2 Basic Theory of Value Distribution

In this section, we introduce and review some basic facts and notations in complex analysis and value distribution which will be used throughout the rest of the thesis. For the sake of brevity, proofs are omitted because they are standard and can be found in [12, 13, 14, 15, 16].

In Nevanlinna's value distribution theory, the following Poisson-Jensen's formula plays a very important role.

Theorem 2.1 (Poisson-Jensen's formula) *Let $0 < R < \infty$ and f be meromorphic in $|z| < R$ and a_μ and b_ν be the zeros and poles of f in $|z| < R$, $1 \le \mu \le M$, $1 \le \nu \le N$, respectively. If $z = re^{i\theta}$, $0 \le r < R$, and $f(z) \ne 0, \infty$, then we have*

$$\begin{aligned}\log|f(z)| =& \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi \\ &+ \sum_{\mu=1}^{M} \log\left|\frac{R(z-a_\mu)}{R^2 - \overline{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z-b_\nu)}{R^2 - \overline{b}_\nu z}\right|.\end{aligned}$$

By taking $z = 0$ in Theorem 2.1, we get the Jensen's formula.

Theorem 2.2 (Jensen's formula) *Under the assumption of Theorem 2.1, if $f(0) \ne 0, \infty$, then we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|a_\mu|} + \sum_{\nu=1}^{N} \log\frac{R}{|b_\nu|}.$$

The assumption $f(0) \ne 0, \infty$ in Theorem 2.1 can be eliminated. In fact, for $0 \le r < \infty$, let $n(r, f)$ denote the number of poles of f in $|z| \le r$ counting multiplicities. Consider the Laurent expansion of f at the origin

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots .$$

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Note that $\lambda = n(0, \frac{1}{f}) - n(0, f)$. Consider the function

$$g(z) = \begin{cases} f(z)(\frac{R}{z})^{\lambda} & \text{if } z \neq 0 \\ c_{\lambda} R^{\lambda} & \text{if } z = 0, \end{cases}$$

then we have the generalized Jensen's formula.

Theorem 2.3 (generalized Jensen's formula) *Under the assumption of Theorem 2.1 without the condition $f(0) \neq 0, \infty$, then we have*

$$\log |\, c_{\lambda} \,| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \, f(Re^{i\varphi}) \, \right| d\varphi - \sum_{\mu=1}^{M} \log \frac{R}{|\, a_{\mu} \,|} - n(0, \frac{1}{f}) \log R$$
$$+ \sum_{\nu=1}^{N} \log \frac{R}{|\, b_{\nu} \,|} + n(0, f) \log R,$$

where c_{λ} is the first non-zero coefficient of the Laurent expansion of f at 0.

From now on, meromorphic function means meromorphic in the whole complex plane. First of all, we introduce the positive logarithmic function.

Definition 2.4 *For $x \geq 0$,*

$$\log^+ x = \max\{\log x, 0\} = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } o \leq x < 1. \end{cases}$$

Obviously, $\log^+ x$ is a continuous non-negative increasing function on $[0, \infty)$ satisfying $\log x = \log^+ x - \log^+ \frac{1}{x}$ and $|\log x| = \log^+ x + \log^+ \frac{1}{x}$.

Let f be a meromorphic function, Nevanlinna [1] introduced the following notations.

Definition 2.5 *For $0 < r < \infty$,*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

Definition 2.6 *For $0 < r < \infty$,*

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

where $n(t,f)$ denotes the number of poles of f in the disc $|\, z \,| \leq t$ counting multiplicities. $N(r,f)$ is called the counting function of f.

For $0 \leq r < \infty$, $n(r,f)$ denotes the number of poles of $f(z)$ in $|z| \leq r$ counting multiplicities; $\overline{n}(r,f)$ denotes the number of poles of $f(z)$ in $|z| \leq r$ ignoring multiplicities; $n_{k)}(r, 1/f)$ (resp.$n_{(k}(r, 1/f)$) denotes the number of zeros of $f(z)$ in $|z| \leq r$ with order $\leq k$(resp.$\geq k$) counting multiplicities; $\overline{n}_{k)}(r, 1/f)$ (resp.$\overline{n}_{(k}(r, 1/f)$) denotes the number of zeros of $f(z)$ in $|z| \leq r$ with order $\leq k$(resp.$\geq k$) ignoring multiplicities.

