4 Meromorphic Functions satisfying some Functional Equations

In this section, we consider two non-constant meromorphic functions f and g satisfying the functional equation

$$\frac{(f')^k}{f(f-1)} = \frac{(g')^k}{g(g-1)}, \tag{4.1}$$

and study the relationship between f and g.

First, we consider the simplest case that $k = 1$.

Theorem 4.1 *Let f and g be non-constant meromorphic functions satisfying* (4.1) *with $k = 1$, that is,*

$$\frac{f'}{f(f-1)} = \frac{g'}{g(g-1)}. \tag{4.2}$$

Then f and g have one of the following relations:

(i) $f = g$;

(ii) $\frac{1}{f} + \frac{1}{g} = 2$;

(iii) $\frac{1}{f} - \frac{c}{g} = 1 - c$ *for some constant* $c \neq -1, 0, 1$.

Proof. Let f and g be non-constant meromorphic functions satisfying (4.2), we can write

$$\frac{f'}{f-1} - \frac{f'}{f} = \frac{g'}{g-1} - \frac{g'}{g},$$

and integrate to obtain

$$\frac{f-1}{f} = c\,\frac{g-1}{g}\ ,$$

where $c \neq 0$ is a const.

Case 1. If $c = 1$, $1 - \frac{1}{f} = 1 - \frac{1}{g}$, then $f = g$.

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Case 2. If $c = -1$, $1 - \frac{1}{f} = -1 + \frac{1}{g}$, then $\frac{1}{f} + \frac{1}{g} = 2$.

Case 3. If $c \neq -1, 1$, then $\frac{1}{f} - \frac{c}{g} = 1 - c$, for some constant $c \neq -1, 0, 1$. ❑

Note that (ii) and (iii) can occur. For (ii), let f be an arbitrary given meromorphic function and $g = \frac{f}{2f-1}$. Then it is easy to see that f and g satisfy (4.2) and $\frac{1}{f} + \frac{1}{g} = 2$. Similarly, for (iii), let f be an arbitrary given meromorphic function and $g = \frac{cf}{(c-1)f+1}$. Then we can see easily that f and g satisfy (4.2) and $\frac{1}{f} - \frac{c}{g} = 1 - c$.

Corollary 4.2 *Let f and g be non-constant meromorphic functions satisfying* (4.2). *Then f and g have the same characteristic function, i.e.,$T(r,f) = T(r,g) + O(1)$. In particular, they have the same order.*

Proof. By Theorem 4.1, there are only three possibilities:

Case 1. If $f = g$, then $T(r,f) = T(r,g)$.

Case 2. If $\frac{1}{f} + \frac{1}{g} = 2$, then $T(r, \frac{1}{f}) = T(r, 2 - \frac{1}{g})$, which implies that $T(r,f) = T(r,g) + O(1)$.

Case 3. If $\frac{1}{f} - \frac{c}{g} = 1 - c$, then $\frac{1}{f} = 1 - c + \frac{c}{g}$, which implies that $T(r, \frac{1}{f}) = T(r, 1 - c + \frac{c}{g})$. Again, we have $T(r,f) = T(r,g) + O(1)$.

Therefore, in any case, we have $T(r,f) = T(r,g) + O(1)$, and f and g have the same order. ❑

Corollary 4.3 *Let f and g be non-constant polynomials satisfying* (4.2). *Then $f = g$.*

Proof. Assume that

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0, \quad n \geq 1$$

and

$$g(z) = b_m z^m + b_{m-1} z^{m-1} + ... + b_0, \quad m \geq 1$$

By Theorem 4.1, there are only three possibilities. If (ii) of Theorem 4.1 holds, then $f+g=2fg$. By comparing their degree, we obtain a contradiction. Hence, it can not be occurred. By the same argument, so is (iii) of Theorem 4.1. Therefore, $f=g$.

❑

Corollary 4.4 *Let f and g be non-constant rational functions satisfying* (4.2). *Then f and g have one of the following relations:*

(i) $f=g$;

(ii) $\frac{1}{f}+\frac{1}{g}=2$;

(iii) $\frac{1}{f}-\frac{c}{g}=1-c$ *for some constant* $c\neq -1,0,1$.

