# Characterizing output processes of $E_{m} / E_{k} / 1$ queues 

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#### Abstract

Our goal is to study which conditions of the output process of a queue preserve the Increasing Failure Rate (IFR) property in the interdeparture time. We found that the interdeparture time does not always preserve the IFR property, even if the interarrival time and service time are both Erlang distributions with IFR. We give a theoretical analysis and present numerical results of $E_{m} / E_{k} / 1$ queues. We show, by numerical examples, that the interdeparture time of $E_{m} / E_{k} / 1$ retains the IFR property if $m \geq k$.


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## 1. Introduction

The queueing model studied in this paper can be viewed as a communication channel where information is encoded into the times of arrivals of packets with noise which sometimes amounts to a virus, and needs to be decoded from the departure times. A decryption process consists of several stages to determine the virus. The departure time may be considered as the time instant that information is delivered to the user. Such channels are of interest in computer and communication security. Thus, users of this system may be concerned in estimating explicitly the mean system down time or mean time of malfunction during a specified time interval, i.e., the probability that the system is down more than $x$ minutes at an instant. This was the question raised in [7] concerning the Increasing Failure Rate (IFR) or decreasing failure rate (DFR) properties, which are to be defined in Definition 2.6. The purpose of this paper is to show that an effective approximation method can be given to evaluate the reliability function of the system and that this approximation can be applied to various Markovian queueing systems with phase-type structure.

From past studies, we know that a $G I / G / 1$ queue has output instants that form a renewal process if and only if the arrival process is Poisson and the services times are exponential. When the output process is not renewal, researchers studied the correlation structure of the output process. This is because departure processes of non- $M / G / 1$ queues are difficult to characterize. In this paper, our goal is to investigate which properties will be preserved for the departure process if the queue is no longer an $M / M / 1$ queue. We will consider a $P H / G / 1$ queue and construct the Laplace-Stieltjes transform (LST) of the interdeparture time distribution, where $P H$ denotes a phasetype distribution. The phase-type distribution can be a good approximation for general distributions. We analyze the

[^0]stochastic properties, such as IFR for the interdeparture time. Because of the computational complexity of $\mathrm{PH} / \mathrm{G} / 1$ and the stationary probability density of the number of customers in the system, we restrict our numerical examples to $E_{m} / E_{k} / 1$ queues, where both the interarrival time and service time are Erlang distributions. We consider that, under certain conditions, the interdeparture time is IFR in the $E_{m} / E_{k} / 1$ queue and illustrate it with numerical results.

Since 1950, many researchers have studied the output process of queueing models. They focused on the distribution of the interdeparture time and the correlation structure of the output process. Daley [2] showed that, among $G I / M / 1$ systems, the only one that has a renewal output process is again the $M / M / 1$ system, and they studied the correlation structure for $G I / G / 1$ queues. Laslett [6] has shown that no $G I / M / 1 / C$ system with finite positive $C$ has a renewal output.

The phase-type distribution and phase-type renewal processes were introduced by Neuts [9], who formed the substrata for the definition of the $N$-process and the Markov-modulated Poisson process (MMPP); see Fischer and Meier-Hellstern [3]. Saito [10] investigated the departure process of an $N / G / 1$ queue, where the arrival process is an $N$-process. Saito focused on the interdeparture times of an $N / D / 1$ queue and showed that correlation of interarrival times is likely to be preserved in interdeparture times of an $N / G / 1$ queue. Luh [8] provided a recursive procedure to calculate the Laplace transform for the joint distribution for $n$ consecutive interdeparture times. Yeh and Chang [12] characterized the departure process of a single server queue from the embedded Markov renewal process at departures. Results obtained include the LST of the stationary distribution function of interdeparture times and a recursive formula for the covariance of interdeparture times.

In this paper, we use an alternative approach to analyze the performance of the output process. We consider the stochastic order relations of the interdeparture time, such as IFR. This is important for the comparison of "new" and "residual" life times. The stochastic order relations of the output processes from one server may be important indices for the input processes of another server in the network. In this study, we investigate conditions under which the interdeparture time will preserve the IFR property. We illustrate this by numerical experiments to verify the main result.

The remaining parts of the paper are organized as follows. In Section 2, we introduce the model of the $P H / G / 1$ queue and derive the LST of the interdeparture time. We define some performance indices of the departure process. In Section 3, we consider the $E_{m} / E_{k} / 1$ queue and discuss the performance of the departure process. In Section 4, we present some numerical results by using Matlab and studying the IFR property of the interdeparture time for $E_{m} / E_{k} / 1$ queues. In Section 5, we draw conclusions and studies for further investigation.

## 2. The model

### 2.1. Description and notation

Consider a first-in first-out (FIFO) single server queue of $P H / G / 1$ type. The service times of all customers are independent and identically distributed, with a distribution function $H(\cdot)$ of finite mean $h$. The Laplace-Stieltjes transform of $H(\cdot)$ is denoted by $\tilde{H}(\cdot)$. The arrival process is a phase-type renewal process, and its interarrival time distribution $F(\cdot)$ has the irreducible representation $(\boldsymbol{\alpha}, \mathbf{T})$ with $m$ phases and mean rate $\lambda . \boldsymbol{\alpha}$ is a probability row vector and the $m \times m$ matrix $\mathbf{T}$ is nonsingular, with negative diagonal elements and nonnegative off-diagonal elements. The vector $\mathbf{T}^{0}$ is nonnegative and satisfies $\mathbf{T e}+\mathbf{T}^{0}=\mathbf{0}$, where $\mathbf{e}=(1, \ldots, 1)^{t}$ is a column $m$-vector of ones. The interarrival time distribution can be written as $F(x)=1-\boldsymbol{\alpha} \exp (\mathbf{T} x)$ e, for $x \geq 0$. The Laplace-Stieltjes transform $\tilde{F}(s)$ of $F(\cdot)$ is given by

$$
\begin{aligned}
\tilde{F}(s) & =\int_{0}^{\infty} \exp (-s x) \mathrm{d} F(x) \\
& =\boldsymbol{\alpha}(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0}, \quad \text { for } \operatorname{Re}(s) \geq 0
\end{aligned}
$$

Let $\left\{\tau_{n}: n \geq 0\right\}$ denote the successive departure epochs with $\tau_{0}=0$, and define $X_{n}$ and $J_{n}$ to be the number of customers in the system and the state of the arrival phase just after $\tau_{n}$, respectively. Set $D_{n}:=\tau_{n+1}-\tau_{n}$. Then, the sequence $\left\{\left(X_{n}, J_{n}, D_{n}\right): n \geq 0\right\}$ forms a semi-Markov sequence on the state space $\{0,1, \ldots\} \times\{1, \ldots, m\}$. The semi-Markov process is positive recurrent when the traffic intensity $\rho=h \lambda<1$. The transition probability matrix $\mathbf{Q}(\cdot)$ is given by

$$
\mathbf{Q}(x)=\left[\begin{array}{lllll}
\mathbf{B}_{0}(x) & \mathbf{B}_{1}(x) & \mathbf{B}_{2}(x) & \mathbf{B}_{3}(x) & \cdots \\
\mathbf{A}_{0}(x) & \mathbf{A}_{1}(x) & \mathbf{A}_{2}(x) & \mathbf{A}_{3}(x) & \cdots \\
& \mathbf{A}_{0}(x) & \mathbf{A}_{1}(x) & \mathbf{A}_{2}(x) & \cdots \\
& & \mathbf{A}_{0}(x) & \mathbf{A}_{1}(x) & \cdots \\
& & & \mathbf{A}_{0}(x) & \cdots \\
& & & & \ddots
\end{array}\right], \quad x \geq 0,
$$

where, for $k \geq 0, \mathbf{A}_{k}(x)$ and $\mathbf{B}_{k}(x)$ are the $m \times m$ matrices of mass functions defined by

$$
\begin{align*}
& {\left[\mathbf{A}_{k}(x)\right]_{i j}=\operatorname{Pr}\left\{X_{n+1}-X_{n}=k-1, J_{n+1}=j, D_{n} \leq x \mid X_{n} \geq 1, J_{n}=i\right\},}  \tag{2.1}\\
& {\left[\mathbf{B}_{k}(x)\right]_{i j}=\operatorname{Pr}\left\{X_{n+1}-X_{n}=k, J_{n+1}=j, D_{n} \leq x \mid X_{n}=0, J_{n}=i\right\},}  \tag{2.2}\\
& \mathbf{B}_{k}(x)=\int_{0}^{x} \exp \{\mathbf{T}(x-t)\} \mathbf{T}^{0} \boldsymbol{\alpha} \mathbf{A}_{k}(t) \mathrm{d} t . \tag{2.3}
\end{align*}
$$