Definition 2.7 *For $0 < r < \infty$, the function $T(r,f)$ defined by*

$$T(r,f) = m(r,f) + N(r,f)$$

is called the (Nevanlinna) characteristic function of f.

It is clear that $T(r,f)$ is a non-negative increasing function and a convex function of $\log r$. Let f be given in Theorem 2.1. It follows from the integration by parts in Riemann-Stieltjes integral, we have

$$\sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} = \int_0^R \frac{n(t, \frac{1}{f}) - n(0, \frac{1}{f})}{t} dt$$

and

$$\sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} = \int_0^R \frac{n(t,f) - n(0,f)}{t} dt.$$

On the other hand, the generalized Jensen's formula can be rewritten as

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\, f(Re^{i\varphi}) \,|\, d\varphi + \sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} + n(0,f)\log R \\ =&\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\, \frac{1}{f(Re^{i\varphi})} \,\right| d\varphi + \sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} + n(0, \frac{1}{f})\log R + \log |\, c_\lambda \,|\,. \end{aligned}$$

Therefore, we obtain

$$m(R,f)+N(R,f)=m(R,\frac{1}{f})+N(R,\frac{1}{f})+\log|\,c_{\lambda}\,|\,,$$

that is,

$$T(R,f)=T(R,\frac{1}{f})+\log|\,c_{\lambda}\,|\,,$$

which is another form of the generalized Jensen's formula and is also known as the Nevanlinna-Jensen's formula.

Theorem 2.8 (Nevanlinna-Jensen's formula) *Let f be a meromorphic function , then, for $r>0$,*

$$T(r,f)=T(r,\frac{1}{f})+\log|\,c_{\lambda}\,|\,,$$

where c_{λ} is the first non-zero coefficient of the Laurent expansion of f at 0.

By the Nevanlinna-Jensen's formula, we can get the Nevanlinna's first fundamental theorem.

Theorem 2.9 (Nevanlinna's First Fundamental Theorem) *Let f be a meromorphic function and a be a finite complex number. Then, for $r>0$, we have*

$$T(r,\frac{1}{f-a})=T(r,f)+\log|c_{\lambda}|+\varepsilon(a,r),$$

where c_{λ} is the first non-zero coefficient of the Laurent expansion of $\frac{1}{f-a}$ at 0, and

$$|\varepsilon(a,r)|\leq\log^{+}|a|+\log 2.$$

Usually, Nevanlinna's first fundamental theorem is written as

$$T(r,\frac{1}{f-a})=T(r,f)+O(1).$$

Now, we come to the most important theorem in the theory of value distribution, namely, Nevanlinna's second fundamental theorem.

Theorem 2.10 (Nevanlinna's Second Fundamental Theorem) *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}$, $1 \le j \le q$, be q distinct finite values $(q \ge 2)$. Then*

$$m(r, f) + \sum_{j=1}^{q} m(r, \frac{1}{f - a_j}) \le 2T(r, f) - N_1(r) + S(r, f),$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$ and

$$S(r, f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f - a_j}) + O(1).$$

Given $a \in \mathbb{C}$, by Nevanlinna's first fundamental theorem,

$$m(r, \frac{1}{f - a}) = T(r, f) - N(r, \frac{1}{f - a}) + O(1).$$

Hence, Nevanlinna's second fundamental theorem can be rewritten as follows.

Theorem 2.11 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values $(q \ge 3)$. Then*

$$(q - 2)T(r, f) < \sum_{j=1}^{q} N(r, \frac{1}{f - a_j}) - N_1(r) + S(r, f),$$

where $N_1(r)$ and $S(r, f)$ are given as in Theorem 2.10.

Note that, in Theorem 2.11, if some $a_j = \infty$, then $N(r, \frac{1}{f - a_j})$ should be read as $N(r, f)$.

Let $n_1(t) = 2n(t, f) - n(t, f') + n(t, \frac{1}{f'})$ and let $\overline{n}(t, f)$ denote the number of distinct poles of f in $|z| \le t$. Define

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r,$$

which is called the reduced counting function of f. Note that, if z_0 is a pole of f of order k in $|z| \le t$, then z_0 is counted $k - 1$ times by $n_1(r)$. Similarly, for a finite value a, if z_0 is a zero of $f-a$ of order k in $|z| \le t$, then z_0 is also counted $k-1$ times by $n_1(r)$. Hence,

$$\sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) \le \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}).$$

Therefore, we have the third form of Nevanlinna's second fundamental theorem.

