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Existence and nonexistence of global solutions of some system of semilinear wave equations

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Abstract

An initial boundary value problem for systems of semilinear wave equations in a bounded domain is considered. We prove the global existence, uniqueness and blow-up of solutions by energy methods and give some estimates for the lifespan of solutions.

Keywords: Global solutions; Blow-up; Lifespan; Wave equations

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 1$, with a boundary $\partial\Omega$ of class C^2 and let $T > 0$. In this paper we shall consider the global existence and blow-up of solutions of an initial boundary value problem for a system of nonlinear wave equations in a bounded domain $\Omega \times [0, T)$, say

$$(u_i)_{tt} - \Delta u_i + m_i^2 u_i + f_i(u_1, u_2) = 0, \quad i = 1, 2, \tag{1.1}$$

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

here $u = (u_1, u_2)$.

The existence and uniqueness of solutions of the Cauchy problem for a single wave equation

$$u_{tt} - \Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \quad n \geqslant 3,$$

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have been discussed by several authors during the past 30 years; see for example [1,3,7,21,24,25] and the references therein. And some blow-up results can be found in [4,6,9,14,23]. For an initial boundary value problem, some global existence and blow-up results are given by [10–12]. Reed [20] proposed an interesting problem for the following system of equations:

$$(u_1)_{tt} - \Delta u_1 + m_1^2 u_1 = -4\lambda(u_1 + \alpha u_2)^3 - 2\beta u_1 u_2^2,$$

$$(u_2)_{tt} - \Delta u_2 + m_2^2 u_2 = -4\alpha\lambda(u_1 + \alpha u_2)^3 - 2\beta u_1^2 u_2. \tag{1.4}$$

As a model it describes the interaction of scalar fields u_1, u_2 of masses m_1, m_2, respectively. This system defines the motion of charged mesons in an electro-magnetic field which was first introduced by Segal [22]. Later, Jörgens [8], Makhankov [16], and Medeiros and Menzala [18] studied such systems to find the existence of weak solutions of the mixed problem in a bounded domain. Further generalizations are also given in [17,19] by using Galerkin methods. Recently, the existence of global and nonglobal solutions of a particular system was discussed in [13]. In [2], some results concerning existence and nonexistence of global solutions of a Cauchy problem for a hyperbolic system of Hamiltonian type in a unbounded domain is given by using weighted Strichartz estimates.

In this paper, we shall discuss the existence, uniqueness and blow-up properties of solutions in $C^2(0,T,L^2(\Omega)) \cap C^1(0,T,H_0^1(\Omega))$ for a system (1.1)–(1.3) in a bounded domain Ω in $\mathbb{R}^n$. The paper is organized as follows. In Section 2, we derive a priori estimates on solutions of the linear problem. Then we obtain the local existence Theorem 2.4 by using successive approximation methods. In Section 3, we shall prove the global existence result in Theorem 3.3. We also show the triviality of the solution when the initial data are zero functions. In Section 4, we first define an energy function $E(t)$ by (3.1) and show that it is a constant function of t which will follow immediately from some essential identities that will be used later for estimating the lifespan T. Then we obtain Theorem 4.4, which shows blow-up of solutions under some restrictions. Estimates for the blow-up time T are also given. In the last section, we give a uniqueness result in Theorem 5.1 under further assumptions on f. In this paper, we extend the result of Li [12] to the system of Hamiltonian type. In Examples 3.7 and 4.6, we also give a partial classification of global existence and blow-up of solutions in the problem (1.2)–(1.4) which is proposed in [20, p. 121].

2. Local existence results

In this section we shall discuss local existence of solutions for (1.1)–(1.3) by the method of successive approximations. We first give some notations below. Let

$$H1 = C^1(0,T,L^2(\Omega)) \cap C^0(0,T,H_0^1(\Omega)),$$

$$H2 = C^2(0,T,L^2(\Omega)) \cap C^1(0,T,H_0^1(\Omega)),$$

with the norms

$$\|u\|_{H1} = \sup_{0\leqslant t\leqslant T} (\|u_t\|_2 + \|\nabla u\|_2)$$

and

$$\|u\|_{H2} = \sup_{0\leqslant t\leqslant T} (\|u_{tt}\|_2 + \|\nabla u_t\|_2 + \|\nabla u\|_2).$$

We say $h\in W^{1,1}(0,T,L^2(\Omega))$, to mean that

$$h\in L^1(0,T,L^2(\Omega)) \quad \text{and} \quad h_t\in L^1(0,T,H_0^1(\Omega)).$$

Definition. A function $u=(u_1,u_2)\in H1\times H1$ is called a weak solution of the initial boundary value problem (1.1)–(1.3), if

$$\int_\Omega [(u_i)_t(t)\eta_i(t) - (u_i)_t(0)\eta_i(0)]\,\mathrm{d}x$$

$$= \int_0^t \int_\Omega [-\nabla u_i\cdot\nabla\eta_i + (u_i)_t(\eta_i)_t - m_i^2 u_i\eta_i + f_i(u)\eta_i]\,\mathrm{d}x\,\mathrm{d}s$$

holds for $\eta=(\eta_1,\eta_2)\in H1\times H1$.

Assume that

(A1) $f_i:\mathbb{R}^2\to\mathbb{R}$ is continuously differentiable such that for each $u=(u_1,u_2)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we have $u_i f_i(u)\in L^1(\Omega)$, $i=1,2$, and $F(u)\in L^1(\Omega)$, where

$$F(u) = \int_0^{u_1} f_1(s,u_2)\,\mathrm{d}s + \int_0^{u_2} f_2(0,s)\,\mathrm{d}s.$$

(A2) $f_i: H_0^1(\Omega)\times H_0^1(\Omega)\to L^2(\Omega)$, $i=1,2$, satisfies a local Lipschitz condition, i.e., for any $\rho>0$, there exists a positive constant $C(\rho)$ such that

$$\|f_i(u)-f_i(v)\|_{L^2} \leqslant C(\rho)\|u-v\|_{H_0^1\times H_0^1},$$

for $u,v\in H_0^1(\Omega)\times H_0^1(\Omega)$ with $\|u\|_{H_0^1\times H_0^1}, \|v\|_{H_0^1\times H_0^1}\leqslant\rho$.