Furthermore, we define the following transform matrices

$$
\begin{aligned}
& \mathbf{A}(x)=\sum_{k=0}^{\infty} \mathbf{A}_{k}(x), \quad \mathbf{B}(x)=\sum_{k=0}^{\infty} \mathbf{B}_{k}(x), \\
& \tilde{\mathbf{A}}(s)=\int_{0}^{\infty} \exp (-s x) \mathrm{d} \mathbf{A}(x), \quad \tilde{\mathbf{B}}(s)=\int_{0}^{\infty} \exp (-s x) \mathrm{d} \mathbf{B}(x) .
\end{aligned}
$$

From Eq. (2.3), it can be shown by a little algebra that

$$
\tilde{\mathbf{B}}(s)=(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \boldsymbol{\alpha} \tilde{\mathbf{A}}(s) .
$$

Now we consider the stationary probability density of the number of customers in the system just after a departure. Write it as $\boldsymbol{\pi}=\left(\pi_{0}, \pi_{1}, \ldots\right)$, where $\boldsymbol{\pi}_{i}=\left(\pi_{i 1}, \pi_{i 2}, \ldots, \pi_{i m}\right), i \geq 0$, with
$\pi_{i j}=\operatorname{Pr}\{$ a departure leaves $i$ customers in the system behind where the arrival process is in phase $j\}$.
The stationary transition probability matrix $\mathbf{Q}(\infty)$ and $\pi$ satisfy

$$
\pi \mathbf{Q}(\infty)=\pi, \quad \pi \mathbf{e}=1
$$

The vector $\pi_{0}$ can be obtained by the method called "matrix-geometric" solutions at departure points. The embedded Markov renewal process, obtained by considering the queue length and the state of the arrival phase, has a transition probability matrix of $M / G / 1$ type. General procedures for solving the probability distributions of queues of this type have been developed by Neuts [9].

With the stationary probability at the departure points, we are able to give the distribution of interdeparture time and investigate its dependence structure.

### 2.2. Departure processes

Next, we will look into the departure process of the $P H / G / 1$ queue, and describe the departure process in terms of the interdeparture intervals. Recall the successive departure time denoted by $\left\{\tau_{n}: n \geq 0\right\}$ with $\tau_{0}=0$. Consider the stationary sequence of positive random variables $\left\{D_{n}\right\}$ with finite mean $E\left[D_{n}\right]$; the corresponding distribution function is given by $D(x)=\operatorname{Pr}\left\{D_{n} \leq x\right\}$, which, in view of stationarity, is the same as that of $D_{n}$ (all $n$ ). Define its Laplace-Stieltjes transforms by $\tilde{D}(s)$.

Theorem 2.1. For the $P H / G / 1$ queue, the LST of the interdeparture time is given by

$$
\begin{cases}\boldsymbol{\pi}_{0}(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \tilde{H}(s)+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) \tilde{H}(s) & \text { if } \rho<1 \\ \tilde{H}(s) & \text { if } \rho \geq 1\end{cases}
$$

Proof. If $\rho \geq 1$, it is straightforward that $\tilde{D}(s)=\tilde{H}(s)$ due to no idle period almost surely. On the other hand, suppose that $\rho<1$. According to (2.1) and (2.2), we write

$$
\begin{aligned}
D(x)= & \operatorname{Pr}\left\{D_{n} \leq x\right\} \\
= & \sum_{i=0}^{\infty} \sum_{j=1}^{m} \operatorname{Pr}\left\{D_{n} \leq x \mid\left(X_{n}, J_{n}\right)=(i, j)\right\} \operatorname{Pr}\left\{\left(X_{n}, J_{n}\right)=(i, j)\right\} \\
= & \sum_{j=1}^{m}\left(\sum_{k=0}^{\infty} \sum_{l=1}^{m} \operatorname{Pr}\left\{X_{n+1}-X_{n}=k, J_{n+1}=l, D_{n} \leq x \mid\left(X_{n}, J_{n}\right)=(0, j)\right\}\right) \operatorname{Pr}\left\{\left(X_{n}, J_{n}\right)=(0, j)\right\} \\
& +\sum_{i=1}^{\infty} \sum_{j=1}^{m}\left(\sum_{k=0}^{\infty} \sum_{l=1}^{m} \operatorname{Pr}\left\{X_{n+1}-X_{n}=k-1, J_{n+1}=l, D_{n} \leq x \mid\left(X_{n}, J_{n}\right)=(i, j)\right\}\right) \\
& \times \operatorname{Pr}\left\{\left(X_{n}, J_{n}\right)=(i, j)\right\} \\
= & \sum_{j=1}^{m} \pi_{0 j}\left(\sum_{k=0}^{\infty} \sum_{l=1}^{m}\left[\mathbf{B}_{k}(x)\right]_{j l}\right)+\sum_{i=1}^{\infty} \sum_{j=1}^{m} \pi_{i j}\left(\sum_{k=0}^{\infty} \sum_{l=1}^{m}\left[\mathbf{A}_{k}(x)\right]_{j l}\right) \\
= & \sum_{j=1}^{m} \pi_{0 j}\left(\sum_{l=1}^{m}[\mathbf{B}(x)]_{j l}\right)+\sum_{i=1}^{\infty} \sum_{j=1}^{m} \pi_{i j}\left(\sum_{l=1}^{m}[\mathbf{A}(x)]_{j l}\right) \\
= & \boldsymbol{\pi}_{0}\{\mathbf{B}(x) \mathbf{e}\}+\sum_{i=1}^{\infty} \pi_{i}\{\mathbf{A}(x) \mathbf{e}\} \\
= & \pi_{0}\{\mathbf{B}(x) \mathbf{e}\}+\left\{\sum_{i=1}^{\infty} \pi_{i}\right\} \mathbf{A}(x) \mathbf{e} .
\end{aligned}
$$

In terms of LST, we rewrite it and give

$$
\begin{aligned}
\tilde{D}(s) & =\boldsymbol{\pi}_{0}\{\tilde{\mathbf{B}}(s) \mathbf{e}\}+\left\{\sum_{i=1}^{\infty} \boldsymbol{\pi}_{i}\right\}\{\tilde{\mathbf{A}}(s) \mathbf{e}\} \\
& =\boldsymbol{\pi}_{0}\left\{(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \boldsymbol{\alpha} \tilde{\mathbf{A}}(s) \mathbf{e}\right\}+\left\{\sum_{i=1}^{\infty} \boldsymbol{\pi}_{i}\right\}\{\tilde{H}(s) \mathbf{e}\} \\
& =\boldsymbol{\pi}_{0}\left\{(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \boldsymbol{\alpha} \tilde{H}(s) \mathbf{e}\right\}+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) \tilde{H}(s) \\
& =\boldsymbol{\pi}_{0}(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \tilde{H}(s)+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) \tilde{H}(s) .
\end{aligned}
$$

This completes the proof.
Remarks: Denote by $I_{n}$ the idle time between serving the $(n-1)$ th and $n$th customers. When the server is busy, we have $I_{n}=0$. Depending on a service condition, the interdeparture time $D_{n}$ of customer $n$ can be written as

$$
\begin{equation*}
D_{n}=I_{n}+S_{n}, \tag{2.4}
\end{equation*}
$$

where $S_{n}$ is the service time of the $n$th customer. Thus, $D(\cdot)$ can be written as the convolution of $I(\cdot)$ and $H(\cdot)$. Apparently, $I(\cdot)$ is the idle time distribution at the departure epochs. $I(\cdot)$ can be considered similarly to $F(\cdot)$, with a stationary probability $\boldsymbol{\pi}_{0}$ instead of $\boldsymbol{\alpha}$, i.e., $I(\cdot)$ is the $P H$-distribution with representation $\left(\boldsymbol{\pi}_{0}, \mathbf{T}\right)$. Its LST is denoted by $\tilde{I}(s)$. Therefore, (2.4) implies

$$
\tilde{D}(s)=\tilde{I}(s) \tilde{H}(s)
$$

Hence, it explicitly yields

$$
\tilde{I}(s)=\boldsymbol{\pi}_{0}(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0}+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) .
$$

### 2.3. Performance analysis of the departure processes

In order to characterize the departure process, we present the Laplace-Stieltjes transform of a corresponding interdeparture time distribution. We obtain other interesting performance measures from its Laplace-Stieltjes transform $\tilde{D}(s)$. First, the moments of the interdeparture distribution are important descriptors of the output process. We recall a general result given in the following lemma.