Note that (ii) and (iii) can occur. For (ii), let $f=\frac{1}{z}$ and $g=\frac{1}{2-z}$. Then it is easy to see that f and g satisfy (4.2) and $\frac{1}{f}+\frac{1}{g}=2$. Similarly, for (iii), let $f=\frac{1}{z}$ and $g=\frac{c}{z-1+c}$. Then we can see easily that f and g satisfy (4.2) and $\frac{1}{f}-\frac{c}{g}=1-c$.

Theorem 4.5 *Let f and g be non-constant rational functions satisfying* (4.2). *If there exists a point z_0 such that $f(z_0)=g(z_0)=\infty$. Then $f=g$.*

Proof. If $f(z_1)=0$, then by Corollary 4.4, we get $g(z_1)=0$, and conversely. By computing the residues of both sides in (4.2), $f(z_1)$ and $g(z_1)$ have the same multiplicity at z_1. Therefore, f and g share 0 CM. Similarly, f and g share 1 CM.

By Corollary 3.6 and the hypothesis, we get $f=g$. ❑

Definition 4.6 *Let f and g be non-constant meromorphic functions, a_1 and a_2 be two arbitrary complex numbers. If $f-a_1$ and $g-a_2$ have the same zeros (counting multiplicity), then we say that f and g share the pair (a_1,a_2)* CM.

Theorem 4.7 *Let f and g be non-constant meromorphic functions satisfying* (4.2)*. Then*

(i) *If (ii) of Theorem 4.1 holds, then f and g share* 0, 1 CM, *and share the pairs* $(\frac{1}{2}, \infty)$, $(\infty, \frac{1}{2})$ CM.

(ii) *If (iii) of Theorem 4.1 holds, then f and g share* 0, 1 CM, *and share the pairs* $(\frac{1}{1-c}, \infty)$, $(\infty, \frac{c}{c-1})$ CM.

Proof.

First, it follows from Theorem 4.1 that f and g share 0 and 1 CM.

Now, we claim that f and g share the pair $(\frac{1}{2}, \infty)$ CM if (ii) of Theorem 4.1 occurs. Clearly, $f(z_1) = \frac{1}{2} \Leftrightarrow g(z_1) = \infty$. Let $f(z_1) = \frac{1}{2}$ and $g(z_1) = \infty$ with multiplicities p and q, resp. Write

$$f(z) = \frac{1}{2} + (z - z_1)^p\, h_1(z), \qquad h_1(z_1) \neq 0$$

and

$$g(z) = (z - z_1)^{-q}\, h_2(z), \qquad h_2(z_1) \neq 0,$$

then

$$\frac{f'(z)}{f(z)(f(z) - 1)} = (z - z_1)^{p-1}\varphi_1(z), \qquad \varphi_1(z_1) \neq 0,$$

and

$$\frac{g'(z)}{g(z)(g(z) - 1)} = (z - z_1)^{q-1}\varphi_2(z), \qquad \varphi_2(z_1) \neq 0.$$

By the hypothesis, we get $p = q$, this implies f and g share the pair $(\frac{1}{2}, \infty)$ CM. Clearly, $f(z_1) = \infty \Leftrightarrow g(z_1) = \frac{1}{2}$, by the same argument. Therefore, f and g share the pair $(\infty, \frac{1}{2})$ CM.

Finally, we claim that f and g share the pair $(\frac{1}{1-c}, \infty)$ CM if (iii) of Theorem 4.1 occurs. Clearly, $f(z_2) = \frac{1}{1-c} \Leftrightarrow g(z_2) = \infty$. Let $f(z_2) = \frac{1}{1-c}$ and $g(z_2) = \infty$ with multiplicities p and q, resp. Write

$$f(z) = \frac{1}{1 - c} + (z - z_2)^p\, \psi_1(z), \qquad \psi_1(z_2) \neq 0$$

and

$$g(z) = (z - z_2)^{-q}\,\psi_2(z), \quad \psi_2(z_2) \neq 0,$$

then

$$\frac{f'(z)}{f(z)(f(z)-1)} = (z - z_2)^{p-1}\xi_1(z), \quad \xi_1(z_2) \neq 0,$$

and

$$\frac{g'(z)}{g(z)(g(z)-1)} = (z - z_2)^{q-1}\xi_2(z), \quad \xi_2(z_2) \neq 0.$$

By the hypothesis, we get $p = q$. Hence f and g share the pair $(\frac{1}{2-c}, \infty)$ CM. Clearly, $f(z_2) = \infty \Leftrightarrow g(z_2) = \frac{c}{c-1}$, by the same argument. Therefore, f and g share the pair $(\infty, \frac{c}{c-1})$ CM. ❑

Next, we consider the case that $k = 2$ in (4.1). In the proof of a result in [11], one can find that it contains the following result. For completeness, we give a proof.