Theorem 2.12 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values ($q \ge 3$). Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

where $S(r,f)$ is given as in Theorem 2.10.

In Nevanlinna's second fundamental theorem, the remainder term $S(r,f)$ is a complicated object which can be estimated by using the method of logarithmic derivative. It turns out that $S(r,f)$ is small comparing to $T(r,f)$. In order to make it clear, we need the concept of the growth of meromorphic function.

Classically, we use the maximum modulus to measure the growth of an entire function.

Definition 2.13 *Let f be a meromorphic function. The order λ of f is defined to be*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and the lower order μ of f is defined to be

$$\mu = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Let f be an entire function. Define, for $r \ge 0$,

$$M(r,f) = \max_{|z \le r|} |f(z)|.$$

Then the relation between $M(r,f)$ and $T(r,f)$ is given as follows.

Theorem 2.14 *Let $0 \le r < R < \infty$ and f be an entire function, we have*

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f).$$

In particular,

$$T(r,f) \le \log^+ M(r,f) \le 3T(2r,f).$$

By Theorem 2.14, the order and lower order of an entire function are unambiguous. Now, we can state the properties of $S(r,f)$.

Lemma 2.15 *Let f be a non-constant meromorphic function. If f is of finite order, then*

$$m(r, \frac{f'}{f}) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$m(r, \frac{f'}{f}) = O(\log(rT(r,f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

Theorem 2.16 *Let f be a non-constant meromorphic function and $S(r,f)$ be defined in Theorem 2.10. If f is of finite order, then*

$$S(r,f) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$S(r,f) = O(\log(rT(r,f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

In the thesis, we will denote by $S(r,f)$ any quantity satisfy $S(r,f) = o(T(r,f))$ as $r \to \infty$ if f is of finite order, and $S(r,f) = o(T(r,f))$ as $r \to \infty, r \notin E$ if f is of infinite order, where E is a set of finite measure.

By Lemma 2.15, $m(r, \frac{f'}{f}) = S(r,f)$. Moreover, Milloux [2] proved the following.

Theorem 2.17 *Let f be a non-constant meromorphic function and k be a positive integer and let*

$$\Psi(z) = \sum_{i=1}^{k} a_i(z) f^{(i)}(z),$$

where $a_1(z), a_2(z), \ldots, a_k(z)$ are small functions of f. Then

$$m(r, \frac{\Psi}{f}) = S(r, f).$$

Now, we record some well-known results on four or five value problem. First, we need some definitions.

Definition 2.18 *Let f and g be non-constant meromorphic functions and $a \in \mathbb{C}_\infty$. We say that*

(i) *f and g share a CM (counting multiplicities) if $f(z) - a = 0$ and $g(z) - a = 0$ have the same number of zeros with the same multiplicities.*

(ii) *f and g share a IM (ignoring multiplicities) if $f(z) - a = 0$ and $g(z) - a = 0$ have the same number of zeros ignoring multiplicities.*

Example 2.19 *e^z and e^{-z} share $0, 1, -1, \infty$ CM.*

In 1929, R. Nevanlinna [1] proved the following remarkable results.

Theorem 2.20 ([1], 5IM) *If f and g are two meromorphic functions and share five distinct values in $\mathbb{C}_\infty$, then $f \equiv g$.*

Theorem 2.21 ([1], 4CM) *If f and g are two meromorphic functions and share four distinct values a_1, a_2, a_3 and a_4 CM, then f is a Möbius transformation of g, two of the values, say a_1 and a_2, must be lacunary, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1979 and 1983, G.G. Gundersen [3, 4] proved the following results.

Theorem 2.22 ([3], 3CM+1IM=4CM) *If f and g are two meromorphic functions and share three values* CM *and share a fourth value* IM, *then they share all four values* CM *and, hence, Theorem 2.21 holds.*

Theorem 2.23 ([4], 2CM+**2**IM=**4**CM) *Let f and g are two meromorphic functions sharing four values a_1, a_2, a_3 and a_4. If f and g share a_1, a_2* CM, *and a_3, a_4* IM, *then f and g share all four values* CM *and, hence, Theorem 2.21 holds.*

Also, C.C. Yang and H.X Yi [5] proved the following results.