Lemma 2.2. Let $D$ denote the interdeparture time. Then, we have

$$
E\left[D^{n}\right]=\left.(-1)^{n} \frac{\mathrm{~d}^{n} \tilde{D}(s)}{\mathrm{d} s^{n}}\right|_{s=0}
$$

Thus, we can derive the mean interdeparture time and its variance from its Laplace-Stieltjes transform in Lemma 2.2. The mean interdeparture time determines the throughput of the system. The variance provides a measure of the variability of the output process.

Let $I$ denote the idle time, i.e., the limit of $I_{n}$ as $n \rightarrow \infty$. In (2.4), because $I_{n}$ and $S_{n}$ are independent, we have $E[D]=E[I]+E[S]$ and $\operatorname{Var}[D]=\operatorname{Var}[I]+\operatorname{Var}[S]$ by taking the limit as $n \rightarrow \infty$. Consequently, we have the following theorem.

Theorem 2.3. For the stationary PH/G/l queue, the mean idle time is

$$
E[I]=\pi_{0}(-\mathbf{T})^{-1} \mathbf{e}
$$

and the mean interdeparture time is

$$
E[D]=\pi_{0}(-\mathbf{T})^{-1} \mathbf{e}+h
$$

Proof. By differentiating $\tilde{D}(s)$ with respect to $s$, we have

$$
\begin{equation*}
\tilde{D}^{\prime}(s)=\pi_{0}\left\{-(s \mathbf{I}-\mathbf{T})^{-2} \mathbf{T}^{0} \tilde{H}(s)+(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \tilde{H}^{\prime}(s)\right\}+\left(1-\pi_{0} \mathbf{e}\right) \tilde{H}^{\prime}(s) . \tag{2.5}
\end{equation*}
$$

Hence, we have $E[D]=-\tilde{D}^{\prime}(0)=\pi_{0}(-\mathbf{T})^{-1} \mathbf{e}+h$ and $E[I]=\pi_{0}(-\mathbf{T})^{-1} \mathbf{e}$.
Theorem 2.4. For a stationary PH/G/l queue, the variance of an idle time is

$$
\operatorname{Var}[I]=2 \boldsymbol{\pi}_{0}(-\mathbf{T})^{-2} \mathbf{e}-\left[\boldsymbol{\pi}_{0}(-\mathbf{T})^{-1} \mathbf{e}\right]^{2}
$$

and the variance of the interdeparture time is

$$
\operatorname{Var}[D]=2 \boldsymbol{\pi}_{0}(-\mathbf{T})^{-2} \mathbf{e}-\left[\boldsymbol{\pi}_{0}(-\mathbf{T})^{-1} \mathbf{e}\right]^{2}+\operatorname{Var}[S] .
$$

Proof. By Eq. (2.5), we obtain

$$
\tilde{D}^{\prime \prime}(s)=\boldsymbol{\pi}_{0}\left\{2(s \mathbf{I}-\mathbf{T})^{-3} \mathbf{T}^{0} \tilde{H}(s)-2(s \mathbf{I}-\mathbf{T})^{-2} \mathbf{T}^{0} \tilde{H}^{\prime}(s)+(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0} \tilde{H}^{\prime \prime}(s)\right\}+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) \tilde{H}^{\prime \prime}(s) .
$$

Hence, we have the variance of the interdeparture time given by

$$
\begin{align*}
\operatorname{Var}[D] & =E\left[D^{2}\right]-E^{2}[D]=\tilde{D}^{\prime \prime}(0)-\left[-\tilde{D}^{\prime}(0)\right]^{2} \\
& =2 \pi_{0}(-T)^{-2} \mathbf{e}-\left[\pi_{0}(-T)^{-1} \mathbf{e}\right]^{2}+\operatorname{Var}[S] \tag{2.6}
\end{align*}
$$

where $\operatorname{Var}[S]=\tilde{H}^{\prime \prime}(0)-h^{2}$ is the variance of the service time. Then, $\operatorname{Var}[I]=2 \pi_{0}(-\mathbf{T})^{-2} \mathbf{e}-\left[\boldsymbol{\pi}_{0}(-\mathbf{T})^{-1} \mathbf{e}\right]^{2}$.
With the independence of $I$ and $S$, it shows that, for a $G I / G / 1$ system, we have $\operatorname{Var}[D] \geq \operatorname{Var}[S]$, which reconfirms the results attained by Daley [2] that the equality holds if and only if the system is $D / D / 1$ without an idle time.

Now, we consider the squared coefficient of variation of the interdeparture time $c_{D}^{2}$, which is defined by

$$
c_{D}^{2}=\frac{\operatorname{Var}[D]}{E^{2}[D]}
$$

The squared coefficient of variation is a measure of the variability of the random variable $D$. For example, the deterministic distribution has $c_{D}^{2}=0$, the exponential distribution has $c_{D}^{2}=1$, and the Erlang- $k$ distribution has the intermediate value $c_{D}^{2}=1 / k$.

For a stationary queue, the departure rate will be the same as the arrival rate. That is, equivalently, $E[D]=\frac{1}{\lambda}$. From Theorem 2.4, we have the squared coefficient of variation of the interdeparture time as follows.

Corollary 2.5. For the stationary $P H / G / 1$ queue, we have the squared coefficient of variation of the interdeparture time

$$
c_{D}^{2}=\lambda^{2}\left\{2 \pi_{0}(-T)^{-2} \mathbf{e}-\left[\pi_{0}(-T)^{-1} \mathbf{e}\right]^{2}+\operatorname{Var}(S)\right\} .
$$

Proof. Under the stability assumption, the mean departure rate is $\lambda$ and, by (2.6), we have the result.
In previous studies, many researchers have analyzed the correlation of departure processes in various models such as $M / M / 1, M / E_{k} / 1$, and $M / G / 1$ in Jenkins [4], King [5], and Takagi and Nishi [11], respectively. Nevertheless, we intend to investigate which properties will be preserved for the departure process if the queue is no longer an $M / M / 1$ queue. In particular, we will pay attention to the property of IFR. Both important characteristics will be discussed after introducing the stochastic order relations.

We briefly review a few notions of the stochastic order relations. First of all, we recall the concept of failure rates for the lifetime distribution of an item. Consider the stationary interdeparture time, a nonnegative random $D$, as the lifetime of an item.

Let $X$ be a nonnegative random variable with distribution $F_{X}(\cdot)$ and density $f_{X}(\cdot)$ when it exists. The survival probability of $X$ is given by

$$
\bar{F}_{X}(x)=1-F_{X}(x)=P\{X>x\}, \quad x \geq 0 .
$$

The failure rate or hazard rate is defined by

$$
\begin{equation*}
r(x)=\frac{f_{X}(x)}{\bar{F}_{X}(x)}, \quad x \geq 0 \tag{2.7}
\end{equation*}
$$

Referring to Buzacott and Shanthikumar [1], we give the following definitions.
Definition 2.6. A non-negative random variable $X$ is said to have an increasing failure rate (IFR) if the failure rate $r(x)$ is non-decreasing in $x$, and it is said to have a decreasing failure rate (DFR) if $r(x)$ is non-increasing in $x$.

Theorem 2.7. For an Erlang-k distribution with mean $\frac{1}{\mu}$, the failure rate is increasing from zero to $k \mu$, which the means that the Erlang-k distribution is IFR.

Proof. Consider a gamma density function $g(t)$ which is given by

$$
g(t)=\frac{\theta^{\beta} t^{\beta-1}}{\Gamma(\beta)} \mathrm{e}^{-\theta t}, \quad t \geq 0,
$$

where the shape parameter $\beta$ and the scale parameter $\theta$ are both positive. $\Gamma(\beta)$ is the complete gamma function defined by $\Gamma(\beta)=\int_{0}^{\infty} \mathrm{e}^{-t} t^{\beta-1} \mathrm{~d} t, \beta>0$. Characterizing the failure rate of the gamma density, we observe that the failure rate function is increasing from zero to $\theta$ for $\beta>1$ and is decreasing from infinity to $\theta$ for $\beta<1$. The gamma distribution with $\beta=1$ (i.e., the exponential distribution) has a constant failure rate $\theta$. Since an Erlang- $k$ distribution is a special case of the gamma density function with $\beta=k>1$, we have proved the theorem.