Theorem 4.8 *Let f and g be non-constant meromorphic functions satisfying* (4.1) *with $k = 2$, that is,*

$$\frac{(f')^2}{f(f-1)} = \frac{(g')^2}{g(g-1)}. \tag{4.3}$$

Then f and g have one of the following relations:

(i) $f = g$;

(ii) $f = 1 - g$;

(iii) $(f^2 + g^2) - (1 + c)(f + g) + 2cfg + \frac{(1+c)^2}{4} = 0$, *for some constant $c \neq -1, 1$.*

Proof. *Put $F = 2f - 1$ and $G = 2g - 1$. Then from* (4.3) *we obtain*

$$\frac{(F')^2}{F^2 - 1} = \frac{(G')^2}{G^2 - 1}, \tag{4.4}$$

We denote by A the common function in (4.3)*. Then, we have*

$$\frac{(F')^2}{A} = F^2 - 1 \quad \textit{and} \quad \frac{(G')^2}{A} = G^2 - 1 \tag{4.5}$$

Differentiate the functions in (4.5), we obtain

$$\frac{2F'F''A-(F')^2A'}{A^2}=2FF' \quad \text{and} \quad \frac{2G'G''A-(G')^2A'}{A^2}=2GG'.$$

As $F'\neq 0$, $G'\neq 0$, we have

$$\frac{2F''A-F'A'}{A^2}=2F \quad \text{and} \quad \frac{2G''A-G'A'}{A^2}=2G.$$

Add them side by side to obtain

$$\frac{2(F''+G'')A-(F'+G')A'}{A^2}=2(F+G). \tag{4.6}$$

Multiply (4.6) *by $F'+G'$ to obtain*

$$\frac{2(F''+G'')(F'+G')A-(F'+G')^2A'}{A^2}=2(F+G)(F'+G'). \tag{4.7}$$

Integrate (4.7) *to get*

$$\frac{(F'+G')^2}{A}=(F+G)^2+c', \tag{4.8}$$

where c' is a constant. From (4.5) *and* (4.8) *we have*

$$\frac{F'G'}{A}=FG+c, \tag{4.9}$$

where $c=\frac{c'}{2}+1$. Eliminating A, F' and G' from (4.5) *and* (4.9) *we have*

$$(F^2-1)(G^2-1)=(FG+c)^2,$$

that is,

$$F^2+2cFG+G^2=1-c^2. \tag{4.10}$$

Eliminating F and G from $F=2f-1$, $G=2g-1$ and (4.10), *we have*

$$(f^2+g^2)-(1+c)(f+g)+2cfg+\frac{(c+1)^2}{4}=0. \tag{4.11}$$

Now, we have the following cases:

Case 1. *If $c=-1$, then $f=g$;*

Case 2. *If $c=1$, then $f=1-g$;*

Case 3. *If $c\neq -1,1$, then $(f^2+g^2)-(1+c)(f+g)+2cfg+\frac{(c+1)^2}{4}=0$.* ❑

Corollary 4.9 *Let f and g be non-constant meromorphic functions satisfying* (4.3). *Then f and g have the same characteristic function, i.e.,$T(r,f) = T(r,g) + O(1)$. In particular,they have the same order.*

Proof. By Theorem 4.8, there are only three possibilities:
Case 1. If $f = g$, then $T(r,f) = T(r,g)$;
Case 2. If $f = 1-g$, then $T(r,f) = T(r,1-g)$, this implies $T(r,f) = T(r,g)+O(1)$;
Case 3. If $c \neq -1, 1$, we divide (4.11) by g^2 to obtain

$$(\frac{f}{g})^2 + 1 - (1+c)(\frac{f}{g^2} + \frac{1}{g}) + 2c\frac{f}{g} + \frac{(1+c)^2}{4g^2} = 0.$$

We get $2T(r,g) = 2T(r,\frac{f}{g}) + O(1)$. By symmetry, divide (4.11) by f^2, we also have

$$2T(r,f) = 2T(r,\frac{g}{f}) + O(1).$$

By Theorem 2.9, we know that $T(r,\frac{g}{f}) = T(r,\frac{f}{g}) + O(1)$, then we have $T(r,f) = T(r,g) + O(1)$.