(A3)

$$\frac{\partial f_1}{\partial u_2} = \frac{\partial f_2}{\partial u_1}.$$

Note that the functions of the form

$$f_1(u_1,u_2) = u_1^{s-1}u_2^s + u_1^p, \qquad f_2(u_1,u_2) = u_1^s u_2^{s-1} + u_2^q,$$

satisfy the assumptions (A1)–(A3) where $1 < s,p,q\leqslant n/(n-2)$ for $n\geqslant 3$ or $s,p,q>1$ for $n=1,2$. The functions of mixed type in (1.4) also satisfy (A1)–(A3) when $n=3$.

Remark. For brevity, we only consider a system of two equations. In fact, a system of k equations ($k\geqslant 2$) can be similarly investigated and here (A3) is replaced by assuming that $\partial f_i/\partial u_j = \partial f_j/\partial u_i$ for $i\neq j$, $1\leqslant i,j\leqslant k$.

Before proving the existence theorem for nonlinear equations, we need the existence result for linear wave equations which is given by Lions and Magenes [15, p. 95] and Haraux [5, p. 31].

Lemma 2.1. *Assume that* $f \in W^{1,1}(0,T,L^2(\Omega))$ *and that* $u_1 \in H_0^1(\Omega)$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. *Then the linear problem for the scalar equation*

$$u_{tt} - \Delta u + f(t,x) = 0,$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega,$$

$$u(x,t) = 0 \quad on\ \partial\Omega \times (0,T), \tag{2.1}$$

has a unique solution $u \in H2$.

Lemma 2.2 (A priori estimate). *Let* u *be a solution of* (2.1). *Then we have the inequality*

$$\|Du\|_2(t) \leqslant \|Du\|_2(0) + \int_0^t \|f\|_2(r)\,\mathrm{d}r, \tag{2.2}$$

where $Du = (u_t, \nabla u)$ *and* $\|Du\|_2^2(t) = \int_\Omega (u_t^2 + |\nabla u|^2)\,\mathrm{d}x$.

Proof. Multiplying by u_t both sides of (2.1) and then integrating over Ω, we have

$$\int_\Omega u_t(u_{tt} - \Delta u)\,\mathrm{d}x = -\int_\Omega u_t f\,\mathrm{d}x. \tag{2.3}$$

By the Divergence Theorem, we get

$$\int_\Omega u_t(u_{tt} - \Delta u)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u_t^2 + |\nabla u|^2)\,\mathrm{d}x. \tag{2.4}$$

Combining (2.3) and (2.4), using Hölder's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|_2(t) \leqslant \|f\|_2. \tag{2.5}$$

Hence (2.2) follows at once by integrating (2.5) from 0 to t.

Remark. The continuous dependence of the solutions of (1.1)–(1.3) on the initial data can be obtained by Lemma 2.2 and Gronwall's inequality.

Theorem 2.3 (Local existence). *Let* $\phi_i \in H_0^1(\Omega)$ *and* $\psi_i \in L^2(\Omega)$ *for* $i = 1,2$, *then there exists a solution* u *of* (1.1)–(1.3) *in* $H1 \times H1$.

Proof. Due to the fact that $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $H_0^1(\Omega)$ and $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, it suffices to consider this problem for $\phi_i \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\psi_i \in H_0^1(\Omega)$ for $i = 1,2$. Let $\{u^m\}_{m \geqslant 1}$ be a sequence of solutions obtained by considering the following linear problems:

$$(u_i^{m+1})_{tt} - \Delta u_i^{m+1} = -m_i^2 u_i^m - f_i(u^m), \quad i = 1, 2. \tag{2.6}$$

$$u^{m+1}(x,0) = \phi(x), \qquad u_t^{m+1}(x,0) = \psi(x), \quad x \in \Omega,$$

$$u^{m+1}(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.7}$$

with the initial function $u^1(x,t) \equiv \phi(x)$.

The existence and uniqueness of the solution $u^m \in H2 \times H2$ of (2.6), (2.7) is guaranteed by Lemma 2.1 since the right-hand side of (2.6) is in $W^{1,1}(0,T,L^2(\Omega))$ due to (A1) and (A2).

By Lemma 2.2, we have

$$\|Du_i^{m+1}\|_2(t) \leqslant \|Du_i^{m+1}\|_2(0) + \int_0^t \|m_i^2 u_i^m + f_i(u^m)\|_2(r)\,\mathrm{d}r. \tag{2.8}$$

Denote by

$$\beta_i = \|Du_i^{m+1}\|_2(0) = (\|\psi\|_2^2 + \|\nabla\phi\|_2^2)^{1/2}, \quad i = 1, 2. \tag{2.9}$$

And let

$$\beta = \beta_1 + \beta_2. \tag{2.10}$$

For $m \geqslant 1$ and $i = 1, 2$, define

$$G_{m,i} = m_i^2 \|u_i^m\|_2 + \|f_i(u^m)\|_2. \tag{2.11}$$

Let

$$H^k(t) \equiv \|Du^k\|_2(t) = \|Du_1^k\|_2(t) + \|Du_2^k\|_2(t), \quad k \geqslant 1, \tag{2.12}$$

where $Du^k = (Du_1^k, Du_2^k)$, for $u^k = (u_1^k, u_2^k)$.

We see that

$$G_{m,1} + G_{m,2} \leqslant C\|Du^m\|_2(t). \tag{2.13}$$

From (2.8), we have

$$\begin{aligned} \|Du_i^2\|_2(t) &\leqslant \beta_i + \int_0^t (m_i^2\|\phi_i\| + \|f_i(\phi)\|_2)\,\mathrm{d}r \\ &\leqslant \beta_i + G_{1,i}t. \end{aligned} \tag{2.14}$$

Note that by (2.13), we have

$$H^2(t) \leqslant \beta + Ct\|Du^1\|_2(t). \tag{2.15}$$

Define

$$\|u\|_{\infty,\tau} = \sup\{\|Du\|_2(t) \mid 0 \leqslant t \leqslant \tau\}.$$

Let M be a positive constant such that $M > \beta$. Then $H^1(t) \leqslant M$, for $0 \leqslant t \leqslant \tau$, or $\|u^1\|_{\infty,\tau} \leqslant M$. Thus from (2.15), we have

$$H^2(t) \leqslant \beta + CtM \leqslant M \quad \text{for } 0 \leqslant t \leqslant \tau, \tag{2.16}$$

provided that $\tau = (M - \beta)/CM$.