Note that the failure rate of the exponential distribution $r(x)=f(x) / \bar{F}(x)=\mu \exp (-\mu x) / \exp (-\mu x)=\mu$ is a constant. However, we are interested in the interdeparture time, whether or not it preserves the IFR property. We obtain the distribution $D(x)$ from its LST $\tilde{D}(s)$ by taking its inversion of LST and calculating $r(x)$ of the interdeparture time distribution. Because of the complexity of the $P H / G / 1$ queue, we restrict our study to the $E_{m} / E_{k} / 1$ queue, which is described in the next section. We examine conditions under which the interdeparture time is IFR for the $E_{m} / E_{k} / 1$ queue, where the service time and interarrival time are both Erlang distributions with increasing failure rates.

### 2.4. Departure processes of a $P H / D / 1$ queue

This subsection focuses on a deterministic service time. In this case, we have the distribution function of the service time

$$
H(x)= \begin{cases}1 & \text { if } x \geq h \\ 0 & \text { if } x<h\end{cases}
$$

which has Laplace-Stieltjes transform $\tilde{H}(s)=\exp (-s h)$. Then, by Theorem 2.1, we have that the LST of the interdeparture time is given by

$$
\begin{equation*}
\tilde{D}(s)=\left[\boldsymbol{\pi}_{0}(s \mathbf{I}-\mathbf{T})^{-1} \mathbf{T}^{0}+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right)\right] \exp (-s h) . \tag{2.8}
\end{equation*}
$$

Therefore, we have the variance and the squared coefficient of variance of the interdeparture time, which are

$$
\operatorname{Var}[D]=2 \pi_{0}(-\mathbf{T})^{-2} \mathbf{e}-\left[\pi_{0}(-\mathbf{T})^{-1} \mathbf{e}\right]^{2}
$$

and

$$
c_{D}^{2}=\lambda^{2}\left\{2 \pi_{0}(-T)^{-2} \mathbf{e}-\left[\pi_{0}(-T)^{-1} \mathbf{e}\right]^{2}\right\} .
$$

Now, we consider the failure rate $r(x)$ of the interdeparture time. It is written in the following theorem.
Theorem 2.8. For the stationary $P H / D / 1$ queue, the failure rate $r(x)$ is given by

$$
r(x)= \begin{cases}r_{I}(x-h) & \text { if } x>h  \tag{2.9}\\ \frac{1-\pi_{0} \mathbf{e}}{\pi_{0} \mathbf{e}} & \text { if } x=h \\ 0 & \text { if } x<h\end{cases}
$$

where $r_{I}(\cdot)$ is the failure rate of the idle time distribution.
Proof. Suppose that $x>h$. From (2.7) and by setting $D=I+h$, where $h$ is a constant, we have

$$
\begin{aligned}
r(x) & =\frac{\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{Pr}\{D \leq x\}}{\operatorname{Pr}\{D>x\}} \\
& =\frac{\frac{\mathrm{d}}{\mathrm{~d} x} \operatorname{Pr}\{I \leq x-h\}}{\operatorname{Pr}\{I>x-h\}} \\
& =r_{I}(x-h) .
\end{aligned}
$$

Since the service time is a constant $h$, the probability of $x<h$ is zero. If $x=h$, then

$$
r(h)=r_{I}(0)=\frac{1-\pi_{0} \mathbf{e}}{\pi_{0} \mathbf{e}} .
$$

Hence, we have completed the proof.
We have given the failure rate $r(x)$ of $P H / D / 1$. In general, we know that, if the $\lim _{x \rightarrow h^{+}} r(x)=\lim _{x \rightarrow 0^{+}} r_{I}(x)<$ $\frac{1-\pi_{0} \mathrm{e}}{\pi_{0} \mathrm{e}}$, the interdeparture time is not IFR. In this regard, the interdeparture time of the $M / D / 1$ queue is not IFR. We will discuss this in the next section.

## 3. Performance analysis of departure processes of the $E_{m} / E_{k} / \mathbf{1}$ queue

### 3.1. Laplace-Stieltjes transform

In this section, we deal with the output process of the $E_{m} / E_{k} / 1$ queue with arrival rate $\lambda$ and mean service time $1 / \mu$. The Erlang- $k$ distribution with mean $\frac{1}{\lambda}$ is a special case of $P H$-distributions. It may be presented by
$\boldsymbol{\alpha}=(1,0, \ldots, 0)$, and the $k \times k$ matrix $\mathbf{T}$ is given by

$$
\mathbf{T}=\left[\begin{array}{ccccc}
-k \lambda & k \lambda & & & \\
& -k \lambda & k \lambda & & \\
& & \cdots & \cdots & \\
& & & & -k \lambda
\end{array}\right]
$$

The probability density function can be written as

$$
f(x)=\frac{(k \lambda)^{k}}{(k-1)!} x^{k-1} \exp (-k \lambda x), \quad \text { for } x \geq 0
$$

and the Laplace-Stieltjes transform $\tilde{F}(s)$ is given by

$$
\begin{equation*}
\tilde{F}(s)=\left(\frac{k \lambda}{s+k \lambda}\right)^{k} . \tag{3.1}
\end{equation*}
$$

By Theorem 2.1, we may obtain the LST of the interdeparture time.
Lemma 3.1. Let $\tilde{D}(s)$ be the LST of the stationary interdeparture time distribution for the $E_{m} / E_{k} / 1$ queue. This produces

$$
\begin{equation*}
\tilde{D}(s)=\sum_{j=1}^{m} \boldsymbol{\pi}_{0}(j)\left(\frac{m \lambda}{s+m \lambda}\right)^{m+1-j}\left(\frac{k \mu}{s+k \mu}\right)^{k}+\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right)\left(\frac{k \mu}{s+k \mu}\right)^{k} . \tag{3.2}
\end{equation*}
$$

Proof. By Theorem 2.1 and (3.1), it is easy to attain the result.

### 3.2. Performance analysis

Now, we consider some important performance measures of the interdeparture time for the $E_{m} / E_{k} / 1$ queue. Applying formulas for the moments given by LST in Lemma 2.2, we obtain the mean of the interdeparture time distribution and have the following results.

Lemma 3.2. For the stationary $E_{m} / E_{k} / 1$ queue, we have

$$
E[D]=\sum_{j=1}^{m} \frac{\pi_{0}(j)(m+1-j)}{m \lambda}+\frac{1}{\mu} .
$$

Proof. Given a distribution function of $E_{m}$ and

$$
-\mathbf{T}^{-1}=\left[\begin{array}{ccccc}
\frac{1}{m \lambda} & \frac{1}{m \lambda} & \frac{1}{m \lambda} & \cdots & \frac{1}{m \lambda} \\
& \frac{1}{m \lambda} & \frac{1}{m \lambda} & \cdots & \frac{1}{m \lambda} \\
& & \cdots & \cdots & 1 \\
& & & & \frac{1}{m \lambda}
\end{array}\right],
$$

we have

$$
-\mathbf{T}^{-1} \mathbf{e}=\left(\frac{m}{m \lambda}, \frac{m-1}{m \lambda}, \ldots, \frac{1}{m \lambda}\right)^{t} .
$$

By Theorem 2.3 and $h=1 / \mu$, we have proved the result.

