7 p-Adic Series

We all familiar with the classical results on real series, for example, in [1]. In this section, we will study the p-adic series and compare the analogue between the p-adic case and the classical case. We have defined the convergence of p-adic sequences and p-adic Cauchy sequences in the previous sections. As in the classical case, we can also define the p-adic series. We will take it for granted.

Now, given a sequence $\{a_n\}$ in $\mathbb{Q}_\infty$. In the classical case, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$, however, the converse is not true, for example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In the p-adic case, it can't be happened. In fact, we have

Theorem 7.1 *A p-adic series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$. Moreover,*

$$\left| \sum_{n=1}^{\infty} a_n \right|_p \leq \max_n |\, a_n \,|_p \, .$$

Proof. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Let

$$S_n = \sum_{k=1}^{n} a_k$$

and

$$\lim_{n\to\infty} S_n = x.$$

Then we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0.$$

To prove the converse, suppose that $\lim_{n\to\infty} a_n = 0$. Then, for every $\epsilon > 0$, there exists a positive integer N such that $|\, a_n \,|_p < \epsilon$ for any $n \geq N$. We have

$$|\, S_m - S_n \,|_p = |\, a_{n+1} + ... + a_m \,|_p \leq \max_{n+1 \leq i \leq m} |\, a_i \,|_p < \epsilon.$$

Therefore, $\{S_n\}$ is a Cauchy sequence in $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$, $\{S_n\}$ converges, i.e. , $\sum_{n=1}^{\infty} a_n$ converges.

Moreover,

$$\left| \sum_{n=1}^{\infty} a_n \right|_p = \lim_{n\to\infty} |\, S_n \,|$$
$$= \lim_{n\to\infty} |\, a_1 + ... + a_n \,|_p$$
$$\leq \lim_{n\to\infty} \max\{|\, a_1 \,|_p \,, ..., |\, a_n \,|_p\}$$
$$\leq \max_n |\, a_n \,|_p \,.$$

❑

As usual, we have some examples as follows.

Example 7.2 *Let $x \in \mathbb{Z}_p$ and $\{\alpha_n\}$ be the canonical representation of x given in Theorem 7.1. If we write $\alpha_n = \sum_{i=0}^{n} a_i p^n$, then the p-adic series $\sum_{n=0}^{\infty} \alpha_n p^n$ converges to x, i.e. $x = \sum_{n=0}^{\infty} \alpha_n p^n$.*

Example 7.3 *There exists a series $\sum_{n=1}^{\infty} a_n$ with $a_n \in \mathbb{Q}$ for all $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{Q}_p$, for all prime p. In fact, let $\{p_1, p_2, \ldots\}$ be the sequence of all primes, and define*

$$a_n = p_1^n \cdots p_n^n, \; n = 1, 2, \ldots .$$

Given an arbitrary prime p. Then we have

$$\lim_{n\to\infty} |\, a_n \,|_p = \lim_{n\to\infty} |\, p_1^n \cdots p_n^n \,|_p = \lim_{n\to\infty} p^{-n} = 0.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{Q}_p$ by Theorem 7.1.

Example 7.4 *Let p_1 and p_2 be distinct primes. Then, obviously, the series $\sum_{n=1}^{\infty} p_1^n$ converges in $\mathbb{Q}_{p_1}$ and diverges in $\mathbb{Q}_{p_2}$. Similarly, $\sum_{n=1}^{\infty} p_2^n$ converges in $\mathbb{Q}_{p_2}$ and diverges in $\mathbb{Q}_{p_1}$.*

Example 7.5 *Given an arbitrary prime p, $\sum_{n=0}^{\infty} n!$ converges in $\mathbb{Q}_p$. In fact, it is known that*

$$|\, n! \,|_p = p^{-\frac{n-S_n}{p-1}} \le p^{-\frac{n}{p-1}+(\frac{\log n}{\log p}+1)},$$

where $S_n = a_0 + \cdots + a_k$ if

$$n = a_0 + a_1 p + \cdots + a_k p^k \text{ and } 0 \le a_i \le p-1 \text{ for all } 0 \le i \le k.$$

Since

$$\lim_{n\to\infty}(\frac{n}{p-1} - \frac{\log n}{\log p} + 1) = \lim_{n\to\infty}[n(\frac{1}{p-1} - \frac{\log n}{n \log p} + \frac{1}{n})] = +\infty,$$

we obtain that

$$\lim_{n\to\infty} p^{-\frac{n}{p-1}+(\frac{log n}{log p}+1)} = 0.$$

Then

$$\lim_{n\to\infty} |\, n! \,|_p = 0,$$

and hence $\sum_{n=0}^{\infty} n!$ converges in $\mathbb{Q}_p$ by Theorem 7.1.

