5 Meromorphic Functions with Few Poles

In this section, we will prove our another main results concerning two meromorphic functions with few poles sharing four distinct small functions. The origional result was proved by K.Ishizaki and N.Toda [11]. Here, we will discuss some variant forms of their results. In fact K.Ishizaki and N.Toda [11] proved the following results.

Lemma 5.1 ([11]) *Let $f(z)$ be a transcendental meromorphic function in the complex plane. For five distinct elements $a_1, ..., a_5 \in \mathbf{S}(f) \cup \{\infty\}$, then*

$$3T(r,f) \leq \sum_{i=1}^{5} \overline{N}(r, \frac{1}{f-a_i}) + 2\{\overline{N}(r, \frac{1}{f-a_j}) + \overline{N}(r, \frac{1}{f-a_k})\} + S(r,f),$$

for some j, k, $1 \leq j \neq k \leq 5$.

Lemma 5.2 ([11]) *Let $f(z)$ be a transcendental meromorphic function in the complex plane and let $a_1, ..., a_5$ be five distinct elements of $\mathbf{S}(f) \cup \{\infty\}$. Then we have the following inequality:*

$$2T(r,f) \leq \sum_{j=1}^{5} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f).$$

Lemma 5.3 ([11]) *Let $f(z)$ be a transcendental meromorphic function in $|z| < \infty$. Then for any q distinct elements $a_1, ..., a_q \in \mathbf{S}(f)$, $2 \leq q < \infty$ the following inequality holds:*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} N_m(r, \frac{1}{f-a_j}) + m\overline{N}(r,f) + S(r,f),$$

where m is the number of elements of a maximal linearly independent subset of $\{a_1, ..., a_q\}$.

Lemma 5.4 ([11]) *If $\Delta_f \neq 0$, we have the inequality*

$$2T(r,f) < N_{1)}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, \frac{1}{f-a}) + \overline{N}(r, \frac{1}{f-b}) + 2\overline{N}(r,f) + S(r,f).$$

Lemma 5.5 ([11]) *If $\Delta_f = 0$, then following relations hold:*
(a)$N_{1)}(r, \frac{1}{f}) = S(r, f)$, (b)$\overline{N}_{(2}(r, \frac{1}{f-1}) = S(r, f)$, (c)$\overline{N}(r, f) = S(r, f)$.

Lemma 5.6 ([11]) *Let f and g be two transcendental meromorphic functions in the complex plane satisfying*

$$\overline{N}(r, f) = S(r, f) \quad \text{and} \quad \overline{N}(r, g) = S(r, g).$$

We suppose that there are four distinct meromorphic functions $a_1, ..., a_4$ in $\mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying the condition

$$E(f = a_j) = E(g = a_j), \quad 1 \leq j \leq 4.$$

Then the following relations hold:
(a)$T(r, g) = T(r, f) + S(r, f)$ *and* $S(r, g) = S(r, f)$.
(b)*If $f \neq g$, then*

$$2T(r, f) = \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f - a_j}) + S(r, f).$$

Theorem 5.7 ([11]) *Let f and g be transcendental meromorphic functions in the complex plane. Suppose that there are four distinct elements $a_1, a_2, a_3, a_4 \in \mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying*

$$E(f = a_j) = E(g = a_j) \quad (j = 1, 2, 3, 4) \tag{5.1}$$

and suppose that there exist $a_j, a_k \in \{a_1, a_2, a_3, a_4\} (j \neq k)$ such that

$$a = \frac{c - a_j}{a_k - a_j} \quad \text{and} \quad b = \frac{d - a_j}{a_k - a_j}$$

are constants, where $\{c, d\} = \{a_1, a_2, a_3, a_4\} - \{a_j, a_k\}$.

If either

(i) $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *or* (ii) $\overline{N}(r, g) \leq uT(r, g) + S(r, g)$

holds for some constant $u \in [0, 1)$, then $f = g$.