Lemma 3.3. For a stationary $E_{m} / E_{k} / 1$ queue, we have the variance of the interdeparture time $\operatorname{Var}[D]$ which is

$$
\sum_{j=1}^{m} \frac{\pi_{0}(j)(m+1-j)^{2}}{(m \lambda)^{2}}+\sum_{j=1}^{m} \frac{\pi_{0}(j)(m+1-j)}{(m \lambda)^{2}}-\left(\sum_{j=1}^{m} \frac{\pi_{0}(j)(m+1-j)}{m \lambda}\right)^{2}+\frac{1}{k \mu^{2}}
$$

Proof. Given $\mathbf{T}$ of $E_{m}$, we have

$$
(-\mathbf{T})^{-2}=\frac{1}{(m \lambda)^{2}}\left[\begin{array}{ccccc}
1 & 2 & 3 & \cdots & m \\
& 1 & 2 & \cdots & m-1 \\
& & \cdots & \cdots & 1
\end{array}\right] .
$$

Hence, we have

$$
(-\mathbf{T})^{-2} \mathbf{e}=\frac{1}{2(m \lambda)^{2}}\left(m^{2}+m,(m-1)^{2}+(m-1), \ldots, 1+1\right)^{t} .
$$

By Theorem 2.4, the result is derived immediately.
Lemma 3.4. For the stationary $E_{m} / E_{k} / 1$ queue, we have $E[D]=\frac{1}{\lambda}$, which implies

$$
\sum_{j=1}^{m} \pi_{0}(j)(m+1-j)=m\left(1-\frac{\lambda}{\mu}\right) .
$$

Proof. This is trivial because, under the stability condition $\frac{\lambda}{\mu}<1$, the departure rate will be the same as the arrival rate. From Lemma 3.2, we have

$$
E[D]=\sum_{j=1}^{m} \frac{\pi_{0}(j)(m+1-j)}{m \lambda}+\frac{1}{\mu}=\frac{1}{\lambda}
$$

implying

$$
\sum_{j=1}^{m} \pi_{0}(j)(m+1-j)=m\left(1-\frac{\lambda}{\mu}\right) .
$$

Lemma 3.4 gives an average of the remaining phases of arrival to enter the system. It also provides a simple calculation for obtaining $c_{D}^{2}$.

Theorem 3.5. For a stationary $E_{m} / E_{k} / 1$ model, we have the squared coefficient of variation of the interdeparture time $c_{D}^{2}=\frac{\operatorname{Var}[D]}{E^{2}[D]} \leq 1$.

Proof. Let $\rho=\frac{\lambda}{\mu}$. By Lemmas 3.3 and 3.4, we have

$$
\begin{aligned}
c_{D}^{2} & =\frac{1}{m^{2}} \sum_{j=1}^{m} \pi_{0}(j)(m+1-j)^{2}+\frac{1}{m}(1-\rho)-(1-\rho)^{2}+\frac{1}{k} \rho^{2} \\
& \leq(1-\rho)+\frac{1}{m}(1-\rho)-(1-\rho)^{2}+\frac{1}{k} \rho^{2} \\
& =\rho(1-\rho)+\frac{1}{m}(1-\rho)+\frac{1}{k} \rho^{2} \leq 1 .
\end{aligned}
$$

By a different approach, Buzacott and Shanthikumar [1] showed that, for any $G I / G / 1$ queue with DMRL (decreasing mean residual life) interarrival time, the upper bound of $c_{D}^{2}$ is $c_{A}^{2}(1-\rho)+\rho^{2} c_{S}^{2}+\rho(1-\rho)$, and when the interarrival time has the property IMRL (increasing mean residual life), the lower bound of $c_{D}^{2}$ is $c_{A}^{2}(1-\rho)+\rho^{2} c_{S}^{2}+\rho(1-\rho)$, where $c_{A}^{2}$ and $c_{S}^{2}$ are the squared coefficient of variation of the interarrival time
and service time, respectively. Since Erlang distributions are DMRL and hyper-exponential distributions are IMRL, it is easy to check with the upper bound condition of the $E_{m} / E_{k} / 1$ queue that has $c_{D}^{2} \leq 1$. With the lower bound condition, it is easy to show that the $H_{m} / H_{k} / 1$ queue has $c_{D}^{2} \geq 1$, in which the interarrival time and service time are both hyper-exponential distributions with $c_{A}^{2}$ and $c_{S}^{2}$ being greater than one.

### 3.3. Stochastic properties

In this subsection, we consider the failure rate for the stationary interdeparture time of $E_{m} / E_{k} / 1$ queues. We find that the stationary interdeparture time does not preserve the property of IFR, even if the interarrival time and the service time are both IFR. We will consider conditions under which the interdeparture time preserves the property of IFR for the $E_{m} / E_{k} / 1$ queue.

From Eq. (3.2), we employ the method of partial fractions that separates it in terms of $\lambda$ and $\mu$, respectively. We have

$$
\begin{equation*}
\tilde{D}(s)=\sum_{j=1}^{m} a_{j}\left(\frac{m \lambda}{s+m \lambda}\right)^{j}+\sum_{i=1}^{k} b_{i}\left(\frac{k \mu}{s+k \mu}\right)^{i}, \tag{3.3}
\end{equation*}
$$

where $a_{j}$ and $b_{i}$ are coefficients associated with each term of $\lambda$ and $\mu$. We will discuss the method of partial fractions and how to obtain $a_{j}$ and $b_{i}$ in numerical examples.

Taking the inverse transform of Eq. (3.3), we have that the density function of the interdeparture time is in the form

$$
d(x)=\sum_{j=1}^{m} a_{j} \frac{(m \lambda)^{j}}{(j-1)!} x^{j-1} \exp (-m \lambda x)+\sum_{i=1}^{k} b_{i} \frac{(k \mu)^{i}}{(i-1)!} x^{i-1} \exp (-k \mu x) .
$$

The coefficients $a_{j}$ and $b_{i}$ are attained in accordance with the method of partial fractions and the stationary probability $\pi_{0}$ that a departure leaves the system empty with respect to the arrival phases. Because there are no general properties of $a_{j}$ and $b_{i}$, we consider the initial value of the failure rate $r(x)$ of the interdeparture time distribution, and we have the following theorem.

Theorem 3.6. For a stationary $E_{m} / E_{k} / 1$ queue, the initial value of the failure rate $r(x)$ of the interdeparture time distribution is

$$
\lim _{x \rightarrow 0^{+}} r(x)= \begin{cases}\left(1-\pi_{0} \mathbf{e}\right) \mu & \text { if } k=1 \\ 0 & \text { if } k \geq 2\end{cases}
$$

Proof. With the phase parameter of service time distribution $k \geq 2$, it is easy to verify that $\lim _{x \rightarrow 0^{+}} r(x)=$ $\lim _{x \rightarrow 0^{+}} d(x)=0$. But, when the service time is an exponential distribution, $\lim _{x \rightarrow 0^{+}} d(x)$ is a constant, namely $\left(1-\pi_{0} \mathbf{e}\right) \mu$, which is not equal to zero.

Now we consider the final value of the failure rate $r(x)$ of the interdeparture time distribution. Since the final value of failure rates of an Erlang- $k$ distribution converges to $\theta$ as $x \rightarrow \infty$, where $\theta=k \mu$, does the failure rate of the interdeparture time distribution $r(x)$ converge as $x \rightarrow \infty$, and what is the limit?

Theorem 3.7. For the stationary $E_{m} / E_{k} / 1$ queue, the final value of the failure rate $r(x)$ of the interdeparture time distribution is given by

$$
\lim _{x \rightarrow \infty} r(x)= \begin{cases}m \lambda & \text { if } m \lambda \leq k \mu, \\ k \mu & \text { if } m \lambda>k \mu, \\ (k<m \rho) \\ (k<m \rho) .\end{cases}
$$

Equivalently, it gives

$$
\lim _{x \rightarrow \infty} r(x)=\min \{m \lambda, k \mu\} .
$$

Proof. Taking the limit of $r(x)$, we have

$$
\lim _{x \rightarrow \infty} r(x)=\lim _{x \rightarrow \infty} \frac{d(x)}{\bar{D}(x)}=\lim _{x \rightarrow \infty}-\frac{d^{\prime}(x)}{d(x)}
$$

where

$$
\begin{aligned}
d^{\prime}(x)= & \sum_{j=2}^{m} a_{j} \frac{(m \lambda)^{j}}{(j-2)!} x^{j-2} \exp (-m \lambda x)-\sum_{j=1}^{m} a_{j} \frac{(m \lambda)^{j+1}}{(j-1)!} x^{j-1} \exp (-m \lambda x) \\
& +\sum_{i=2}^{k} b_{i} \frac{(k \mu)^{i}}{(i-2)!} x^{i-2} \exp (-k \mu x)-\sum_{i=1}^{k} b_{i} \frac{(k \mu)^{i+1}}{(i-1)!} x^{i-1} \exp (-k \mu x) .
\end{aligned}
$$

To characterize the limit of the failure rate, we distinguish it between the cases $m \lambda>k \mu$ and $m \lambda \leq k \mu$, then we have

$$
\lim _{x \rightarrow \infty} r(x)= \begin{cases}m \lambda & \text { if } m \lambda \leq k \mu, \\ k \mu & \text { if } m \lambda>k \mu, \\ (k<m \rho)\end{cases}
$$

Is the failure rate of the interdeparture time increasing in $x$ ? The answer is not easy to provide by analysis in general, because of the complexity of the stationary density $\boldsymbol{\pi}_{0}$. Instead, we discuss the failure rate property in the next section and examine it by numerical method using Matlab. We will display some numerical results in the next section.