Therefore, in any case, we have $T(r,f) = T(r,g) + O(1)$. In particular, f and g have the same order. ❑

Corollary 4.10 *Let f and g be non-constant polynomials satisfying* (4.3). *Then f and g have the same degree and satisfying:*

(i) $f = g$;

(ii) $f + g = 1$;

Proof. First, we prove that f and g have the same degree. If (i) and (ii) of Theorem 4.8 holds, then f and g have the same degree. Now, assume that (iii) of Theorem 4.8 occurs the degree of f is n and the degree of g is m, $n \neq m$. Then

$$(f^2 + g^2) - (1+c)(f+g) + 2cfg + \frac{(1+c)^2}{4}$$

is a non-constant polynomial of degree $\max\{2n, 2m\} \geq 2$ which is a contradiction. Therefore, (iii) of Theorem 4.8 can't be occurred.

Then, we have only the possibility $f = g$ or $f + g = 1$ by Theorem 4.8.

❑

Note that (ii) can occur. Let $f(z) = z$ and $g(z) = 1 - z$. Then it is easy to see that f and g satisfy (4.3) and $f + g = 1$. For rational function, (ii) can occur. For example, let $f(z) = \frac{1}{z}$ and $g(z) := 1 - \frac{1}{z}$. Then it is easy to see that f and g satisfy (4.3) and $f + g = 1$.

Finally, we consider the case that $k \geq 3$ in (4.1).

Definition 4.11 *Let f be a non-constant meromorphic function and let S be a subset in the extended complex plane $\mathbb{C}_\infty$. Define*

$$E_f(S) = \bigcup_{a \in S} \{z | f(z) - a = 0\},$$

where a zero of $f - a$ with multiplicity m counts m times in $E_f(S)$. If g is another meromorphic function and $E_f(S) = E_g(S)$, then we say that f and g share the set S CM.

The main properties of such f and g is as follows.

Theorem 4.12 *Let f and g be non-constant meromorphic functions satisfying* (4.1) *with $k \geq 3$, that is,*

$$\frac{(f')^k}{f(f-1)} = \frac{(g')^k}{g(g-1)}.$$

Then f and g share ∞ CM, *and the set $\{0, 1\}$* CM.

Proof. Denote by

$$F = \frac{(f')^k}{f(f-1)} \quad \textit{and} \quad G = \frac{(g')^k}{g(g-1)}.$$

If z_0 is a pole of f of multiplicity p, then F has a pole at z_0 of multiplicity $(k-2)p+k \geq 4$, so is G by assumption. Now, we claim that g has a pole at z_0 of multiplicity p

Assume that z_0 is a zero of g of multiplicity q. If $q=1$, then G has a simple pole which is impossible. If $q \geq 2$, then G has a zero at z_0 of multiplicity $q(k-1)-k \geq 0$, which is also impossible. Therefore, z_0 is not a zero of g. Similarly, z_0 is not a 1-point of g. Hence g must have a pole at z_0 of multiplicity q. By the same argument as in the beginning, $p=q$. Conversely, if g has a pole at z_0 of multiplicity p, then f has a pole at z_0 of multiplicity p. So f and g share ∞ CM.

Now assume that z_0 is a zero of $f(z)$ of multiplicity p. If $p=1$, then F has a simple pole at z_0. If $p \geq 2$, then F has a zero at z_0 of multiplicity $p(k-1)-k \geq 0$. As in the first part of proof, g can not have a pole at z_0. So $g(z_0)=0$ or 1. If z_0 is a zero of $g(z)$ with multiplicity q, as above, either G has a simple pole at z_0 or G has a zero at z_0 of multiplicity $(k-1)q-k$. Since $F=G$, it must be $p=q$. If z_0 is a 1-point of g of multiplicity q, then either G has a simple at z_0 or G has a zero at z_0 of multiplicity $(k-1)q-k$. Again, $p=q$.