Theorem 5.8 ([11]) *Let f and g be transcendental meromorphic functions in the complex plane. Suppose that there are four distinct elements $a_1, a_2, a_3, a_4 \in \mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying*

$$E(f = a_j) = E(g = a_j) \quad (j = 1, 2, 3, 4),$$

and suppose that for any $j, k (1 \le j \ne k \le 4)$, at least one of

$$a = \frac{c - a_j}{a_k - a_j} \quad \text{and} \quad b = \frac{d - a_j}{a_k - a_j}$$

is not constant, where $\{c, d\} = \{a_1, a_2, a_3, a_4\} - \{a_j, a_k\}$.

If $\overline{N}(r, f)$ and $\overline{N}(r, g)$ satisfy one of the following conditions (a), (b), (c) *and* (d)*:*

(a) $\overline{N}(r, f) = S(r, f)$ *and* $\overline{N}(r, g) = S(r, g)$*;*

(b) $\overline{N}(r, f) \ne S(r, f)$, $\overline{N}(r, g) = S(r, g)$ *and* $\overline{N}(r, f) \le uT(r, f) + S(r, f)$ *for some* $u \in (0, \frac{1}{19})$*;*

(c) $\overline{N}(r, g) \ne S(r, g)$, $\overline{N}(r, f) = S(r, f)$ *and* $\overline{N}(r, g) \le vT(r, g) + S(r, g)$ *for some* $v \in (0, \frac{1}{19})$*;*

(d) $\overline{N}(r, f) \ne S(r, f)$, $\overline{N}(r, g) \ne S(r, g)$ *and*

$$\overline{N}(r, f) \le uT(r, f) + S(r, f), \quad \overline{N}(r, g) \le vT(r, g) + S(r, g), \tag{5.2}$$

for some $u, v \in (0, 1)$ satisfying either

(i) $0 < u < \frac{1}{19}$ *and* $0 < v < \frac{2-19u}{20-19u}$ *or* (ii) $0 < v < \frac{1}{19}$ *and* $0 < u < \frac{2-19v}{20-19v}$,

then $f = g$.

Now, by adopting the approach of K.Ishizaki and N.Toda, we can improve Theorem 5.7 and Theorem 5.8 as follows:

Theorem 5.9 *Let f and g be transcendental meromorphic functions in the complex plane. Suppose that there are four distinct elements $a_1, a_2, a_3, a_4 \in \mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying*

$$E(f = a_j) = E(g = a_j) \quad (j = 1, 2, 3, 4).$$

Suppose that $\overline{N}(r, f)$ and $\overline{N}(r, g)$ satisfy one of the following conditions (a), (b), (c) *and* (d)*:*

(a) $\overline{N}(r, f) = S(r, f)$ *and* $\overline{N}(r, g) = S(r, g)$.

(b) $\overline{N}(r, f) \neq S(r, f)$, *and there exist u and v satisfying*

$$0 \leq v < \frac{1}{10} \quad \text{and} \quad 0 \leq u < \frac{1 - 10v}{19(1 - v)}$$

such that

$$\overline{N}(r, g) \leq vT(r, g) + S(r, g) \quad \text{and} \quad \overline{N}(r, f) \leq uT(r, f) + S(r, f).$$

(c) $\overline{N}(r, g) \neq S(r, g)$, *and there exist u and v satisfying*

$$0 \leq u < \frac{1}{10} \quad \text{and} \quad 0 \leq v < \frac{1 - 10u}{19(1 - u)}$$

such that

$$\overline{N}(r, f) \leq uT(r, f) + S(r, f) \quad \text{and} \quad \overline{N}(r, g) \leq vT(r, g) + S(r, g).$$

(d) $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$ *and there exist u and v satisfying either*

(i) $0 \leq u < \frac{1}{10}$ *and* $0 \leq v < \frac{1-10u}{19(1-u)}$ *or*

(ii) $0 \leq v < \frac{1}{10}$ *and* $0 \leq u < \frac{1-10v}{19(1-v)}$ *such that*

$$\overline{N}(r, f) \leq uT(r, f) + S(r, f) \quad \text{and} \quad \overline{N}(r, g) \leq vT(r, g) + S(r, g).$$

Then $f = g$.

Proof. For any j, k $(1 \le j \ne k \le 4)$, let

$$a = \frac{c - a_j}{a_k - a_j} \quad and \quad b = \frac{d - a_j}{a_k - a_j},$$

where $\{c, d\} = \{a_1, a_2, a_3, a_4\} - \{a_j, a_k\}$. If a and b are both constants for some $1 \le j \ne k \le 4$, then the result follows from Theorem 5.7.