### 3.4. Failure rate analysis of $E_{m} / D / 1$ queues

In this subsection, we consider a deterministic service case, since $F(\cdot)$ converges to a constant as $k \rightarrow \infty$. According to Section 2, we can obtain several performance indices for the departure process. We examine whether or not the interdeparture time of the $E_{m} / D / 1$ queue has a non-decreasing failure rate. By Theorem 2.8, the failure rate $r(x)$ of a stationary $E_{m} / D / 1$ queue is given by (2.9). The LST of $I(\cdot)$ is given by $\tilde{I}(s)=\sum_{j=1}^{m} \pi_{0}(j)\left(\frac{m \lambda}{s+m \lambda}\right)^{m+1-j}+$ $\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right)$, which is derived from Theorem 2.1. Let $i(x)$ be the probability density function of $I(x)$. We have

$$
\lim _{x \rightarrow 0^{+}} I(x)=1-\pi_{0} \mathbf{e}
$$

and

$$
\lim _{x \rightarrow 0^{+}} i(x)=\pi_{0}(m) m \lambda .
$$

Since $r_{I}(x)=\frac{i(x)}{1-I(x)}$, we have

$$
\begin{equation*}
\lim _{x \rightarrow 0^{+}} r_{I}(x)=\frac{\boldsymbol{\pi}_{0}(m) m \lambda}{\boldsymbol{\pi}_{0} \mathbf{e}} \tag{3.4}
\end{equation*}
$$

For $m=1$, namely the $M / D / 1$ queue, we have $I(x)=1-(1-\rho) \exp (-\lambda x)$ for $x>0$. Then, $\lim _{x \rightarrow 0^{+}} r_{I}(x)=\lambda$. When we let $\lambda=\rho$ with service rate $\mu=1$, we have $\lim _{x \rightarrow h^{+}} r(x)=\lambda<\rho /(1-\rho)$. This implies that the output process for the $M / D / 1$ queue does not have the IFR property.

If $m \geq 2$, we have $\lim _{x \rightarrow h^{+}} r(x)=\lim _{x \rightarrow 0^{+}} r_{I}(x)=\frac{\pi_{0}(m) m \lambda}{\pi_{0} \mathrm{e}}$ by Eq. (3.4). We will investigate whether or not $\lim _{x \rightarrow h^{+}} r(x)$ is larger than $\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) / \boldsymbol{\pi}_{0} \mathbf{e}$ by numerical method in the next section.

## 4. Numerical examples and discussion

We know that the interdeparture time for the $M / M / 1$ queue has an exponential distribution and it is IFR with a constant failure rate. Is the failure rate of the interdeparture time for the $E_{m} / E_{k} / 1$ queue non-decreasing? When the arrival process is not a Poisson process, the analysis of the output process becomes more complex. In this section, we will give the results by displaying graphs of $r(x)$ in Matlab and examining the failure rate of various $E_{m} / E_{k} / 1$ queues.

We will show how we calculate the failure rate of the interdeparture time by numerical examples. We take advantage of the computer to visualize the performance of the output process. Let the service time distribution be Erlang- $k$ with mean service rate 1.0 for $k=1,2,3,5,10,20$. The arrival process is Erlang- $m$ with arrival rate $0.1,0.5$, and 0.9 for light, median and heavy traffic. To characterize the property of the failure rate, we categorize the stationary $E_{m} / E_{k} / 1$ queues into the following cases.

## 4.1. $M / E_{k} / 1$ queues

In this case, we set $m=1$ and $k=2,3,5,10,20$. Since $k \geq 2$, we have that the initial value of $r(x)$ is zero from Theorem 3.6. From Theorem 3.7, we have that the failure rate $r(x)$ converges to $\lambda$ because $k>1>\rho$. We take the $M / E_{2} / 1$ queue with $\rho=0.1$ as an example, explained in the following.

Example 1. $M / E_{2} / 1$ queue with $\rho=0.1$.
First, from Lemma 3.1, we have that the LST of the interdeparture time $\tilde{D}(s)$ is given by

$$
\tilde{D}(s)=(1-\rho) \frac{\lambda}{s+\lambda}\left(\frac{2 \mu}{s+2 \mu}\right)^{2}+\rho\left(\frac{2 \mu}{s+2 \mu}\right)^{2} .
$$

By (3.3), we use the method of partial fractions. Taking $\lambda=0.1$ and $\mu=1$, we have

$$
\begin{equation*}
\tilde{D}(s)=\frac{360}{361} \frac{0.1}{s+0.1}-\frac{18}{361} \frac{2}{s+2}+\frac{1}{19}\left(\frac{2}{s+2}\right)^{2} . \tag{4.1}
\end{equation*}
$$

By the inverse of the LST (4.1), we have

$$
D(x)=1-\left[\frac{360}{361} \mathrm{e}^{-0.1 x}+\frac{1}{361} \mathrm{e}^{-2 x}+\frac{2}{19} x \mathrm{e}^{-2 x}\right], \quad x \geq 0
$$

The expected value is $E[D]=10.0$ and the variance is $\operatorname{Var}[D]=99.5$. Then the squared coefficient of variation is $c_{D}^{2}=0.995<1$. Thus, we have the failure rate $r(x)=\frac{d(x)}{1-D(x)}$ by definition. Taking the limit of $r(x)$ as $x \rightarrow 0^{+}$and $x \rightarrow \infty$, we have the initial value 0 and the final value 0.1 of $r(x)$. We draw the graph of $r(x)$ using Matlab and find that $r(x)$ is not non-decreasing in $x$. The graph is given in Fig. 1 .

We plot $r(x)$ for $k=2,3,5,10,20$ for $M / E_{k} / 1$ with traffic loads $\rho=0.5$ and 0.9 in Figs. 2 and 3, respectively. From Figs. 1-3, we found that, for every $M / E_{k} / 1$ queue, there exists a value $c$ such that the failure rate $r(x)$ increases from zero to $c$ then decreases and converges to $\lambda$. From this, we know that the failure rate of interdeparture time for $M / E_{k} / 1$ queues is not IFR.

## 4.2. $E_{m} / M / 1$ queues

In this case, let $k=1$ and $m=2,3,5,10,20$. Since $k=1$, we have that the initial value of $r(x)$ is $\left(1-\pi_{0} \mathbf{e}\right) \mu$ from Theorem 3.6. From Theorem 3.7, we have that the failure rate $r(x)$ converges to $\min \{m \lambda, \mu\}$. We take the $E_{2} / M / 1$ queue with $\rho=0.5$ as an example, explained in the following.

Example 2. $E_{2} / M / 1$ queue with $\rho=0.5$.
From Lemma 3.1, we have that the LST of the interdeparture time $\tilde{D}(s)$ is given by

$$
\tilde{D}(s)=\pi_{0}(1)\left(\frac{2 \lambda}{s+2 \lambda}\right)^{2} \frac{\mu}{s+\mu}+\pi_{0}(2) \frac{2 \lambda}{s+2 \lambda} \frac{\mu}{s+\mu}+\left(1-\pi_{0} \mathbf{e}\right) \frac{\mu}{s+\mu} .
$$

When $\lambda=0.5$ and $\mu=1$, we have $2 \lambda=\mu$. In this example, we do not need to use the method of partial fractions.
Taking $\lambda=0.5$ and $\mu=1$, we have

$$
\begin{equation*}
\tilde{D}(s)=\pi_{0}(1)\left(\frac{1}{s+1}\right)^{3}+\pi_{0}(2)\left(\frac{1}{s+1}\right)^{2}+\left(1-\pi_{0} \mathbf{e}\right) \frac{1}{s+1} . \tag{4.2}
\end{equation*}
$$



Fig. 1. $r(x)$ of $M / E_{k} / 1$ with $\rho=0.1$.