That is,

$$\|u^2\|_{\infty,\tau} \leqslant M. \tag{2.17}$$

Suppose that $\|u^m\|_{\infty,\tau} \leqslant M$. By adding (2.8) and by using (2.13), we have

$$\begin{aligned} H^{m+1}(t) &\leqslant \beta + (G_{m,1} + G_{m,2})t \\ &\leqslant \beta + Ct\|Du^m\|_2(t) \leqslant M, \quad 0 \leqslant t \leqslant \tau. \end{aligned} \tag{2.18}$$

Thus $\|u^{m+1}\|_{\infty,\tau} \leqslant M$. Therefore, we have

$$\|u^m\|_{\infty,\tau} \leqslant M \quad \text{for all } m \geqslant 1. \tag{2.19}$$

Next we claim that $\{u^m\}_{m\geqslant 1}$ is a Cauchy sequence in $H1 \times H1$. Let $z^m = u^{m+1} - u^m$. From (2.6) and (2.7), we see that

$$(z_i^m)_{tt} - \Delta(z_i^m) = -m_i^2 z_i^{m-1} - (f_i(u^m) - f_i(u^{m-1})), \tag{2.20}$$

$$z^m(x,0) = 0, \quad z_t^m(x,0) = 0, \quad x \in \Omega, \qquad z^m(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{2.21}$$

As in the previous argument, we see that

$$\begin{aligned} \|Dz^m\|_2(t) \leqslant{}& \|Dz^m\|_2(0) \\ &+ \sum_{i=1}^{2} \int_0^t \{m_i^2 \|z_i^{m-1}\|_2 + \|f_i(u^m) - f_i(u^{m-1})\|_2\}\, \mathrm{d}r. \end{aligned} \tag{2.22}$$

From (2.21), $\|Dz^m\|_2(0) = 0$. By (2.19), (A2) and by Sobolev's inequality, we obtain

$$\|D(z^m)\|_2(t) \leqslant K \int_0^t \|D(z^{m-1})\|_2(r)\, \mathrm{d}r, \quad 0 \leqslant t \leqslant \tau, \tag{2.23}$$

where K is a constant depending on m_1, m_2 and the Sobolev constant. Thus by induction we have

$$\|z^m\|_{\infty,\tau} \leqslant K\tau \|z^{m-1}\|_{\infty,\tau} \leqslant \cdots \leqslant (K\tau)^{m-1} \|z^1\|_{\infty,\tau}. \tag{2.24}$$

Thus for any positive integer p and $K\tau \in (0,1)$, we get

$$\begin{aligned} \|u^{m+p} - u^m\|_{\infty,\tau} &\leqslant [(K\tau)^{m+p-2} + \cdots + (K\tau)^{m-1}]\|u^2 - u^1\|_{\infty,\tau} \\ &\leqslant \frac{(K\tau)^{m-1}}{1 - K\tau} \|u^2 - u^1\|_{\infty,\tau} \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

Thus the Cauchy sequence $\{u^m\}_{m\geqslant 1}$ converges in $H1\times H1$ and the limit function $u=\lim_{m\to\infty}u^m$ in $H1\times H1$ is a local solution of (1.1)–(1.3).

3. Global existence

In this section, we shall show the local uniqueness and the global existence of solutions u for the problem (1.1)–(1.3). Before doing this, we shall prove that $\|Du\|_2(t)$ is uniformly bounded by a constant (independent of t) for all $0\leqslant t\leqslant T\leqslant\infty$.

We first define an energy function $E(t)$ by

$$E(t)=\frac{1}{2}\sum_{i=1}^{2}\int_{\Omega}(|\nabla u_i|^2+(u_i)_t^2+m_i^2u_i^2)\,\mathrm{d}x+\int_{\Omega}F(u)\,\mathrm{d}x, \tag{3.1}$$

where F is given in (A1).

Lemma 3.1. *Let u be a solution of* (1.1)–(1.3). *Then*

$$E(t)=\frac{1}{2}\sum_{i=1}^{2}\int_{\Omega}(|\nabla\phi_i|^2+\psi_i^2+m_i^2\phi_i^2)\,\mathrm{d}x+\int_{\Omega}F(\phi)\,\mathrm{d}x. \tag{3.2}$$

Proof. We see that $\mathrm{d}E/\mathrm{d}t=0$ by using the Divergence Theorem, (1.1)–(1.3). Thus $E(t)=E(0)$ for $t>0$, i.e., we have (3.2).

Lemma 3.2. *Let u be a solution of* (1.1)–(1.3). *Assume that*
(A4)

$$2F(\xi)+\sum_{i=1}^{2}m_i^2\xi_i^2\geqslant 0 \quad \textit{for } \xi=(\xi_1,\xi_2)\in\mathbb{R}^2.$$

Then we have

$$\|Du\|_2^2(t)\leqslant E(0) \quad \textit{for all } t\geqslant 0. \tag{3.3}$$

Proof. Eq. (3.3) follows at once from (3.2) and (A4).

Theorem 3.3 (Global existence). *If* (A1)–(A4) *holds, then there exists a global solution u of* (1.1)–(1.3).

Proof. We first claim the local uniqueness of solutions of (1.1)–(1.3). Let u and u^* be two solutions of (1.1)–(1.3) and let $w=u-u^*$, then we get, for $i=1,2$,

$$(w_i)_{tt}-\Delta w_i=-m_i^2(u_i-u_i^*)-(f_i(u)-f_i(u^*)). \tag{3.4}$$

Multiplying (3.4) by $(w_i)_t$ and then integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2^2\leqslant\|(w_i)_t\|_2\{\|m_i^2(u_i-u_i^*)\|_2+\|f_i(u)-f_i(u^*)\|_2\}, \tag{3.5}$$

for $i=1,2$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2 \leqslant \|m_i^2(u_i-u_i^*)\|_2 + \|f_i(u)-f_i(u^*)\|_2. \tag{3.6}$$

Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|_2 \leqslant \sum_{i=1}^{2}\{m_i^2\|(u_i-u_i^*)\|_2 + \|f_i(u)-f_i(u^*)\|_2\}. \tag{3.7}$$

Note that by Lemma 3.2 we have

$$\|Dw\|_2^2(t) \leqslant E(0) \quad \text{for } t \geqslant 0. \tag{3.8}$$

By using the Lipschitz condition on f and Sobolev's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|_2(t) \leqslant C\|Dw\|_2(t), \tag{3.9}$$

where C is some constant.