Now, we assume that at least one of a or b is not constant and $f \ne g$. The proof divides into four cases.

Case 1. If $\overline{N}(r, f) = S(r, f)$ and $\overline{N}(r, g) = S(r, g)$.

Subcase 1-1. There exists j, $1 \le j \le 4$, such that

$$\overline{N}(r, \frac{1}{f - a_j}) = S(r, f). \tag{5.3}$$

We may assume that $j = 1$ in (5.3). Then, by Lemma 5.1 for $a_5 = \infty$ and Lemma 5.6(b), we have

$$\begin{aligned} 3T(r, f) &\le \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f - a_j}) + \overline{N}(r, f) + 2\{\overline{N}(r, \frac{1}{f - a_1}) + \overline{N}(r, f)\} + S(r, f) \\ &\le \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f - a_j}) + 2\overline{N}(r, \frac{1}{f - a_1}) + S(r, f) \\ &= 2T(r, f) + S(r, f), \end{aligned}$$

which implies that $T(r, f) = S(r, f)$. This is a contradiction.

Subcase 1-2. For all j, $1 \le j \le 4$,

$$\overline{N}(r, \frac{1}{f - a_j}) \ne S(r, f). \tag{5.4}$$

We shall arrive at a contradiction by five steps.

Step 1. For each j, either

(i) $\overline{N}_{(2}(r, \frac{1}{f-a_j}) = S(r, f)$ or (ii) $N_{1)}(r, \frac{1}{f-a_j}) = S(r, f)$.

We may assume that $j = 1$. Put

$$F = \frac{f - a_1}{a_2 - a_1}, \quad a = \frac{a_3 - a_1}{a_2 - a_1} \quad and \quad b = \frac{a_4 - a_1}{a_2 - a_1}.$$

Then 0, 1, a and b are different from each other, $a, b \in \mathbf{S}(f) \cap \mathbf{S}(g)$ and at least one of a, b is not constant by assumption. Moreover, F is transcendental and we have

$$\begin{aligned}
&T(r,F) = T(r,f) + S(r,f), && S(r,F) = S(r,f),\\
&\overline{N}(r,\frac{1}{F}) = \overline{N}(r,\frac{1}{f-a_1}) + s(r,f), && N_{1)}(r,\frac{1}{F}) = N_{1)}(r,\frac{1}{f-a_1}) + s(r,f),\\
&\overline{N}(r,\frac{1}{F-1}) = \overline{N}(r,\frac{1}{f-a_2}) + s(r,f), && \overline{N}_{(2}(r,\frac{1}{F-1}) = \overline{N}_{(2}(r,\frac{1}{f-a_2}) + s(r,f),\\
&\overline{N}(r,\frac{1}{F-a}) = \overline{N}(r,\frac{1}{f-a_3}) + s(r,f), && \overline{N}(r,\frac{1}{F-b}) = \overline{N}(r,\frac{1}{f-a_4}) + s(r,f),\\
&\overline{N}(r,F) = S(r,F). &&
\end{aligned} \tag{5.5}$$

We shall show that if $\Delta_F \neq 0$ then (i) holds, and that if $\Delta_F = 0$ then (ii) holds. First we treat the case when $\Delta_F \neq 0$. By applying Lemma 5.4 to F, we obtain from (5.5) that

$$\begin{aligned}
2T(r,f) + S(r,f) &= 2T(r,F)\\
&\leq N_{1)}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{F-a}) + \overline{N}(r,\frac{1}{F-b}) + 2\overline{N}(r,F) + S(r,F)\\
&= N_{1)}(r,\frac{1}{f-a_1}) + \sum_{j=2}^{4}\overline{N}(r,\frac{1}{f-a_j}) + S(r,f).
\end{aligned}$$

By Lemma 5.6(b) as $f \neq g$, we have

$$\begin{aligned}
2T(r,f) + \overline{N}_{(2}(r,\frac{1}{f-a_1}) &\leq \sum_{j=1}^{4}\overline{N}(r,\frac{1}{f-a_j}) + S(r,f)\\
&= 2T(r,f) + S(r,f).
\end{aligned}$$

