,
W_m: the average wage in zone m in unit of \$/year,
α, β: parameters representing the income shares of commodities and land,
d_{ij}: the distance between zone i and zone j,
η: transport costs per two units distance of commuter which are composed of user costs incurred by operating vehicles,
P_i: commodity unit price at zone i,
δ: a parameter measuring the positive effect from the service activities to the residential area.

The parameter μ, sensitivity to differences in attractiveness, represents the amount of information about the location available to residents, or represents the dispersion in residents' tastes. It is negatively related to the variance of the unsystematic utility. Consequently, it indicates the degree of unanimity of residential response to the zone's attractiveness. A high value of μ corresponds to a homogeneous population, and a low value of μ expresses an uniform, inhomogeneous population (Anas 1990).

The residential potential is the maximum population that can be sustained at the locality in the long run. It is analogous to AS carrying capacity and is different from the existing residential population. The residential potential is defined by

$$Z'_i = \sum_m^k \nu \cdot J_m \cdot M_{im} = \sum_m^k \nu \cdot J_m \cdot \frac{\exp^{\mu V_{im}}}{\sum_j^k \exp^{\mu V_{jm}}} , \tag{4}$$

where k is the total number of zones in the study region, v is the average number of persons supported by one job, and J_m is the total numbers of employment in zone i which is the sum of service and export employment. The symbol M_{im} represents the relative residential attractiveness for the people working in zone m and residing in zone i. It is a logit function essentially based on the residential utility level V_{ij}. The difference between the residential potential and the existing population is the major factor explaining the change of population.

The residential potential is the sum of weighted labor demand of all zones for the residents of zone i. Each zone's demand is weighted by the relative residential attractiveness. The aggregate weighted labor demand can also be interpreted as the job accessibility of zone i. The evolution of the population is composed of the natural change and net immigration. Parameter $\bar{n}$ represents net birth rate. Another important term, net immigration, is defined to be proportional to the potential population growth, which is the difference between the potential population and the existing resident.

$$\frac{dZ_i}{dt} = \tau \cdot (Z_i' - Z_i) + \bar{n} \cdot Z_i \ , \tag{5}$$

where the symbol τ is interpreted as the speed of the adjustment from actual population level to its potential level. There will be a population growth in zone i, if the residential potential is higher than the existing population in zone i. The population is in a dynamic adjustment process based on the gap between the residential potential and the existing population.

2.2.3. Employment. The variable V_i is a function of firm's profit, π_i. It represents the utility of industry located at zone i,

$$V_{im}^S = \log(\pi_{im}^S) \ ; \quad V_i^E = \log(\pi_i^E) \ . \tag{6}$$

The superscription E and S expresses export and service activity. The locational attractiveness of zone i to service activities with market in zone m is measured by

$$\exp^{\mu' V_{im}^S} = \left[C_3 \cdot \left(\frac{(P_i^S - t^S \cdot d_{im}) \cdot A_i^S}{W_i^a \cdot R_i^b} \right)^{\frac{1}{1-a-b}} \right]^{\mu'} \ . \tag{7}$$

The parameter a and b present the share of labor and land in the production function. The parameter μ' is a positive dispersion parameter of the distribution. The locational attractiveness of zone i to export activities is measured by

$$\exp^{\mu' V_i^E} = \left[C_3 \cdot \left(\frac{(P_i^E - t^E \cdot \phi_i^E) \cdot A_i^E}{W_i^a \cdot R_i^b} \right)^{\frac{1}{1-a-b}} \right]^{\mu'} \ . \tag{8}$$

The net agglomeration effect is essential in the locational attractiveness for firms, which is the major factor in a firm's location decision.