After integrating (3.9), we obtain

$$\|Dw\|_2(t) \leqslant \mathrm{e}^{C\tau}\|Dw\|_2(0) \quad \text{for } 0 \leqslant t \leqslant \tau. \tag{3.10}$$

Hence $\|Dw\|_2(t)=0$ for $0 \leqslant t \leqslant \tau$, and we have proved local uniqueness of the solution of (1.1)–(1.3).

A global solution of (1.1)–(1.3) can be obtained in the usual manner because of (3.3). Once we have a local solution u in $[0,\tau)$, we then set

$$\bar{\phi}(\cdot) = u\left(\cdot,\frac{\tau}{2}\right) \in H_0^1(\Omega), \quad \bar{\psi}(\cdot) = u_t\left(\cdot,\frac{\tau}{2}\right) \in L^2(\Omega),$$

then we have a local solution $\bar{u}$ of (1.1)–(1.3) on $[\tau/2, 3\tau/2)$. By the local uniqueness of solutions, we have $u=\bar{u}$ on $[\tau/2,\tau)$. Now we have extended the solution u up to $[0,3\tau/2)$. Continuing in this way, we then obtain a global solution of (1.1)–(1.3).

Let

$$A(t) = \int_\Omega (u_1^2(x,t)+u_2^2(x,t))\,\mathrm{d}x. \tag{3.11}$$

In the following, we shall prove the triviality of the solution provided that the initial data are zero functions. We first derive an essential equality which will be used later.

Lemma 3.4. *Let u be a solution of (1.1)–(1.3) with $u_i \in C^2(\mathbb{R}^+, H_0^1(\Omega))$, for $i=1,2$. Then we have*

$$\int_\Omega \sum_{i=1}^{2}|\nabla u_i|^2\,\mathrm{d}x = E(0) - \frac{A''(t)}{4}$$

$$-\int_\Omega \left(F(u) + \sum_{i=1}^{2}\left(m_i^2u_i^2 + \frac{u_if_i(u)}{2}\right)\right)\mathrm{d}x, \tag{3.12}$$

or

$$2\int_\Omega \sum_{i=1}^{2} (u_i)_t^2 \,\mathrm{d}x = \frac{A''(t)}{2} + 2E(0) + \int_\Omega \left(\sum_{i=1}^{2} u_i f_i(u) - 2F(u) \right) \mathrm{d}x. \tag{3.13}$$

Proof. By differentiating (3.13) once and twice, respectively, we obtain

$$A'(t) = 2\int_\Omega \sum_{i=1}^{2} u_i (u_i)_t \,\mathrm{d}x \tag{3.14}$$

and

$$A''(t) = 2\int_\Omega \sum_{i=1}^{2} (u_i)_t^2 \,\mathrm{d}x - 2\int_\Omega \sum_{i=1}^{2} (|\nabla u_i|^2 + m_i^2 u_i^2 + u_i f_i(u)) \,\mathrm{d}x. \tag{3.15}$$

By (3.2), we then obtain (3.12). It follows at once that (3.13) holds by using (3.15) in (3.12).

Theorem 3.5. *Let u be a solution of* (1.1)–(1.3) *with $u_i \in C^2(\mathbb{R}^+, H_0^1(\Omega))$, for $i = 1, 2$. Assume that*

$$2F(\xi) + \sum_{i=1}^{2} \left(2m_i^2 + \frac{1}{\lambda_1} \right) \xi_i^2 + \xi_i f_i(\xi) \geqslant 0 \quad \text{for all } \xi \in \mathbb{R}^2, \tag{3.16}$$

where

$$\lambda_1 = \inf \left\{ \frac{\|\nabla u\|_2}{\|u\|_2} \,\middle|\, u \in H_0^1(\Omega), u \neq 0 \right\}.$$

If $\phi = \psi = 0$ in Ω, then the only global solution of (1.1)–(1.3) *is the trivial solution.*

Proof. From the assumptions (3.16) and (3.2), we have $E(0) = 0$. From (3.12), we get

$$2\int_\Omega \sum_{i=1}^{2} |\nabla u_i|^2 \,\mathrm{d}x = -\frac{A''(t)}{2} - \int_\Omega \left(2F(u) + \sum_{i=1}^{2} (2m_i^2 u_i^2 + u_i f_i(u)) \right) \mathrm{d}x. \tag{3.17}$$

By using Poincaré's inequality in (3.17) and (3.16), we get $A''(t) \leqslant 0$. That is $A(t)$ is concave down. Since $A(0) = 0$, $A'(0) = 0$, we then obtain $A(t) \leqslant 0$ for $t \geqslant 0$. Hence $A(t) \equiv 0$ for $t \geqslant 0$.

Example 3.6. Consider a particular system (1.1)–(1.3) in $\mathbb{R}^3$ with

$$f_1(u_1, u_2) = \gamma_1 u_1 u_2^2, \quad f_2(u_1, u_2) = \gamma_2 u_1^2 u_2, \tag{3.18}$$

here $\gamma_1, \gamma_2 > 0$.

Without loss of generality, we may assume that $\gamma_1 = \gamma_2 = 1$ by changing the scales. Now $F(u_1, u_2) = \frac{1}{2} u_1^2 u_2^2$, and (A1)–(A4) are satisfied. By Theorem 3.3, there is a global solution u of (1.1)–(1.3) where f is given by (3.18).

Example 3.7. Consider the problem (1.2)–(1.4) in $\mathbb{R}^3$. Assume that $\lambda \geqslant 0$, $\beta \geqslant 0$ and α is any real number. Now $F(u_1,u_2) = \lambda(u_1 + \alpha u_2)^4 + \beta u_1^2 u_2^2$, and conditions (A1)–(A4) are satisfied. By Theorem 3.3, there is a global solution u of (1.2)–(1.4).