This implies that

$$\overline{N}_{(2}(r,\frac{1}{f-a_1}) = S(r,f).$$

Secondly, we treat the case that $\Delta_F = 0$. By applying Lemma 5.5(a) to F, we obtain from (5.5) that

$$N_{1)}(r,\frac{1}{f-a_1}) + S(r,f) = N_{1)}(r,\frac{1}{F}) = S(r,F) = S(r,f),$$

that is,

$$N_{1)}(r, \frac{1}{f-a_1}) = S(r,f).$$

Therefore, we complete the proof of Step 1. Similarly, we can prove for $j = 2, 3, 4$.

Step 2. There is at most one j, $1 \le j \le 4$, such that

$$N_{1)}(r, \frac{1}{f-a_j}) = S(r,f). \tag{5.6}$$

Otherwise, there exists $j \neq k$ such that (5.6) holds, say $j = 1$, $k = 2$. Then

$$N_{1)}(r, \frac{1}{f-a_1}) = S(r,f) \quad \textit{and} \quad N_{1)}(r, \frac{1}{f-a_2}) = S(r,f). \tag{5.7}$$

Let F, a and b be as in Step 1. If $\Delta_F \neq 0$. Then, by Step 1(i), we have

$$N_{(2}(r, \frac{1}{f-a_1}) = S(r,f). \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\overline{N}(r, \frac{1}{f-a_1}) = N_{1)}(r, \frac{1}{f-a_1}) + \overline{N}_{(2}(r, \frac{1}{f-a_1}) = S(r,f),$$

which contradicts with (5.4). Therefore, $\Delta_F = 0$. Then, by Lemma 5.5(b) to F, we have from (5.5) that

$$\overline{N}_{(2}(r, \frac{1}{f-a_2}) + S(r,f) = \overline{N}_{(2}(r, \frac{1}{F-1}) = S(r,F) = S(r,f),$$

that is,

$$\overline{N}_{(2}(r, \frac{1}{f-a_2}) = S(r,f). \tag{5.9}$$

From (5.7) and (5.9), we get

$$\overline{N}(r, \frac{1}{f-a_2}) = N_{1)}(r, \frac{1}{f-a_2}) + \overline{N}_{(2}(r, \frac{1}{f-a_2}) = S(r,f),$$

which contradicts with (5.4). Therefore, Step 2 holds.

Step 3. Either

(i) $\overline{N}_{(2}(r, \frac{1}{f-a_j}) = S(r,f), \;\; 1 \le j \le 4$ or

(ii) $N_{1)}(r, \frac{1}{f-a_j}) = S(r,f)$ for one j, $\;1 \le j \le 4$ and $\overline{N}_{(2}(r, \frac{1}{f-a_k}) = S(r,f)$ for any k, $1 \le k \le 4$ and $k \ne j$.

In this case, it directly follows from Step 1 and Step 2.

Step 4. If (i) of Step 3 holds, then for all $m \in N$,we have the inequality

$$N_m(r, \frac{1}{f-a_j}) \le \overline{N}(r, \frac{1}{f-a_j}) + (m-1)\overline{N}_{(2}(r, \frac{1}{f-a_j}) \tag{5.10}$$

$$= \overline{N}(r, \frac{1}{f-a_j}) + S(r,f) \tag{5.11}$$

for $1 \le j \le 4$.

Using (5.10) and applying Lemma 5.3 for $q = 4$, Lemma 5.6(b), we have

$$\begin{aligned} 3T(r,f) &\le \sum_{j=1}^{4} N_m(r, \frac{1}{f-a_j}) + m\overline{N}(r,f) + S(r,f) \\ &\le \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f) \\ &= 2T(r,f) + S(r,f) \end{aligned}$$

which implies that $T(r,f) = S(r,f)$. This is a contradiction. Therefore, (i) of Step 3 can't be occurred.