The potential employment of zone i, which is predicted by the aggregate weighted demand from all zones for the commodities produced in zone i at a given time, is the maximum employment that can be sustained at zone i in the long run. The potential employment of service activities is generated by local demand within the system. The number of jobs induced by the local demand is the product of a multiplier and the population. I weight this local employment demand by the relative attractiveness and sum them together to derive the amount of the local employment that this zone is able to attract. The potential employment of service activities is measured by

$$J_i^{S'} = \sum_m \lambda^S Z_m \cdot \frac{\exp^{\mu' V_{im}^S}}{\sum_j \exp^{\mu' V_{jm}^S}} , \tag{9}$$

where the parameter λ^S is a multiplier representing the number of jobs generated from the commodities demanded by one person, $\lambda^S Z_m$ is the number of jobs induced by the residents of zone m, and μ' is the sensitivity to differences in attractiveness. A high μ' value reflects that almost all the activities will tend to concentrate in the most attractive zone.

External demand is the aggregate demand for the export commodity, and external employment demand is the number of jobs of exporting activities. The potential employment of exporting activities in a zone is measured by the product of the external employment demand and the relative attractiveness in that zone, which is the share of the external employment demand this zone is able to attract. The potential employment of export activities is measured by

$$J_i^{E'} = D^E \cdot \frac{\exp^{\mu' V_i^E}}{\sum_j \exp^{\mu' V_j^E}} , \tag{10}$$

where D^E is the external employment demand.

Similar to population evolution, the employment growth of these two activities is proportional to the gap between the potential employment and the existing employment. The employment growth is defined by

$$\frac{dJ_i^k}{dt} = \theta^k \cdot (J_i^{k'} - J_i^k) \; ; \quad \text{where } k = E, S . \tag{11}$$

The parameter $\theta^E (\theta^S)$ represents the speed at which export (nonexport) activities respond to changes in the external (local) market. The employment evolution model is to make the aggregate commodity supply satisfy the existing aggregate commodity demand. The share of either local demand or external demand to a certain zone is determined by the relative attractiveness of that zone. The attractiveness of a given zone is measured by the profit that each firm could earn in that zone.

2.2.4. Land rent. The land rent is determined by the simultaneous solutions of the residential land demand function and the industrial land demand function given the market clearing condition. The residential land demand function is:

$$H_i^* = \frac{\beta}{\alpha + \beta} \cdot \frac{W_j - \eta d_{ij}}{R_i} . \tag{12}$$

The industrial land demand function is:

$$B_i^* = \left[\frac{a^a b^{1-a} \cdot P_i \cdot A_i}{W_i^a \cdot R_i^{1-a}}\right]^{C_4} . \tag{13}$$

And the market clearing condition, the sum of land demanded from residents and firms equal the amount of land developed, is:

$$H_i^* Z_i + B_i^{E*} N_i^E + B_i^{S*} N_i^S = G_{i,t-1} , \tag{14}$$

The variable H_i is the amount of land demanded by each resident at i, and B_i is the amount of land demanded by each industry at zone i. The variable $G_{i,t-1}$ is the total amount of developed land in zone i last period; it is exogenous in determining R_i. The term $N_i^E(N_i^S)$ is the numbers of export (service) industries in zone i, and v^E (v^S) is the inverse of the number of employment offered by each exporting (service) firm.
The land rent is measured by

$$R_i - \left[\frac{\left(\frac{a^a b^{1-a} \bar{P}_i^E A_i^E}{W_i^a}\right)^{C_4} \cdot N_i^E + \left(\frac{a^a b^{1-b} P_{im}^S A_i^S}{W_i^a}\right)^{C_4} \cdot N_i^S}{G_{i,t-1}}\right] R_i^{(-bC_4)} \tag{15}$$

$$- \left[\frac{\left(\frac{\beta}{\alpha+\beta}\right)(W_j - \eta d_{ij}) Z_i}{G_{i,t-1}}\right] = 0 .$$

Population, employment, agglomeration effect and land rent are all endogenous variables in this model. Besides, the price of service commodity and wage rate are also endogenously determined. The price of service commodity is determined by the supply from the service industry and the demand from residents. Wage rate is determined by labor supply from residents and labor demand from both industries. The differences between this model and the AS model are as follows: 1) This model draws a clearer picture of the commodity and labor markets than AS or its successors; the growth of both population and employment are derived on the processes of pursuing the dynamic equilibrium of both the labor market and commodity market. 2) The cost of land, costs of commodities and wage are all included in this study. 3) The agglomeration equation formulated in this model is more complete than that of the AS model.