Theorem 2.24 ([5]) *If f and g are two meromorphic functions and share four distinct values a_1, a_2, a_3 and a_4* IM *and if $\overline{N}(r, \frac{1}{f-a_j}) = S(r, f)$ $(j = 1, 2)$, then f and g share a_j $(j = 1, 2, 3, 4)$* CM.

Now we record some well-known results on two meromorphic functions sharing four or five small functions as follows.

Definition 2.25 *Let f be a meromorphic function. A meromorphic function $a(z)$ is said to be a small function with respect to f if $T(r, a) = S(r, f)$.*

Denote by $\mathbf{S}(f)$ the collection of small functions with respect to f. We put for $a \in \mathbf{S}(f) \cup \{\infty\}$

$$E(f = a) = \{z : f(z) - a(z) = 0\},$$

where $f(z) - \infty$ means $\frac{1}{f(z)}$.

We have the generalization of second fundamental theorem for three small functions.

Theorem 2.26 ([5]) *Let f be a non-constant meromorphic function and $a_1(z)$, $a_2(z)$ and $a_3(z)$ are three distinct small function. Then*

$$T(r,f) < \sum_{j=1}^{3} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

By the Theorem 2.26, we have the following consequence.

Corollary 2.27 *Let f and g be transcendental meromorphic functions in the complex plane. Suppose that there are three distinct elements $a_1, a_2, a_3 \in \mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying*

$$E(f = a_j) = E(g = a_j) \ (j = 1, 2, 3),$$

then

$$T(r,f) \leq 3T(r,g) + S(r,f) \ \text{and} \ T(r,g) \leq 3T(r,f) + S(r,g).$$

In particular,

$$T(r,f) = O(T(r,g)), \quad T(r,g) = O(T(r,f)) \quad \text{and} \quad S(r,f) = S(r,g).$$

Proof. By assumption and Theorem 2.26, we have

$$\begin{aligned} T(r,f) &\leq \sum_{j=1}^{3} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f) \\ &= \sum_{j=1}^{3} \overline{N}(r, \frac{1}{g-a_j}) + S(r,f) \\ &\leq \sum_{j=1}^{3} T(r, \frac{1}{g-a_j}) + S(r,f) \\ &= 3T(r,g) + S(r,f), \end{aligned}$$

which implies that,

$$T(r,f) = O(T(r,g)).$$

Similarly, we have $T(r,g) = O(T(r,f))$, and $S(r,f) = S(r,g)$. ❑

For meromorphic functions sharing small functions, the following results are well-known.

Theorem 2.28 ([6], 6IM) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, ..., 6)$ be six distinct small functions. If f and g share a_j $(j = 1, ..., 6)$* IM, *then $f = g$.*

Theorem 2.29 ([7], 5IM) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, ..., 5)$ be five distinct small functions. If f and g share a_j $(j = 1, ..., 5)$* IM, *then $f = g$.*

Theorem 2.30 ([8]) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, ..., 5)$ be five distinct small functions. If f and g share a_j $(j = 1, ..., 4)$* IM, *and $\overline{N}(r, f = a_5 = g) \neq S(r, f)$, then $f = g$.*

Theorem 2.31 ([9], 4CM) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, 2, 3, 4)$ be four distinct small functions. If f and g share a_j $(j = 1, 2, 3, 4)$* CM, *then f is a Möbius transformation of g with small functions as coefficients.*

Theorem 2.32 ([9], 3CM+1IM=4CM) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, 2, 3, 4)$ be four distinct small functions. If f and g share a_j $(j = 1, 2, 3)$* CM *and a_4* IM, *then f is a Möbius transformation of g with small functions as coefficients.*

Theorem 2.33 ([10], 2CM+2IM=4CM) *If f and g are two non-constant meromorphic functions, and a_j $(j = 1, 2, 3, 4)$ be four distinct small functions. If f and g share a_1, a_2* CM *and a_3, a_4* IM, *then f is a Möbius transformation of g with small functions as coefficients.*