Step 5. If (ii) of Step 3 holds, say $j = 1$ and $k = 2,3,4$, then

$$N_{1)}(r, \frac{1}{f-a_1}) = S(r,f) \quad \textit{and} \quad \overline{N}_{(2}(r, \frac{1}{f-a_k}) = S(r,f).$$

As the inequality (5.10) holds for $j = 2, 3, 4$ in this case, we have by Lemma 5.3 for $q = 4$, $1 \le m \le 4$, Lemma 5.6(b) and $\overline{N}(r, f) = S(r, f)$, we obtain

$$\begin{aligned}3T(r, f) &\le \sum_{j=1}^{4} N_m(r, \frac{1}{f - a_j}) + m\overline{N}(r, f) + S(r, f)\\ &\le N_m(r, \frac{1}{f - a_1}) + \sum_{j=2}^{4} N_m(r, \frac{1}{f - a_j}) + S(r, f)\\ &\le N_m(r, \frac{1}{f - a_1}) + \sum_{j=2}^{4} \overline{N}(r, \frac{1}{f - a_j}) + S(r, f)\\ &\le N_m(r, \frac{1}{f - a_1}) - \overline{N}(r, \frac{1}{f - a_1}) + 2T(r, f) + S(r, f),\end{aligned}$$

which says that

$$\begin{aligned}T(r, f) + \overline{N}(r, \frac{1}{f - a_1}) &\le N_m(r, \frac{1}{f - a_1}) + S(r, f)\\ &\le T(r, f) + S(r, f).\end{aligned}$$

Since the inequality $N_m(r, \frac{1}{f-a_1}) \le T(r, f) + S(r, f)$ holds in general. We get

$$\overline{N}(r, \frac{1}{f - a_1}) = S(r, f),$$

which contradicts with (5.4). Therefore, (ii) of Step 3 can't be occurred.

From Subcase 1-1 and 1-2, f must be equal to g.

Case 2. Suppose that $\overline{N}(r, f) \ne S(r, f)$, and there exist u and v satisfying $0 \le v < \frac{1}{10}$ and $0 \le u < \frac{1-10v}{19(1-v)}$ such that

$$\overline{N}(r, g) \le vT(r, g) + S(r, g) \quad \textit{and} \quad \overline{N}(r, f) \le uT(r, f) + S(r, f).$$

Apply Lemma 5.2, to $a_5 = \infty$, we obtain

$$\begin{aligned}2T(r, f) &\le \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f - a_j}) + \overline{N}(r, f) + S(r, f)\\ &\le \overline{N}(r, \frac{1}{f - g}) + \overline{N}(r, f) + S(r, f)\\ &\le T(r, f) + T(r, g) + uT(r, f) + S(r, f),\end{aligned}$$

that is

$$(1-u)T(r,f) \le T(r,g) + S(r.f). \tag{5.12}$$

By symmetry, we obtain the inequality

$$(1-v)T(r,g) \le T(r,f) + S(r.g). \tag{5.13}$$

Since $1-u>0$ and $1-v>0$, we have, from (5.12) and (5.13)

$$S(r,f) = S(r,g). \tag{5.14}$$

Let $\varphi = \frac{f-a_j}{a_k-a_j}$, $1 \le j \le 4$ and $1 \le k \ne j \le 4$. If $\Delta_\varphi = 0$, from Lemma 5.5(c), we get

$$\overline{N}(r,f) + S(r,f) = \overline{N}(r,\varphi) = S(r,\varphi) = S(r,f),$$

that is,

$$\overline{N}(r,f) = S(r,f),$$

which contradicts to $\overline{N}(r,f) \ne S(r,f)$. Hence, $\Delta_\varphi \ne 0$. Now, we apply Lemma 5.4 to obtain

$$\begin{aligned} 2T(r,f) + S(r,f) &= 2T(r,\varphi) \\ &\le N_{1)}(r,\frac{1}{\varphi}) + \overline{N}(r,\frac{1}{\varphi-1}) + \overline{N}(r,\frac{1}{\varphi-a}) + \overline{N}(r,\frac{1}{\varphi-b}) + 2\overline{N}(r,\varphi) + S(r,\varphi) \\ &= N_{1)}(r,\frac{1}{f-a_j}) + \overline{N}(r,\frac{1}{f-a_k}) + \overline{N}(r,\frac{1}{f-c}) + \overline{N}(r,\frac{1}{f-d}) + 2\overline{N}(r,f) + S(r,f) \end{aligned}$$

which implies that

$$\begin{aligned} 2T(r,f) + \overline{N}_{(2}(r,\frac{1}{f-a_j}) &\le \sum_{i=1}^{4} \overline{N}(r,\frac{1}{f-a_i}) + 2\overline{N}(r,f) + S(r,f) \\ &\le \overline{N}(r,\frac{1}{f-g}) + 2\overline{N}(r,f) + S(r,f) \\ &\le T(r,f) + T(r,g) + 2\overline{N}(r,f) + S(r,f). \end{aligned}$$