Fig. 2. $r(x)$ of $M / E_{k} / 1$ with $\rho=0.5$.

By using the "matrix-geometric" method to solve $\boldsymbol{\pi}_{0}$, we have $\boldsymbol{\pi}_{0}=(0.38196601125011,0.23606797749979)$. Taking the inverse of the LST (4.2), we have

$$
D(x)=1-\mathrm{e}^{-x}-0.618034 \cdot x \mathrm{e}^{-x}-0.190983 \cdot x^{2} \mathrm{e}^{-x}, \quad x \geq 0 .
$$

The expected value is $E[D]=2.0$ and the variance is $\operatorname{Var}[D]=2.76393$. Then we have that the squared coefficient of variation is $c_{D}^{2}=0.69098<1$. Furthermore, we have the failure rate

$$
\begin{align*}
r(x) & =\frac{d(x)}{1-D(x)}=\frac{0.381966 \cdot \mathrm{e}^{-x}+0.236068 \cdot x \mathrm{e}^{-x}+0.190983 \cdot x^{2} \mathrm{e}^{-x}}{\mathrm{e}^{-x}+0.618034 \cdot x \mathrm{e}^{-x}+0.190983 \cdot x^{2} \mathrm{e}^{-x}} \\
& =\frac{0.381966+0.236068 x+0.190983 x^{2}}{1+0.618034 x+0.190983 x^{2}} . \tag{4.3}
\end{align*}
$$



Fig. 3. $r(x)$ of $M / E_{k} / 1$ with $\rho=0.9$.


Fig. 4. $r(x)$ of $E_{m} / M / 1$ with $\rho=0.1$.
Taking the limit of $r(x)$ in (4.3) as $x \rightarrow 0^{+}$and $x \rightarrow \infty$, we have the initial value 0.381966 and the final value 1 of $r(x)$. By differentiating $r(x)$ in (4.3), we have

$$
r^{\prime}(x)=\frac{0.072949012 x^{2}+0.236067974 x+0.0000000252}{\left(1+0.618034 x+0.190983 x^{2}\right)^{2}}>0, \quad \text { for } x \geq 0 .
$$

Hence, $r(x)$ is increasing in $x$. That is, the interdeparture time of the $E_{2} / M / 1$ queue with $\rho=0.5$ preserves the IFR property. We draw the graph of $r(x)$ using Matlab and find that $r(x)$ is non-decreasing in $x$. The graph is given in Fig. 5.

Meanwhile, we plot $r(x)$ for $m=2,3,5,10,20$ for $E_{m} / M / 1$ with traffic loads $\rho=0.1$ and 0.9 given in Figs. 4 and 6 , respectively.

We found that the initial value of $r(x)$ is $\left(1-\pi_{0} \mathbf{e}\right) \mu$, not zero. From Figs. 4-6, we have that the failure rate $r(x)$ increases from $\left(1-\pi_{0} \mathbf{e}\right) \mu$ and converges to $\min \{m \lambda, 1\}$. In Fig. 6, the failure rate converges to 1 . No matter whether the traffic load is light, medium or heavy, we found that the interdeparture time of $E_{m} / M / 1$ queues preserves the IFR property in our experiments.


Fig. 5. $r(x)$ of $E_{m} / M / 1$ with $\rho=0.5$.


Fig. 6. $r(x)$ of $E_{m} / M / 1$ with $\rho=0.9$.

## 4.3. $E_{m} / E_{k} / 1$ queues

In this case, we consider the general cases of Erlang distributions. We investigate the failure rate function of the $E_{m} / E_{k} / 1$ queues, where $m$ and $k$ are given by $2,3,5,10,20$. Since service time is assumed to be an Erlang- $k$ distribution with $k \geq 2$, we have that the initial value of $r(x)$ is 0 , from Theorem 3.6. From Theorem 3.7, we have that the failure rate $r(x)$ converges to $\min \{m \lambda, k \mu\}$. We take an $E_{2} / E_{2} / 1$ queue with $\rho=0.9$ as an example, explained in the following.

Example 3. $E_{2} / E_{2} / 1$ queue with $\rho=0.9$.
By Eq. (3.3), we have

$$
\tilde{D}(s)=\pi_{0}(1)\left(\frac{2 \lambda}{s+2 \lambda}\right)^{2}\left(\frac{2 \mu}{s+2 \mu}\right)^{2}+\pi_{0}(2) \frac{2 \lambda}{s+2 \lambda}\left(\frac{2 \mu}{s+2 \mu}\right)^{2}+\left(1-\pi_{0} \mathbf{e}\right)\left(\frac{2 \mu}{s+2 \mu}\right)^{2} .
$$



Fig. 7. $r(x)$ of $E_{2} / E_{k} / 1$ with $\rho=0.1$.


Fig. 8. $r(x)$ of $E_{2} / E_{k} / 1$ with $\rho=0.5$.
Taking $\lambda=0.9, \mu=1$ and $\pi_{0}=(0.06074278417903,0.07851443164195)$, we obtain the LST $\tilde{D}(s)$. By the inverse of the $\operatorname{LST} \tilde{D}(s)$, we have

$$
D(x)=1-\left[10.9337 x \mathrm{e}^{-1.8 x}-95.4413 \mathrm{e}^{-1.8 x}+96.4413 \mathrm{e}^{-2 x}+10.1486 x \mathrm{e}^{-2 x}\right], \quad x \geq 0 .
$$

The expected value is $E[D]=1.1111$ and the variance is $\operatorname{Var}[D]=0.6486$. Then we have that the squared coefficient of variation is $c_{D}^{2}=0.52537<1$. Taking the limit of $r(x)$ as $x \rightarrow 0^{+}$and $x \rightarrow \infty$, we have the initial value 0 and the final value 1.8 of $r(x)$. We draw the graph of $r(x)$ using Matlab and find that $r(x)$ is increasing in $x$. The graph is given in Fig. 9.

We plot $r(x)$ for $k=2,3,5,10,20$ for $E_{2} / E_{k} / 1$ with traffic loads $\rho=0.1,0.5$, and 0.9 in Figs. 7 and 8, respectively.

We have examined the failure rates for $m=2,3,5$ and $k=2,3,5,10,20$. From Figs. 7-9, we found that the failure rate $r(x)$ of $E_{2} / E_{k} / 1$ is affected by $k$ and $\rho$. In this example of $m=2$, the interdeparture times of $E_{2} / M / 1$ and $E_{2} / E_{2} / 1$ preserve the IFR property, no matter whether the traffic load is light, medium or heavy. For $\rho=0.9$, the interdeparture time of $E_{2} / E_{3} / 1$ is not increasing in $x$. From the graphs of $r(x)$ for the cases that are not IFR, we found


Fig. 9. $r(x)$ of $E_{2} / E_{k} / 1$ with $\rho=0.9$.
that there exist two numbers $a$ and $b$, where $a<b$ such that $r(x)$ is increasing in the interval $[0, a]$ and decreasing in [ $a, b$ ]. Finally, the failure rate $r(x)$ increases and converges to $\min \{m \lambda, k \mu\}$. Our conjecture is that if $k \leq m$, then the interdeparture time of $E_{m} / E_{k} / 1$ is IFR in our experiments.

## 4.4. $E_{m} / D / 1$ queues

In this case, we consider a deterministic service case where $\tilde{H}(s)=\exp (-s h)$ and $h=1 / \mu$. In Section 3, we have shown that the interdeparture time of the $E_{m} / D / 1$ queue does not preserve the IFR property and $\lim _{x \rightarrow 0^{+}} r_{I}(x)=0$. Furthermore, the failure rate $r(x)$ will converge to $m \lambda$. Now, we take the $E_{2} / D / 1$ queue with $\rho=0.1$ as an example, explained in the following.