Similarly, if z_1 is a 1-point of f of multiplicity p, then either z_1 is a zero of g of multiplicity p or z_1 is a 1-point of g of multiplicity p. Conversely, we can reverse the roles of f and g. Therefore, f and g share the set $\{0,1\}$ CM.

❑

Corollary 4.13 *Let f and g be non-constant meromorphic functions satisfying (4.1) with $k \geq 3$. Then $T(r,f)=O(T(r,g))$ and $T(r,g)=O(T(r,f))$. In particular, f and g have the same order.*

Proof. By Theorem 2.12 for $q = 3$, and from Theorem 4.12, we have

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f-1}) + S(r,f) \\ &= \overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) + \overline{N}(r,\frac{1}{g-1}) + S(r,f) \\ &\leq 3T(r,g) + S(r,f), \end{aligned}$$

which implies that

$$T(r,f) = O(T(r,g)).$$

Similarly, we get

$$T(r,g) = O(T(r,f)).$$

Therefore, f and g have the same order. ❑

In the following, we consider some interesting simple cases, and obtain some results.

Theorem 4.14 *Let f and g are non-constant polynomials of degree one satisfying* (4.1) *with $k \geq 3$. Then either $f = g$ or $f + g = 1$.*

Proof. Assume that $f(z) = az + b$ and $g(z) = cz + d$, $a \neq 0$ and $c \neq 0$. From

$$\frac{(f')^k}{f(f-1)} = \frac{(g')^k}{g(g-1)},$$

we have

$$\frac{a^k}{(az+b)(az+b-1)} = \frac{c^k}{(cz+d)(cz+d-1)},$$

which implies that

$$A^k[c^2z^2 + (2cd - c)z + (d^2 - d)] = [a^2z^2 + (2ab - a)z + (b^2 - b)], \tag{4.12}$$

where $A = \frac{a}{c}$ is a non-constant. Compare the coefficients of both sides in (4.12), we get

$$A^k = A^2, \tag{4.13}$$

$$A(2d-1)=2b-1, \tag{4.14}$$

and

$$A^k(d^2-d)=(b^2-b), \tag{4.15}$$

Now, we have two cases:

Case 1. If $d=\frac{1}{2}$, then $b=\frac{1}{2}$ by (4.14) and $A\neq 0$, and $A^2=1$ by (4.15) and (4.13). If $A=1$, then $a=c$, which implies that $f=g$. If $A=-1$, then $a=-c$, which implies that $f+g=1$. We are done.

Case 2. If $d\neq\frac{1}{2}$, then $b\neq\frac{1}{2}$ by (4.14) and $A\neq 0$,

we divide the discussion into five subcases:

Subcase 2.1. $d=0$ and $b=0$. Then we get $A=1$ by (4.14), and $f=g$.

Subcase 2.2. $d=0$ and $b=1$. Then we get $A=-1$ by (4.14), and $f+g=1$.

Subcase 2.3. $d=1$ and $b=0$. Then we get $A=-1$ by (4.14), and $f+g=1$.

Subcase 2.4. $d=1$ and $b=1$. Then we get $A=1$ by (4.14), and $f=g$.

Subcase 2.5. $d\neq 0,1$ and $b\neq 0,1$. Then by (4.13), (4.14) and (4.15), we have

$$(\frac{2b-1}{2d-1})^2=\frac{b^2-b}{d^2-d},$$

which implies that $d^2-d=b^2-b$. By (4.15) and (4.13), we get $A^2=1$. Again, we also have $f=g$ or $f+g=1$ by case1.

Therefore, in any case, we always have $f=g$ or $f+g=1$. ❑

Now, in order to get another result for non-constant rational functions f and g satisfying (4.1) with $k\geq 3$, we need some preliminaries.

Lemma 4.15 *Let f be a non-constant rational function. Then, for any positive integer n, $f^n, f^{n-1}, ..., f^2, f, 1$ are linearly independent.*

***Proof*.** By induction. For $n = 1$, if $f, 1$ are linearly dependent, then there exist $a, b \in \mathbb{C}$ not all zeros, such that $af + b = 0$. If $a \neq 0$, then $f = -\frac{b}{a}$ is a constant, which is a contradiction. Now, we assume that $a = 0$, then we get $b = 0$ which is impossible. Hence, $f, 1$ are linearly independent.