4. Blow-up of solutions

In this section, we shall discuss blow-up properties of solutions for a system (1.1)–(1.3). Before doing this, let us give the following two lemmas, which will be used later.

Definition. A solution $u=(u_1,u_2)$ of (1.1)–(1.3) is called a blow-up solution if there exists a finite T^* such that

$$\lim_{t\to T^*-}\left(\int_\Omega (u_1^2+u_2^2)\,\mathrm{d}x\right)^{-1} = 0.$$

Lemma 4.1. *Let $b(t):\mathbb{R}^+ \to \mathbb{R}^+$ be a C^2-function satisfying*

$$b''(t) - 4(\delta+1)b'(t) + 4(\delta+1)b(t) \geqslant 0 \quad \textit{for } t \geqslant 0. \tag{4.1}$$

If

$$b'(0) > r_2 b(0), \tag{4.2}$$

then $b'(t) > 0$ for $t > 0$, where $r_2 = 2(\delta+1) - 2\sqrt{\delta(\delta+1)}$ is the smallest root of the equation $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$.

Proof. Let r_1 be the largest root of $r^2 - 4(\delta+1)r + 4(\delta+1) = 0$. Then (4.1) is equivalent to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - r_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - r_2\right) b(t) \geqslant 0. \tag{4.3}$$

By integrating (4.3) from 0 to t, we get

$$b'(t) \geqslant r_2 b(t) + (b'(0) - r_2 b(0))\mathrm{e}^{r_1 t}. \tag{4.4}$$

By (4.2), we get $b'(t) > 0$ for $t > 0$.

Lemma 4.2. *If $J(t)$ is a nonincreasing function on $[t_0,\infty)$, $t_0 \geqslant 0$, and satisfies the differential inequality*

$$J'(t)^2 \geqslant a + bJ(t)^{2+1/\delta} \quad \textit{for } t \geqslant t_0, \tag{4.5}$$

where $a > 0$ and $b \in \mathbb{R}$, then there exists a finite positive number T^* such that $\lim_{t\to T^*-} J(t) = 0$ and an upper bound for T^* is estimated, respectively, in the following cases:

(i) *when $b < 0$ and $J(t_0) < \min\{1, \sqrt{a/-b}\}$,*

$$T^* \leqslant t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{a/-b}}{\sqrt{a/-b} - J(t_0)}, \tag{4.6}$$

(ii) *when* $b = 0$,

$$T^* \leqslant t_0 + \frac{J(t_0)}{\sqrt{a}}, \tag{4.7}$$

(iii) *when* $b > 0$,

$$T^* \leqslant t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{a}} \{1 - (1 + cJ(t_0))^{-1/2\delta}\}, \tag{4.8}$$

where $c = (a/b)^{2+1/\delta}$.

Proof. (i) Since $\sqrt{c^2 - d^2} \geqslant c - d$ for $c \geqslant d > 0$, we have from (4.5),

$$J'(t) \leqslant -\sqrt{a} + \sqrt{-b}\,J(t) \quad \text{for } t \geqslant t_0. \tag{4.9}$$

Thus we get

$$J(t) \leqslant \left(J(t_0) - \sqrt{-(a/b)}\right) e^{(t-t_0)\sqrt{-b}} + \sqrt{-(a/b)}. \tag{4.10}$$

Hence there exists a positive $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$, and an upper bound of T^* is given by (4.6).

(ii) When $b = 0$, from (4.5), we get

$$J(t) \leqslant J(t_0) - \sqrt{a}(t - t_0) \quad \text{for } t \geqslant t_0.$$

Thus there exists $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$, and an upper bound of T^* is given by (4.7).

(iii) When $b > 0$, we get from (4.5)

$$J'(t) \leqslant -\sqrt{a(1 + (cJ(t))^{2+1/\delta})}, \tag{4.11}$$

where $c = (a/b)^{2+1/\delta}$.

By using the inequality

$$m^q + n^q \geqslant 2^{1-q}(m+n)^q \quad \text{for } m, n > 0 \text{ and } q \geqslant 1, \tag{4.12}$$

with $q = 2 + 1/\delta$, we obtain

$$J'(t) \leqslant -\sqrt{a}\,2^{(-\delta-1)/2\delta}(1 + cJ(t))^{1+1/\delta}. \tag{4.13}$$

By solving the differential inequality (4.13), we obtain

$$J(t) \leqslant \frac{1}{c}\left\{-1 + \left[(1 + cJ(t_0))^{-1/2\delta} + \frac{\sqrt{a}}{\delta c} 2^{-(3\delta+1)/2\delta}(t - t_0)\right]^{-2\delta}\right\}. \tag{4.14}$$

Hence there exists $T^* < \infty$ such that $\lim_{t\to T^*-} J(t) = 0$ and an upper bound of T^* is given by (4.8).

Hereafter we shall consider the blow-up of the solution under the following assumption:

(A5) there exists a positive constant δ such that

$$-\sum_{i=1}^{2} \xi_i f_i(\xi) + (4\delta + 2)F(\xi) \geqslant 0 \quad \text{for all } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Let

$$J(t) = A(t)^{-\delta} \quad \text{for } t > 0. \tag{4.15}$$

By differentiating (4.15) once and twice, respectively, we obtain

$$J'(t) = -\delta A(t)^{-\delta-1} A'(t), \tag{4.16}$$

and

$$J''(t) = \delta A(t)^{-\delta-2}\{(\delta+1)(A'(t))^2 - A(t)A''(t)\}. \tag{4.17}$$

Note that by the Schwarz inequality and the triangle inequality, we obtain

$$(A'(t))^2 \leqslant 4A(t)\int_\Omega ((u_1)_t^2 + (u_2)_t^2)\,\mathrm{d}x. \tag{4.18}$$

From (4.17) and (4.18), we get

$$J''(t) \leqslant -\delta A(t)^{-\delta-1} K(t) \quad \text{for } t > 0, \tag{4.19}$$

where

$$K(t) = A''(t) - 4(\delta+1)\int_\Omega ((u_1)_t^2 + (u_2)_t^2)\,\mathrm{d}x. \tag{4.20}$$