3. Simulation

3.1. Simulating urban evolution

I first program model in MATLAB, which is a computer software. The basic structure of the program is an iterative computation of the change in population and employment density. The simulation region is assumed to be a square and is composed of a triangular lattice as in Fig. 1. I define the initial values of some variables in order to start the simulation. Population and employment are uniformly distributed in the beginning. Each zone in the simulation begins with 60 units of population. All zones have the same initial conditions; the only difference between each zone in the beginning is the location. Different location gives different accessibility. Thus the agglomeration effect which depends on the location and industry size of each zone is varied. Due to the difference in commuting cost, population and activities will redistribute from the starting uniform distributions. Land rent is computed endogenously given the current population, employment and the other variables. Then the population and employment of next time period can be computed given the values of variables, parameters and the land price.

3.2. Simulation experiment

In this study, the population density gradient, estimated from the negative exponential density function, is used to describe overall degree of centralization. This indicator is just a crude aggregate measure of gradient from the center to zones in all directions. The negative exponential density function in the monocentric model is written as

$$Z_i = X_0 e^{-DG \cdot d_{ij}} , \tag{15}$$

where Z_i is the population density at zone i, d_{ij} is the distance between zone i and zone j, zone j is the center zone, X_0 and DG are constants. I take the natural logarithm of the negative exponential density and get:

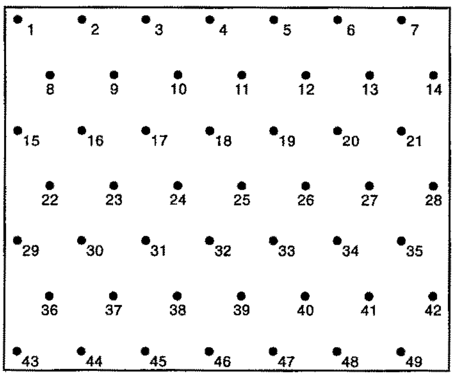

Fig. 1. The numbering of the points on our triangular lattice

$$\ln(Z_i) = \ln(X_0) + DG \cdot d_{ij} \ . \qquad (16)$$

The constant DG represents the percentage rate of decline per distance. The absolute value of DG is the density gradient and X_0 is the theoretical population density of the urban center. The magnitude of the gradient can reflect the degree of centralization. The less the gradient, the less centralized the population. This gradient is a proper indicator to explain the population and land rent distribution as long as the urban structure is monocentric. Once the urban structure transforms to the polycentric pattern, the estimated density gradient is no longer a reasonable estimator, but a low gradient value still expresses that the urban structure is suburbanized.

The sequence of simulations under different commuting costs or the other conditions are presented in Fig. 2. This simulation examines how the population distribution changes, due to an innovation in transport technology in the middle of evolution process. This experiment starts from uniform distribution of both population and employment. The initial population of each zone is 60, the initial export employment equals zero and the initial service employment equals 30 per zone. The density gradients and land rent gradients that resulted from different scenarios are presented in Table 1. The population density distribution at time $t = 15$ (structure A) is shown in Fig. 3. Structure B (Fig. 4) is the structure at $t = 30$ evolves from A ($t = 15$) given all conditions unchanged. Structure C (Fig. 5) is the structure that evolves from A ($t = 15$) at time $t = 30$, with a suddenly lowered transport cost between $t = 16$ and $t = 30$.