Assume that $f^{n-1}, f^{n-2}, ..., f, 1$ are linearly independent for $n \geq 3$. Since $f(z)$ be a non-constant rational function, we may assume that

$$f(z) = \frac{q(z)}{p(z)}, \tag{4.16}$$

where $(p(z), q(z)) = 1$, and $p(z)$, $q(z)$ are complex polynomials. Assume that

$$a_n f(z)^n + a_{n-1} f(z)^{n-1} + ... + a_1 f(z) + a_0 = 0, \tag{4.17}$$

where $a_i \in \mathbb{C}$, $0 \leq i \leq n$. From (4.16) and (4.17), we have

$$a_n (\frac{q(z)}{p(z)})^n + a_{n-1} (\frac{q(z)}{p(z)})^{n-1} + ... + a_1 (\frac{q(z)}{p(z)}) + a_0 = 0, \tag{4.18}$$

which implies that

$$a_n q(z)^n + a_{n-1} q(z)^{n-1} p(z) + ... + a_1 q(z) p(z)^{n-1} + a_0 p(z)^n = 0. \tag{4.19}$$

Now, we divide the proof into three cases:

Case 1. $q(z)$ is a constant. In this case, $p(z)$ is a non-constant polynomial. Choose $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. By (4.19), we get $a_n = 0$. Hence, (4.19) becomes

$$a_{n-1} f(z)^{n-1} + ... + a_1 f(z) + a_0 = 0.$$

By induction hypothesis, we conclude that $a_{n-1} = ... = a_0 = 0$.

Case 2. $p(z)$ is a constant. The leading coefficient of the left-hand side in (4.19) must be zero. i.e. $a_n b^n = 0$, where b is the leading coefficient of $q(z)$. Hence, $a_n = 0$. As in Case 1, we get $a_{n-1} = ... = a_0 = 0$.

Case 3. $p(z)$ and $q(z)$ are not constant. Choose a point z_0 such that $p(z_0) = 0$. Since $(p(z), q(z)) = 1$, then $q(z_0) \neq 0$. This implies that $a_n q(z_0)^n = 0$. Hence, $a_n = 0$. Similarly, we get $a_{n-1} = ... = a_0 = 0$.

In any case $f^n, f^{n-1}, ..., f, 1$ are linearly independent. ❑

Theorem 4.16 *Let f and g be non-constant rational functions satisfying* (4.1) *with* $k \geq 3$. *Then either* $f = g$ *or* $f + g = 1$.

***Proof*.** By Theorem 4.12, we may write

$$\frac{g(g-1)}{f(f-1)} = e^{\alpha}, \tag{4.20}$$

where α is an entire function. Since the left-hand side of (4.20) is a rational function. Then, e^{α} must be a nonzero constant, say $e^{\alpha} = c \neq 0$. From, (4.1) and (4.20), we have

$$(\frac{f'}{g'})^k = \frac{f(f-1)}{g(g-1)} = \frac{1}{c}.$$

Choose $a \in \mathbb{C}$ such that $a^k = c$. We get

$$(\frac{af'}{g'})^k = 1.$$

Let $w = \frac{af'}{g'}$, that is, $w^k = 1$, which implies that w is a constant, and we have

$$g' = \frac{a}{w} f'. \tag{4.21}$$

And integral (4.21) to obtain

$$g = Af + B,$$

where $A = \frac{a}{w}$, and B are constants. Again, from (4.1), we obtain

$$\frac{(f')^k}{f(f-1)} = \frac{(Af')^k}{(Af+B)(Af+B-1)},$$

which implies that

$$(A^2 - A^k)f^2 + (2AB - A + A^k)f + (B^2 - B) = 0.$$

By Lemma 4.15 for $n = 2$, $f^2, f, 1$ are linearly independent, hence

$$A^2 - A^k = 0,$$

$$2AB - A + A^k = 0,$$

$$B^2 - B = 0.$$

By a simple calculation, we get $B = 0$ and $A = 1$ or $B = 1$ and $A = -1$ which implies that $f = g$ or $f + g = 1$, respectively. ❑