By (3.2) and (3.13), we have

$$K(t) = -4(1+2\delta)E(0) + \int_\Omega (8\delta+4)F(u)\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega (4\delta|\nabla u_i|^2 + 4\delta m_i^2 u_i^2 - 2u_i f_i(u))\,\mathrm{d}x. \tag{4.21}$$

By Sobolev's inequality, there is a constant λ_1 such that

$$\int_\Omega |\nabla w|^2\,\mathrm{d}x \geqslant \frac{1}{\lambda_1}\int_\Omega |w|^2\,\mathrm{d}x \quad \text{for } w \in H_0^1(\Omega). \tag{4.22}$$

Combining (4.20), (4.21) and (4.22), we have

$$K(t) \geqslant -4(1+2\delta)E(0) + \int_\Omega (8\delta+4)F(u)\,\mathrm{d}x + \sum_{i=1}^{2}\int_\Omega \left(4\delta\left(\frac{1}{\lambda_1} + m^2\right)u_i^2 - 2u_i f_i(u)\right)\mathrm{d}x \tag{4.23}$$

here $m = \min\{m_1, m_2\}$.

By (A5), we have

$$A''(t) - 4(\delta+1)\int_\Omega ((u_1)_t^2 + (u_2)_t^2)\,\mathrm{d}x \geqslant -4(1+2\delta)E(0). \tag{4.24}$$

We consider three different cases on the sign of the initial energy $E(0)$.

(i) If $E(0) < 0$, then $A''(t) \geqslant -4(1+2\delta)E(0)$, for $t \geqslant 0$. By integration, we have $A'(t) \geqslant A'(0) - 4(1+2\delta)E(0)t$, for $t \geqslant 0$. Thus we get $A'(t) > 0$, for $t > t^*$, where

$$t^* = \max\left\{\frac{A'(0)}{4(1+2\delta)E(0)}, 0\right\}. \tag{4.25}$$

(ii) If $E(0) = 0$, then $A''(t) \geqslant 0$, for $t \geqslant 0$. If $A'(0) > 0$, then $A'(t) > 0$, for $t > 0$.

(iii) If $E(0) > 0$, by the triangle inequality, we have

$$A'(t) \leqslant A(t) + \int_{\Omega} ((u_1)_t^2 + (u_2)_t^2)\,\mathrm{d}x. \tag{4.26}$$

From (4.24) and (4.26), we have the differential inequality

$$A''(t) - 4(\delta+1)A'(t) + 4(\delta+1)A(t) + 4(1+2\delta)E(0) \geqslant 0. \tag{4.27}$$

Let

$$b(t) = A(t) + \frac{(1+2\delta)E(0)}{1+\delta} \quad \text{for } t > 0.$$

Then $b(t)$ satisfies (4.1). By Lemma 4.1, we obtain $A'(t) > 0$ for $t > 0$, provided that

$$A'(0) > r_2 \left(A(0) + \frac{(1+2\delta)E(0)}{1+\delta} \right). \tag{4.28}$$

Consequently, we have

Lemma 4.3. *Assume that* (A5) *holds and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $A'(0) > 0$,
(iii) $E(0) > 0$ *and* (4.28) *holds.*

Then $A'(t) > 0$ *for* $t > t_0$, *where* $t_0 = t^*$ *is given by* (4.25) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Hereafter, we shall find the estimate for the lifespan of $A(t)$. From (4.19) and (4.24), we have

$$J''(t) \leqslant 4\delta(1+2\delta)E(0)A(t)^{-\delta-1} \quad \text{for } t \geqslant t_0. \tag{4.29}$$

Note that $J'(t) < 0$ for $t > t_0$ by Lemma 4.3. Hence multiplying (4.29) by $J'(t)$ and then integrating from t_0 to t, we get

$$J'(t)^2 \geqslant a + bJ(t)^{2+1/\delta} \quad \text{for } t \geqslant t_0, \tag{4.30}$$

where

$$a = \delta^2 A(t_0)^{-2\delta-2}\{A'(t_0)^2 - 8E(0)A(t_0)\}, \tag{4.31}$$

and

$$b = 8\delta^2 E(0). \tag{4.32}$$

Note that $a > 0$ if and only if $E(0) < A'(t_0)^2/8A(t_0)$.

In the case that $E(0) < 0$, we obtain the rough estimate of the upper bound for blow-up time T^* with $T^* \leqslant t_0 - J(t_0)/J'(t_0)$. For the remaining cases, by Lemma 4.2, we obtain the following main result.

Theorem 4.4. *Assume that* (A1)–(A3) *and* (A5) *hold and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $A'(0) > 0$,
(iii) $A'(0)^2/8A(0) > E(0) > 0$ *and* (4.28) *holds.*

Then the solution u blows up at time T^* *in the sense that* $\lim_{t\to T^*-} A(t) = \infty$.

In case (i),

$$T^* \leqslant t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\{1, \sqrt{a/-b}\}$, we have

$$T^* \leqslant t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{a/-b}}{\sqrt{a/-b} - J(t_0)}.$$

In case (ii),

$$T^* \leqslant t_0 + \frac{J(t_0)}{\sqrt{a}}.$$

In case (iii),

$$T^* \leqslant t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{a}} \{1 - (1 + cJ(t_0))^{-1/2\delta}\},$$

where $c = (a/b)^{2+1/\delta}$ with $a = \delta^2 A(t_0)^{-2\delta-2}\{A'(t_0)^2 - 8E(0)A(t_0)\}$ and $b = 8\delta^2 E(0)$. Note that in case (i), $t_0 = t^*$ is given in (4.25) and $t_0 = 0$ in cases (ii) and (iii).

Example 4.5. Consider the system (1.1)–(1.3) in $\mathbb{R}^3$, with $m_i = 0$, $i = 1, 2$, $f_1(u_1, u_2) = -u_1 u_2^2$ and $f_2(u_1, u_2) = -u_1^2 u_2$. Now we have $F(u_1, u_2) = -\frac{1}{2} u_1^2 u_2^2$. The assumption (A5) is satisfied if $0 < \delta \leqslant \frac{1}{2}$. Hence Theorem 4.4 is applicable.