In Tab. 1, the density and land rent of structure C are flatter than those of structure B and structure A. These cross sectional and time serious comparisons show that a decrease in transport cost during the evolution will result in a decentralization of urban structure and that a city with a higher transport cost will have a steeper population density than a city with a lower transport cost. These results correspond to the following evidences. First, density is getting flatter with the passage of time and the improvement of technology. Second, the densities of a sample of Japanese cities are 3.2 times as steep as for the sample of American cities since the transport cost in Japan is much higher (See Mills and Hamilton 1989, Chap. 7). Furthermore, the center density of structure C is less than that of structure B: central-city size declines as commuting cost decreases because the adjustment of the trade off between commuting cost and living cost. When the com-

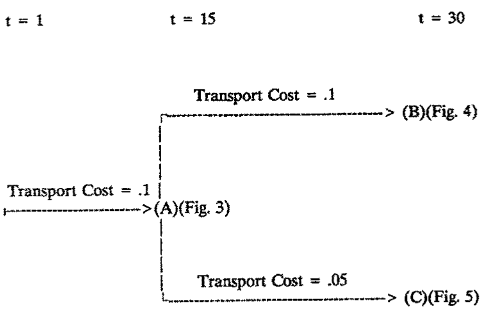

Fig. 2. Change over time in unit transport cost in experiment

Table 1. Density gradient and center zone density of experiment

T Unit of Time	*C* Trans. Cost	*DG* Density Gradient	*CD* Center Density	*RG* Rent Gradient	Stru.
1 – 15	0.1	– 0.0268*	1427	– 0.0353	A
1 – 30	0.1	– 0.0299*	24416	– 0.0279	B
1 – 15	0.1				
16 – 30	0.05	– 0.0153	23780	– 0.0121	C