So

$$\overline{N}_{(2}(r, \frac{1}{f-a_j}) \leq T(r,g) - T(r,f) + 2\overline{N}(r,f) + S(r,f). \tag{5.15}$$

Put $F = \frac{f-a_1}{a_2-a_1}$ and apply Lemma 5.3 to F and $0, 1, a = \frac{a_3-a_1}{a_2-a_1}, b = \frac{a_4-a_1}{a_2-a_1}$. Note that $q = 4$ and the number m in Lemma 5.3 satisfies $1 \leq m \leq 3$ in this case. By (5.10) and (5.15), we have

$$\begin{aligned}
3T(r,f) + S(r,f) &= 3T(r,F) \\
&\leq N_3(r, \frac{1}{F}) + N_3(r, \frac{1}{F-1}) + N_3(r, \frac{1}{F-a}) + N_3(r, \frac{1}{F-b}) + 3\overline{N}(r,F) + S(r,F) \\
&= \sum_{j=1}^{4} N_3(r, \frac{1}{f-a_j}) + 3\overline{N}(r,f) + S(r,f) \\
&\leq \sum_{j=1}^{4} \overline{N}(r, \frac{1}{f-a_j}) + \sum_{j=1}^{4} 2\overline{N}_{(2}(r, \frac{1}{f-a_j}) + 3\overline{N}(r,f) + S(r,f) \\
&\leq \overline{N}(r, \frac{1}{f-g}) + 8(T(r,g) - T(r,f)) + 19\overline{N}(r,f) + S(r,f) \\
&\leq T(r,f) + T(r,g) + 8(T(r,g) - T(r,f)) + 19\overline{N}(r,f) + S(r,f).
\end{aligned}$$

Which becomes

$$10T(r,f) \leq 9T(r,g) + 19\overline{N}(r,f) + S(r,f). \tag{5.16}$$

From (5.13), (5.14) and the hypothesis, we have

$$10T(r,f) \leq \frac{9}{1-v}T(r,f) + 19uT(r,f) + S(r,f),$$

which implies that

$$(10 - \frac{9}{1-v} - 19u)T(r,f) \leq S(r,f).$$

This is a contradiction, because

$$\begin{aligned}
10 - \frac{9}{1-v} - 19u &> 10 - \frac{9}{1-v} - \frac{19(1-10v)}{19(1-v)} \\
&= \frac{10(1-v) - 9 - (1-10v)}{1-v} \\
&= 0.
\end{aligned}$$

Hence, $f = g$ in this case.

Case 3. Suppose that $\overline{N}(r,g) \neq S(r,g)$ and there exist u and v satisfying $0 \le u < \frac{1}{10}$ and $0 \le v < \frac{1-10u}{19(1-u)}$ such that

$$\overline{N}(r,f) \le uT(r,f) + S(r,f) \quad \textit{and} \quad \overline{N}(r,g) \le vT(r,g) + S(r,g).$$

As in Case 2, we obtain the inequality

$$10T(r,g) \le 9T(r,f) + 19\overline{N}(r,g) + S(r,g). \tag{5.17}$$

From (5.12), (5.14) and the hypothesis, we have

$$10T(r,g) \le \frac{9}{1-u}T(r,g) + 19vT(r,g) + S(r,g),$$

which implies that

$$(10 - \frac{9}{1-u} - 19v)T(r,g) \le S(r,g).$$

Since

$$\begin{aligned} 10 - \frac{9}{1-u} - 19v &> 10 - \frac{9}{1-u} - \frac{19(1-10u)}{19(1-u)} \\ &= \frac{10(1-u) - 9 - (1-10u)}{1-u} \\ &= 0. \end{aligned}$$

We get a contradiction. Therefore, $f = g$ in this case.