Example 4.6. Consider the problem (1.2)–(1.4) in $\mathbb{R}^3$. Assume that $\lambda < 0$, $\beta < 0$ and α is any real number. Now we have

$$F(u_1, u_2) = \lambda(u_1 + \alpha u_2)^4 + \beta u_1^2 u_2^2.$$

We see that (A5) is satisfied if $0 < \delta \leqslant \frac{1}{2}$. Thus Theorem 4.4 is applicable.

5. Uniqueness of solutions

In this section, we shall discuss the uniqueness of the solutions of the system (1.1)–(1.3) under the following assumption:

(A6) Assume that there exists $p > 1$ such that

$$\left| \frac{\partial f_i}{\partial u_j}(u_1, u_2) \right| \leqslant k(1 + |u_1|^{p-1} + |u_2|^{p-1}), \quad i, j = 1, 2,$$

holds for all $u_1, u_2 \in \mathbb{R}$.

We are going to prove the following uniqueness result.

Theorem 5.1. *Assume that* (A6) *holds, then the uniqueness of the solutions of* (1.1)–(1.3) *holds either in* $C([0,T],L^\infty(\Omega))$ *for* $p>1$, *or in* $C(0,T,H_0^1(\Omega))\cap C^1(0,T,L^2(\Omega))$ *for* $1<p\leqslant n/(n-2)$, $n\geqslant 3$, *and for* $1<p<\infty$, $n=1,2$.

Proof. Let u and u^* be two solutions of (1.1)–(1.3). Put $w=u-u^*$ and let

$$H(t)=\|Dw_1\|_2^2(t)+\|Dw_2\|_2^2(t).$$

From (1.1)–(1.3) we get

$$(w_i)_{tt}-\Delta w_i=-m_i^2w_i-(f_i(u)-f_i(u^*))\quad \text{for } i=1,2. \tag{5.1}$$

Multiplying (5.1) by $(w_i)_t$ and then integrating over Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2^2=-2\int_\Omega\{m_i^2w_i(w_i)_t+(w_i)_t(f_i(u)-f_i(u^*)\}\,\mathrm{d}x. \tag{5.2}$$

Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2\leqslant\|m_i^2w_i\|_2+\|f_i(u)-f_i(u^*)\|_2. \tag{5.3}$$

By integrating (5.3) from 0 to t, we have

$$\|Dw_i\|_2(t)\leqslant\int_0^t\{\|m_i^2w_i\|_2(s)+\|f_i(u)-f_i(u^*)\|_2(s)\}\,\mathrm{d}s, \tag{5.4}$$

or

$$\|Dw_i\|_2^2(t)\leqslant 2\int_0^t\{m_i^4\|w_i\|_2^2(s)+\|f_i(u)-f_i(u^*)\|_2^2\}\,\mathrm{d}s. \tag{5.5}$$

Then we have

$$H(t)\leqslant 2\int_0^t\sum_{i=1}^2\{m_i^4\|w_i\|_2^2(s)+\|f_i(u)-f_i(u^*)\|_2^2\}\,\mathrm{d}s. \tag{5.6}$$

By (A6) we have

$$\begin{aligned}
&|f_i(u)-f_i(u^*)|\\
&\quad=\left|\int_0^1\nabla f_i(su+(1-s)u^*)\cdot(u-u^*)\,\mathrm{d}s\right|\\
&\quad\leqslant|u-u^*|\int_0^1 k(1+|su_1+(1-s)u_1^*|^{p-1}+|su_2+(1-s)u_2^*|^{p-1})\,\mathrm{d}s\\
&\quad\leqslant k|u-u^*|\{1+2^{p-2}(|u_1|^{p-1}+|u_1^*|^{p-1}+|u_2|^{p-1}+|u_2^*|^{p-1})\}.
\end{aligned} \tag{5.7}$$

Thus we have

$$|f_i(u)-f_i(u^*)|^2\leqslant 2k^2G(x,t)|u-u^*|^2, \tag{5.8}$$

where

$$G(x,t) = 1 + 2^{2(p-1)}(|u_1|^{2p-2} + |u_1^*|^{2p-2} + |u_2|^{2p-2} + |u_2^*|^{2p-2}). \tag{5.9}$$

Hence

$$\|f_i(u) - f_i(u^*)\|_2^2 \leqslant 2k^2 \int_\Omega G(x,t)|u-u^*|^2 \, dx. \tag{5.10}$$

Thus from (5.6) we have

$$H(t) \leqslant 2\int_0^t \left(M\|u-u^*\|_2^2(s) + 2k^2 \int_\Omega G(x,t)|u-u^*|^2 \, dx \right) ds, \tag{5.11}$$

where $M = \max\{m_1^4, m_2^4\}$.

(I) If $u_i, u_i^* \in C([0,T], L^\infty(\Omega))$, let

$$K(T) = \sup_{0 \leqslant t \leqslant T} \left\{ 1 + 2^{2p-2} \left(\|u_1\|^{2p-2} + \|u_1^*\|^{2p-2} + \|u_2\|^{2p-2} + \|u_2^*\|^{2p-2} \right) \right\}.$$

Then we have

$$H(t) \leqslant 2\int_0^t (M + 2k^2 K(T))\|u-u^*\|_2^2(s) \, ds. \tag{5.12}$$

Note that $\|u-u^*\|_2^2(t) \leqslant \lambda H(t)$ for some $\lambda > 0$. Then we obtain

$$H(t) \leqslant 2\lambda(M + 2k^2 K(T)) \int_0^t H(s) \, ds, \tag{5.13}$$

for all $t \in [0,T]$. By Gronwall's inequality, we have $H(t) = 0$ for all $t \in [0,T]$. Hence the uniqueness result holds.

(II) If $u_i, u_i^* \in C(0,T,H_0^1(\Omega)) \cap C^1(0,T,L^2(\Omega))$, we shall discuss the uniqueness of solutions of (1.1)–(1.3) for $1 < p \leqslant n/(n-2)$.