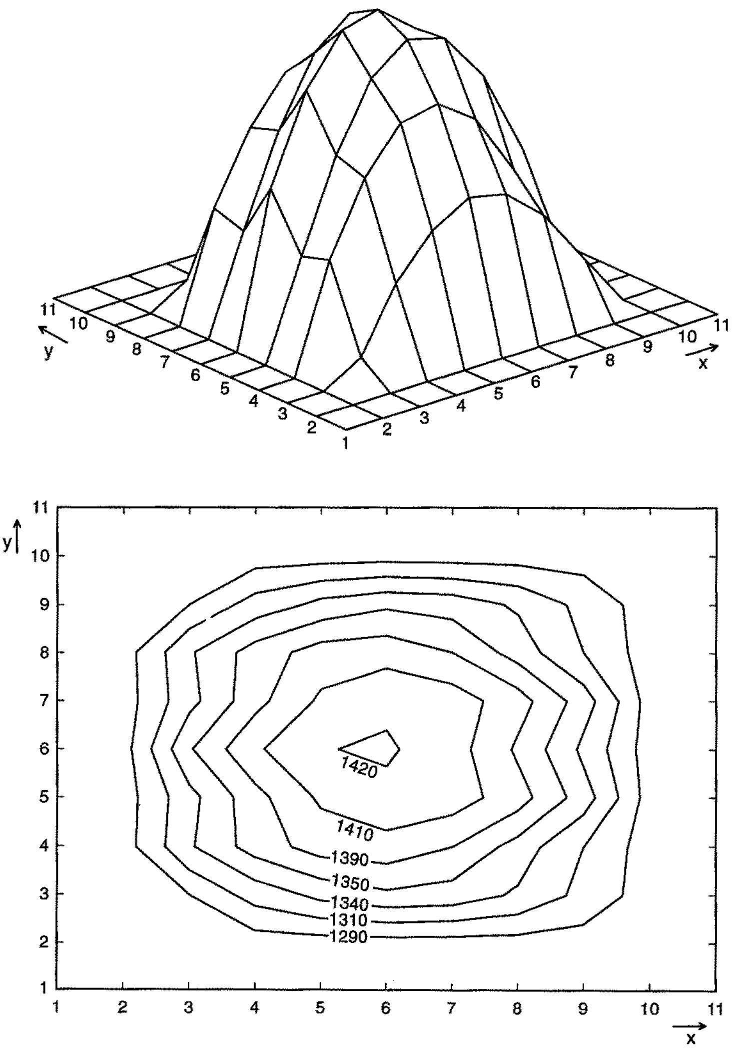

Fig. 3. The population density of structure A in experiment

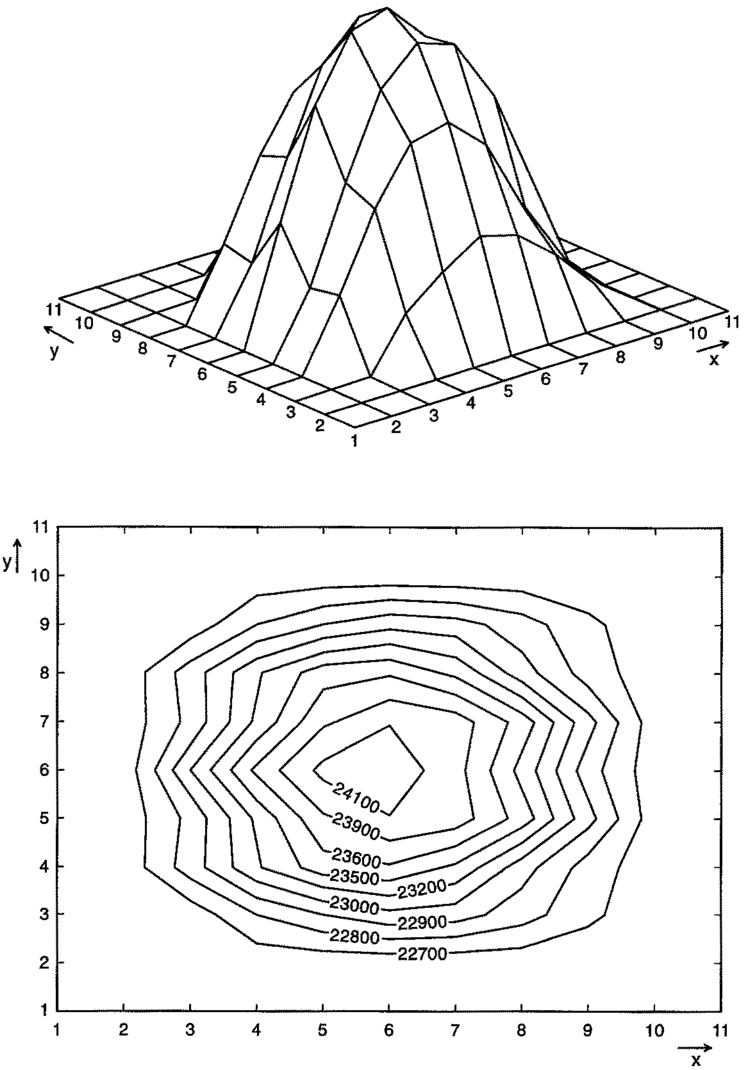

Fig. 4. The population density of structure B in experiment

muting cost declines, the living cost in the central-city appears relatively high compare to surrounding cities. Therefore, the degree of commuting cost is essential in determining the distribution of population and employment.

In Fig. 3, it is obvious that structure A is monocentric. It remains monocentric after 15 units of time (Fig. 4). Central city has the largest population and the highest land rent in the study region. As transportation technology improved during time $t = 16$ to $t = 30$, the cost of transportation between CBD and suburbs is decreasing. Thus the relative location advantage of CBD decreases. In addition, the central-city size is so large that its agglomeration effect is declining and rela-