Example 4. $E_{2} / D / 1$ queue with $\rho=0.1$.
From Eq. (2.8) and Lemma 3.1, we have that the LST of the interdeparture time $\tilde{D}(s)$ of the $E_{2} / D / 1$ queue is given by

$$
\begin{equation*}
\tilde{D}(s)=\left(\pi_{0}(1)\left(\frac{2 \lambda}{s+2 \lambda}\right)^{2}+\pi_{0}(2) \frac{2 \lambda}{s+2 \lambda}+\left(1-\pi_{0} \mathbf{e}\right)\right) \exp (-s h) . \tag{4.4}
\end{equation*}
$$

By the inverse of the LST (4.4), we have

$$
D(x)= \begin{cases}1-\pi_{0}(1)(x-h) \mathrm{e}^{-2 \lambda(x-h)}-\left(\boldsymbol{\pi}_{0} \mathbf{e}\right) \mathrm{e}^{-2 \lambda(x-h)} & \text { if } x>h  \tag{4.5}\\ 1-\boldsymbol{\pi}_{0} \mathbf{e} & \text { if } x=h \\ 0 & \text { if } x<h\end{cases}
$$

Hence, we have that $r(x)$ is the same as (2.9), where $r_{I}(x)$ is given by

$$
\begin{equation*}
r_{I}(x)=\frac{\pi_{0}(1) x \mathrm{e}^{-2 \lambda x}+\pi_{0}(2) \mathrm{e}^{-2 \lambda x}}{\pi_{0}(1) x \mathrm{e}^{-2 \lambda x}+\left(\boldsymbol{\pi}_{0} \mathbf{e}\right) \mathrm{e}^{-2 \lambda x}} \quad \text { for } x>0 . \tag{4.6}
\end{equation*}
$$

Furthermore, we differentiate $r_{I}(x)$, which results in $r_{I}^{\prime}(x)>0$ for $x>0$. Thus, notice that the failure rate of the interdeparture time is increasing when $x>h$, regardless of $\lambda$ and $\mu$.

Now we consider the failure rate at $x=h$. Solving the vector $\boldsymbol{\pi}_{0}$, we have $\boldsymbol{\pi}_{0}=(0.8177650179,0.1644699642)$ when $\lambda=0.1$ and $\mu=1$. Hence, we have $\pi_{0}(2) 2 \lambda=0.032894>1-\boldsymbol{\pi}_{0} \mathbf{e}=0.017765$. That is, $\lim _{x \rightarrow h^{+}} r(x)>$ $\left(1-\pi_{0} \mathbf{e}\right) / \pi_{0} \mathbf{e}$. Thus, we see that the failure rate of the interdeparture time is non-decreasing in $x$.

Table 1
Comparing $\pi_{0}(m) m \lambda$ and $1-\boldsymbol{\pi}_{0} \mathbf{e}$ of $E_{m} / D / 1$ with $\lambda=0.1$ and $\mu=1$

| $m$ | 2 | 3 | 5 | 10 | 20 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\pi_{0}(m) m \lambda$ | 0.032894 | 0.010019 | 0.000790 | $1.01 \times 10^{-6}$ | $1.16 \times 10^{-12}$ |
| $1-\pi_{0} \mathbf{e}$ | 0.017765 | 0.003613 | 0.000172 | + | + |
| Diff. | + | + | + | $+10^{-7}$ |  |

Table 2
Comparing $\pi_{0}(m) m \lambda$ and $1-\pi_{0} \mathbf{e}$ of $E_{m} / D / 1$ with $\lambda=0.5$ and $\mu=1$

| $m$ | 2 | 3 | 5 | 10 |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\pi_{0}(m) m \lambda$ | 0.3534821 | 0.3794649 | 0.3444262 | 0.1851261 |  |
| $1-\pi_{0} \mathbf{e}$ | 0.3232589 | 0.2252732 | + | + | 0.033148 |
| Diff. | + | + | + | 0.00374719 |  |

Table 3
Comparing $\pi_{0}(m) m \lambda$ and $1-\pi_{0} \mathbf{e}$ of $E_{m} / D / 1$ with $\lambda=0.9$ and $\mu=1$

| $m$ | 2 | 3 | 5 | 10 | 20 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\pi_{0}(m) m \lambda$ | 0.1904125 | 0.2857929 | 0.4611315 | 0.8283842 |  |
| $1-\pi_{0} \mathbf{e}$ | 0.8471076 | 0.8076807 | 0.7478285 | 0.6468613 |  |
| Diff. | - | - | + | 0.5232651 |  |

We compare $\boldsymbol{\pi}_{0}(m) m \lambda$ and $\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right)$ for $E_{m} / D / 1$ with $m=2,3,5,10,20$ and traffic loads $\rho=0.1,0.5$, and 0.9. If $\boldsymbol{\pi}_{0}(m) m \lambda-\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) \geq 0$ then $\lim _{x \rightarrow h^{+}} r(x)>\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right) / \boldsymbol{\pi}_{0} \mathbf{e}$, which implies that the failure rate of the interdeparture time is non-decreasing in $x$. The differences of $\boldsymbol{\pi}_{0}(m) m \lambda$ and $\left(1-\boldsymbol{\pi}_{0} \mathbf{e}\right)$, shown only by plus or minus signs, are given in Tables $1-3$, respectively, to visualize the behavior of $r(x)$.

We have shown that the interdeparture time of the $M / D / 1$ queue does not preserve the IFR property in Section 3 . In our experiments, we found that the failure rate of the interdeparture time is increasing when $x>h$. We also have $r(x)=0$ when $x<h$. From Tables 1-3, we found that $\lim _{x \rightarrow h^{+}} r(x)=\lim _{x \rightarrow 0^{+}} r_{I}(x)<\frac{1-\pi_{0} \mathrm{e}}{\pi_{0} \mathrm{e}}$ when $m=2,3,5$ with $\rho=0.9$. That is, they are not IFR. However, for $m=2,3,5,10,20$ with $\rho=0.1, m=2,3,5,10,20$ with $\rho=0.5$ and $m=10,20$ with $\rho=0.9$, we found that $\lim _{x \rightarrow h^{+}} r(x)=\lim _{x \rightarrow 0^{+}} r_{I}(x)>\frac{1-\pi_{0} \mathrm{e}}{\pi_{0} \mathrm{e}}$. Hence, they are IFR.

From the curves of $r(x)$ in the Figures, the failure rate increases (i.e., IFR) in the early stage. Afterwards, the failure rate decreases (i.e., DFR) in the middle stage for the queues for which the interdeparture time is not IFR. However, the failure rate decreases (i.e., DFR) in the final stage for $M / E_{k} / 1$. Except the $M / E_{k} / 1$ queues, the failure rate increases (i.e., IFR) in the final stage for those queues with IFR departure processes.

## 5. Conclusions and future research

### 5.1. Conclusions

In this paper, we derived the Laplace-Stieltjes transform (LST) of the interdeparture time of the $P H / G / 1$ queue and gave some indices for the performance analysis of the departure process of the $P H / G / 1$ queue, such as the moments, the variance, and the squared coefficient of variation. We showed that the $E_{m} / E_{k} / 1$ queue has $c_{D}^{2} \leq 1$.

In particular, we analyzed the failure rate of the stationary interdeparture time. To the best of our knowledge, this has not been studied before in this aspect. We focused on the IFR property of the interdeparture time of the $E_{m} / E_{k} / 1$ queue. Because of the complexity of the stationary probability density $\pi_{0}$, we took advantage of the computer to visualize the performance of the output process. We displayed some numerical results for the $E_{m} / E_{k} / 1$ queues. We found that the interdeparture time does not always preserve the IFR property, even if the interarrival time and service time are Erlang distributions with IFR. But if $k \leq m$, the interdeparture time of $E_{m} / E_{k} / 1$ remains the IFR property in our experiments. When the arrival process is not a Poisson process, the solution of the stationary density $\boldsymbol{\pi}_{0}$ becomes
more complex. However, our conjecture is that the IFR propery of departure at $E_{m} / E_{k} / 1$ is independent of $\boldsymbol{\pi}_{0}$. This only depends on the order of the number of phases in the arrival and service processes.

### 5.2. Future research

We have focused on the continuous time system with a $P H$-type arrival process in this paper. This can be extended to discrete time analysis for the interdeparture time, that is, the interarrival time and service time are the discrete distributions with the IFR property. For example, the uniform distribution, the geometric distribution and the negative binomial distribution are IFR.

We propose some other studies for further investigation. First, it may be extended to more complicated arrival and service distributions with the IFR property, such as Weibull distributions and normal distributions. Second, it is interesting to check conditions under which the departure process preserves the DFR property. For example, does the interdeparture time of the $H_{m} / H_{k} / 1$ queue preserve the DFR property where the arrival and service processes have hyper-exponential distributions?

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