(i) When $1 + 1/n < p \leqslant n/(n-2)$, note that by Hölder's inequality, we have

$$\int_\Omega G(x,t)|u-u^*|^2 \, dx \leqslant \left(\int_\Omega G(x,s)^q \, dx \right)^{1/q} \left(\int_\Omega |u-u^*|^{2r} \, dx \right)^{1/r} dx, \tag{5.14}$$

here $q = r/(r-1)$, $r > 1$.

Now from (5.9), by (4.12), we have

$$G(x,t)^q \leqslant 2^{q-1}(1 + 2^{2(pq-1)} g(x,t)),$$

where

$$g(x,t) = |u_1|^{(2p-2)q} + |u_1^*|^{(2p-2)q} + |u_2|^{(2p-2)q} + |u_2^*|^{(2p-2)q}.$$

Thus

$$\int_\Omega G(x,t)^q \, dx \leqslant 2^{q-1} \left\{ |\Omega| + 2^{2(pq-1)} \int_\Omega g(x,t) \right\} dx. \tag{5.15}$$

Since $1 + 1/n < p \leqslant n/(n-2)$, by choosing $r = n/(n-2)$ or $q = n/2$ in (5.15), we have $1 < (2p-2)q \leqslant 2n/(n-2)$. By Sobolev's inequality, we have

$$\int_\Omega G(x,t)^q \, dx \leqslant 2^{q-1}(|\Omega| + C_1) \equiv C^*, \tag{5.16}$$

where

$$C_1 = \sup_{0\leqslant t\leqslant T} 2^{2(pq-1)}(\|u_1\|_{1,2}^{2(p-1)q} + \|u_1^*\|_{1,2}^{2(p-1)q} + \|u_2\|_{1,2}^{2(p-1)q}\|u_2^*\|_{1,2}^{2(p-1)q}).$$

Thus by using Hölder's inequality in (5.11) and (5.16), we obtain

$$H(t) \leqslant 2\int_0^t (m\|u-u^*\|_2^2(s) + 2k^2(C^*)^{1/q}\|u-u^*\|_{2r}^2(s))\,\mathrm{d}s. \tag{5.17}$$

By Sobolev's inequality again, we obtain

$$H(t) \leqslant \tilde{C}\int_0^t H(s)\,\mathrm{d}s \quad \text{for } 0\leqslant t\leqslant T, \tag{5.18}$$

where $\tilde{C} = 2\lambda(M + 2k^2(C^*)^{1/q})$ and λ is Sobolev constant. Therefore, $H(t)\equiv 0$ for $0\leqslant t\leqslant T$.

(ii) When $1 < p \leqslant 1 + 1/n$, from (5.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2^2 \leqslant 2\int_\Omega \{m_i^2|w_i|\,|(w_i)_t| + |(w_i)_t|\,|(f_i(u) - f_i(u^*)|\}\,\mathrm{d}x. \tag{5.19}$$

By (5.7), we then have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2^2 \leqslant 2\int_\Omega \{m_i^2|w_i|\,|(w_i)_t| + \tilde{G}(x,t)|w|\,|(w_i)_t|\}\,\mathrm{d}x, \tag{5.20}$$

where

$$\tilde{G}(x,t) = k\{1 + 2^{p-2}(|u_1|^{p-1} + |u_1^*|^{p-1} + |u_2|^{p-1} + |u_2^*|^{p-1})\}. \tag{5.21}$$

By Hölder's inequality, we have

$$\int_\Omega \tilde{G}(x,t)|w|\,|(w_i)_t|\,\mathrm{d}x \leqslant \|\tilde{G}\|_\beta\|w\|_\alpha\|(w_i)_t\|_2, \tag{5.22}$$

with $1/\alpha + 1/\beta = 1/2,\ \alpha,\beta > 2$.

Note that by the inequality (4.12), we have

$$\tilde{G}(x,t)^\beta \leqslant k^\beta 2^{\beta-1}\{1 + 2^{p\beta-2}(|u_1|^{(p-1)\beta} + |u_1^*|^{(p-1)\beta} + |u_2|^{(p-1)\beta} + |u_2^*|^{(p-1)\beta})\}.$$

Since $1 < p \leqslant 1 + 1/n$, after suitably choosing α with $2 < \alpha \leqslant 2n/(n-2)$, we can get $1 < (p-1)\beta \leqslant 2n/(n-2)$. Therefore Sobolev's inequality is applicable. Thus we have

$$\left(\int_\Omega \tilde{G}(x,t)^\beta\,\mathrm{d}x\right)^{1/\beta} \leqslant [k^\beta 2^{\beta-1}(|\Omega| + C_2)]^{1/\beta} \equiv \tilde{C}, \tag{5.23}$$

where

$$C_2 = \sup_{0\leqslant t\leqslant T} 2^{p\beta-2}\lambda(\|u_1\|_{1,2}^{(p-1)q} + \|u_1^*\|_{1,2}^{(p-1)q} + \|u_2\|_{1,2}^{(p-1)q}\|u_2^*\|_{1,2}^{(p-1)q}).$$

Again by Sobolev's inequality, we have

$$\|w\|_\alpha \leqslant \lambda\|w\|_{1,2} \quad \text{for } 1 < \alpha \leqslant \frac{2n}{n-2}. \tag{5.24}$$

Thus from (5.22)–(5.24) we get

$$\int_{\Omega} \tilde{G}(x,t)|w|\,|(w_i)_t|\,\mathrm{d}x \leqslant \tilde{C}\lambda\|w\|_{1,2}\|(w_i)_t\|_2, \quad i = 1,2. \tag{5.25}$$

Hence from (5.20) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dw_i\|_2^2 \leqslant 2(m_i^2 + \tilde{C}\lambda)\|w\|_{1,2}\|(w_i)_t\|_2$$

$$\leqslant (m_i^2 + \tilde{C}\lambda)(\|w\|_{1,2}^2 + \|(w_i)_t\|_2^2), \tag{5.26}$$

for $i = 1,2$.

Therefore, we get

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} \leqslant (m_1^2 + m_2^2 + \tilde{C}\lambda)H(t) \quad \text{for } 0 \leqslant t \leqslant T. \tag{5.27}$$

By solving (5.27) with $H(0) = 0$, we obtain $H(t) \equiv 0$ for $0 \leqslant t \leqslant T$. Hence we completed the proof. □

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