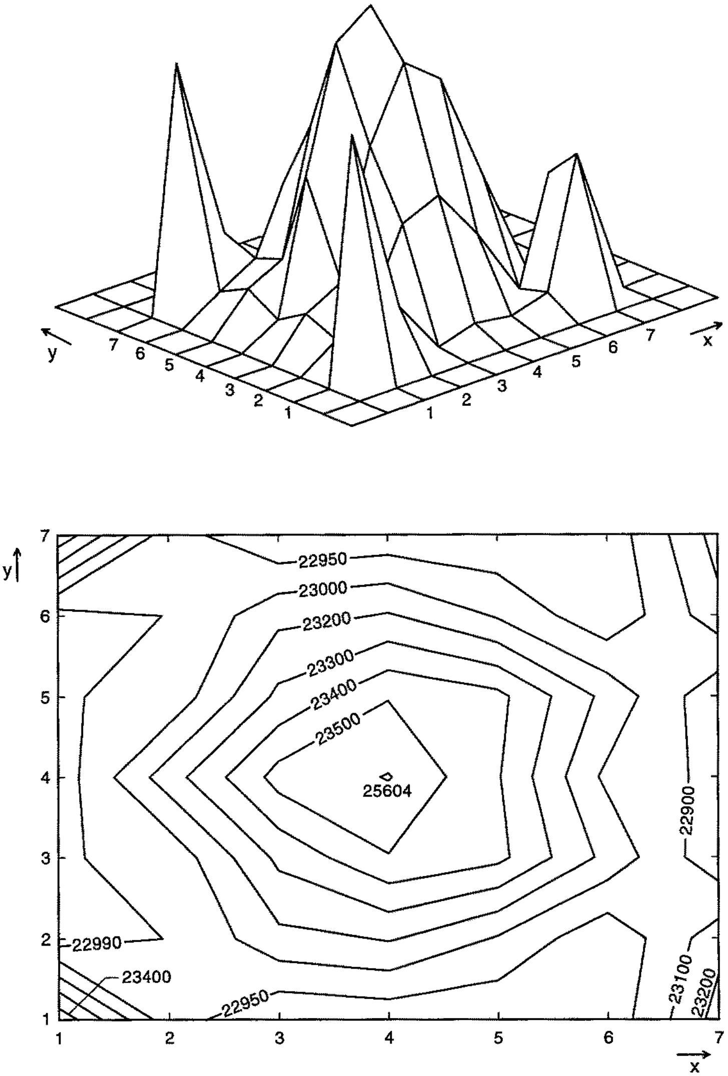

Fig. 5. The population density of structure C in experiment

tively lower than that of other zones. Subcenters tend to emerge under these conditions.

4. Findings

A summary of the results from the simulations is the following:

1) The lower the commuting cost, the flatter the density and land rent gradients, and the smaller the size of CBD. In other words, a reduction in commuting

cost will result in a more suburbanized urban area. The land rent distribution is dispersed along with the decentralized population distribution. Moreover, if commuting cost is zero, the model will predict that population, employment and the land rent are uniformly distributed. These features correspond with the evidence and knowledge presented in Mills and Hamilton (1989) Chap. 7 and Chap. 15. A reduction in commuting cost reduces the premium a resident is willing to pay for a more accessible location. Therefore, a decline in transport cost will make the rent function flatter.

2) An urban area having an improvement in transportation technology, i.e., a reduction of commuting cost, in its history will be more suburbanized than a location with a constant commuting cost. This result is also corresponding to our historical evidence. Historically, the technological improvement has resulted in lower transportation costs in the US. In Table 7.2 of Mills and Hamilton (1989), the average of gradients for population-density functions of four metropolitan areas are getting flatter as time goes by.[2]

3) During the evolution process, some centers emerge outside the CBD. This is due to a reduction in commuting cost and a decline in the central city's agglomeration effect. As the size of central city is getting bigger, the negative agglomeration effect becomes larger relatively to the positive agglomeration effect. Once the negative effect is stronger than the positive effect, the net agglomeration effect is declining. Subcenters are inclined to emerge under the situations that central city is declining and commute cost is reduced.

4) Employment is less suburbanized than population since there is an agglomeration effect in industry production function. People tend to reside outside the center and commute to CBD to work.

5) Most of the large areas in our simulation have flatter gradients than small ones; and older urban areas often have steeper gradients. This observation is similar to the finding from the studies of Mills (1970), Macauley (1984) and Ingram (1982).

6) The simulation results show that the center zone, which is located at the center of the study region, naturally becomes the zone with the highest population density and the highest employment density (CBD) of the region without any scenarios involved during its evolution process. The center zone has higher relative labor-force accessibility and inherits larger economies of agglomeration than the other zones in the region due to its location advantage. However, when its net agglomeration effect is declining caused by congestion, population or other scale diseconomies, the center tends to grow more slowly than the other zones in the region.

This model is expected to be capable of performing the features of both monocentric and polycentric models given different value of parameters and different kind of scenarios.

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2 The four metropolitan areas are Baltimore, Philadelphia, Milwaukee, and Rochester, NY.

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