Case 4. Suppose that $\overline{N}(r,f) \neq S(r,f)$, $\overline{N}(r,g) \neq S(r,g)$ and there exist u and v satisfying either

(i) $0 \le u < \frac{1}{10}$ *and* $0 \le v < \frac{1-10u}{19(1-u)}$ or

(ii) $0 \le v < \frac{1}{10}$ *and* $0 \le u < \frac{1-10v}{19(1-v)}$,

such that

$$\overline{N}(r,f) \le uT(r,f) + S(r,f) \quad \textit{and} \quad \overline{N}(r,g) \le vT(r,g) + S(r,g).$$

Now, adding (5.16) and (5.17), and using the hypothesis, we obtain

$$(1-19u)T(r,f)+(1-19v)T(r,g)\le S(r,f)+S(r,g). \tag{5.18}$$

Subcase 4-1. When $1-19u\ge 0$, and (u,v) satisfies (i).

Since

$$1-19v>1-\frac{19(1-10u)}{19(1-u)}=1-\frac{1-10u}{1-u}=\frac{9u}{1-u}>0,$$

(5.18) gives a contradiction.

Subcase 4-2. When $1-19u<0$, and (u,v) satisfies (i).

From (5.18), we obtain

$$(1-19u)T(r,f)+(1-19v)T(r,g)\le S(r,f)+S(r,g).$$

Multiplying (5.18) by $(1-u)$ to get

$$(1-19u)(1-u)T(r,f)+(1-19v)(1-u)T(r,g)\le S(r,f)+S(r,g).$$

By (5.12), we have

$$(1-19u)T(r,g)+(1-19v)(1-u)T(r,g)\le S(r,g),$$

which implies that

$$\{(1-19u)+(1-19v)(1-u)\}T(r,g)\le S(r,g).$$

Since

$$\begin{aligned}(1-19u)+(1-19v)(1-u)&>(1-19u)+[1-\frac{19(1-10u)}{19(1-u)}](1-u)\\&=(1-19u)+(1-u)-(1-10u)\\&=1-10u\\&>0,\end{aligned}$$

it is also a contradiction.

Subcase 4-3. When $1 - 19v \geq 0$, and (u, v) satisfies (ii).

Since

$$1 - 19u > 1 - \frac{19(1 - 10v)}{19(1 - v)} = 1 - \frac{1 - 10v}{1 - v} = \frac{9v}{1 - v} > 0,$$

(5.18) gives a contradiction.

Subcase 4-4. When $1 - 19v < 0$, and (u, v) satisfies (ii).

From (5.18), we obtain

$$(1 - 19u)T(r, f) + (1 - 19v)T(r, g) \leq S(r, f) + S(r, g).$$

Multiplying (5.18) by $(1 - v)$ to get

$$(1 - 19u)(1 - v)T(r, f) + (1 - 19v)(1 - v)T(r, g) \leq S(r, f) + S(r, g).$$

By (5.13), we have

$$(1 - 19u)(1 - v)T(r, f) + (1 - 19v)T(r, f) \leq S(r, f),$$

which implies that

$$\{(1 - 19u)(1 - v) + (1 - 19v)\}T(r, f) \leq S(r, f).$$

Since

$$\begin{aligned}(1 - 19u)(1 - v) + (1 - 19v) &> [1 - \frac{19(1 - 10v)}{19(1 - v)}](1 - v) + (1 - 19v) \\ &= (1 - v) - (1 - 10v) + (1 - 19v) \\ &= 1 - 10v \\ &> 0,\end{aligned}$$

it is also a contradiction.

Therefore, in any case, we conclude that $f = g$. ❑

Theorem 5.10 *Let f and g be transcendental meromorphic functions in the complex plane. Suppose that there are four distinct elements $a_1, a_2, a_3, a_4 \in \mathbf{S}(f) \cap \mathbf{S}(g)$ satisfying*

$$E(f = a_j) = E(g = a_j), \quad 1 \leq j \leq 4.$$

Define the quantity ρ by

$$\rho = \max\{ \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,f)}{T(r,f)}, \quad \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,g)}{T(r,g)} \}.$$

If $0 \leq \rho < \frac{1}{19}$, then $f = g$.

Proof. If $\rho = 0$, then

$$0 = \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,f)}{T(r,f)} \quad \text{and} \quad 0 = \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,g)}{T(r,g)}.$$

In particular,

$$\overline{N}(r,f) = o(T(r,f)) \quad \text{and} \quad \overline{N}(r,g) = o(T(r,g)).$$

which implies that

$$\overline{N}(r,f) = S(r,f) \quad \text{and} \quad \overline{N}(r,g) = S(r,g).$$

Then the result follows from Theorem 5.9 (a), we get $f = g$.

Now, we deal with $\rho > 0$. Assume that $f \neq g$. Given $\varepsilon > 0$, such that $0 < \rho + \varepsilon < \frac{1}{19}$. Then, by definition of ρ, we have

$$\overline{N}(r,f) < (\rho+\varepsilon)T(r,f) \quad \text{and} \quad \overline{N}(r,g) < (\rho+\varepsilon)T(r,g) \tag{5.19}$$

for r large enough. Also, since $\rho > 0$, we have either

$$\overline{N}(r,f) \neq S(r,f) \quad \text{or} \quad \overline{N}(r,g) \neq S(r,g).$$

Now, we divide the proof into three cases

(a) $\overline{N}(r,f) \neq S(r,f)$, $\overline{N}(r,g) = S(r,g)$;

(b) $\overline{N}(r,f) = S(r,f)$, $\overline{N}(r,g) \neq S(r,g)$;

(c) $\overline{N}(r,f) \neq S(r,f)$, $\overline{N}(r,g) \neq S(r,g)$.

Case 1. Since $\overline{N}(r,f) \neq S(r,f)$ as in (5.16), we get

$$10T(r,f) \leq 9T(r,g) + 19\overline{N}(r,f) + S(r,f), \tag{5.20}$$

Apply Lemma 5.2 to $a_5 = \infty$ and $\overline{N}(r,g) = S(r,g)$, we obtain

$$\begin{aligned} 2T(r,g) &\leq \sum_{j=1}^{4} \overline{N}(r, \frac{1}{g-a_j}) + \overline{N}(r,g) + S(r,g) \\ &\leq \overline{N}(r, \frac{1}{f-g}) + S(r,g) \\ &\leq T(r,f) + T(r,g) + S(r,g), \end{aligned}$$

which implies that

$$T(r,g) \leq T(r,f) + S(r,g). \tag{5.21}$$

Therefore, by Corollary 2.27, we have $S(r,f) = S(r,g)$. Eliminating $T(r,g)$ from (5.20) and (5.21), we have

$$10T(r,f) \leq 9T(r,f) + 19\overline{N}(r,f) + S(r,f),$$

which implies that

$$T(r,f) \leq 19\overline{N}(r,f) + S(r,f).$$

By (5.19), we obtain

$$T(r,f) < 19(\rho + \varepsilon)T(r,f) + S(r,f),$$

for r large enough, which implies that

$$(1 - 19(\rho + \varepsilon))T(r,f) \leq S(r,f).$$

Since

$$1 - 19(\rho + \varepsilon) > 1 - \frac{19}{19} = 0.$$

We get a contradiction. Therefore, $f = g$ in this case.

Case 2. By the same argument as in Case 2, we conclude that $f = g$.

Case 3. Since $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$. From (5.20), we have

$$10T(r, f) \leq 9T(r, g) + 19\overline{N}(r, f) + S(r, f).$$

Similarly, we have

$$10T(r, g) \leq 9T(r, f) + 19\overline{N}(r, g) + S(r, g), \tag{5.22}$$

Adding (5.20) and (5.22), to obtain

$$T(r, f) + T(r, g) \leq 19(\overline{N}(r, f) + \overline{N}(r, g)) + S(r, f) + S(r, g).$$

By (5.19), we have

$$T(r, f) + T(r, g) \leq 19(\rho + \varepsilon)(T(r, f) + T(r, g)) + S(r, f) + S(r, g),$$

for r large enough, which implies that

$$(1 - 19(\rho + \varepsilon))(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g).$$

Since

$$1 - 19(\rho + \varepsilon) > 1 - \frac{19}{19} = 0.$$

We get a contradiction. Therefore, $f = g$ in this case.

❑