Theorem 7.6 (generalized Geometric Series) *The p-adic geometric series*

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \; if \; |\, x \,|_p < 1$$

.

Proof. Since, for $|\, x \,|_p < 1$, we have

$$\lim_{n\to\infty} |\, ax^n \,|_p = \lim_{n\to\infty} |\, a \,|_p \, |\, x \,|_p^n = 0,$$

so $\sum_{n=0}^{\infty} ax^n$ converges by Theorem 7.1. Let

$$S_n = \sum_{k=0}^{n} ax^k, \quad n = 1, 2, \ldots.$$

Thus $S_n - xS_n = a(1 - x^{n+1})$, and

$$S_n = a(\frac{1}{1-x} - \frac{x^{n+1}}{1-x}), \quad n = 1, 2, \ldots.$$

Then we have

$$\lim_{n\to\infty} S_n = \frac{a}{1-x}, \text{ i.e. } \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x} \text{ if } |\, x \,|_p < 1.$$

❏

Theorem 7.7 *Let $\sum_{n=0}^{\infty} a_n$ be a p-adic series. If $\sum_{n=0}^{\infty} a_n$ converges, then so is $\sum_{n=0}^{\infty}(-1)^{n-1}a_n$.*

Proof. Since $\sum_{n=0}^{\infty} a_n$ converges, by Theorem 7.1, $\lim_{n\to\infty} a_n = 0$ which implies that $\lim_{n\to\infty}(-1)^{n-1}a_n = 0$. Again, by Theorem 7.1, $\sum_{n=0}^{\infty}(-1)^{n-1}a_n$ converges. ❏

Remark. Theorem 7.7 is not true in the classical case. For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges, but $\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Definition 7.8 *A series $\sum_{n=0}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=0}^{\infty} |\, a_n \,|_p$ converges. It is called conditionally convergent if $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |\, a_n \,|_p$ diverges.*

Theorem 7.9 *Absolute convergence of $\sum_{n=0}^{\infty} a_n$ implies convergence.*

Proof. Suppose that the series $\sum_{n=0}^{\infty} a_n$ absolutely convergent, i.e. $\sum_{n=0}^{\infty} |\, a_n \,|_p$ converges. Then $\lim_{n\to\infty} |\, a_n \,|_p = 0$. Hence, $\lim_{n\to\infty} a_n = 0$. By Theorem 7.1, $\sum_{n=0}^{\infty} a_n$ converges. ❏

In the classical case, there exist series which converges conditionally, for example, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally. Similarly, the following example shows that, in the p-adic case, there exists a series which converges conditionally.

Example 7.10 *Consider the following consecutive terms of the series:* 1; *p repeated p times; p^2 repeated p^2 times; etc. These terms tend to* 0*, hence the series converges. However,*

$$\sum_{n=0}^{\infty} |\, a_n \,|_p = 1 + p \cdot p^{(-1)} + p^2 \cdot p^{(-2)} + \cdots = \infty.$$

Definition 7.11 *Let $\sum_{n=0}^{\infty} a_n$ be a p-adic series and σ be a bijective mapping. The series $\sum_{n=0}^{\infty} a_{\sigma(n)}$ is called a rearrangement of $\sum_{n=0}^{\infty} a_n$.*

Clearly, if $\sum_{n=0}^{\infty} a_{\sigma(n)}$ is a rearrangement of $\sum_{n=0}^{\infty} a_n$, then $\sum_{n=0}^{\infty} a_n$ is also a rearrangement of $\sum_{n=0}^{\infty} a_{\sigma(n)}$.

As in the classical case, we have

Theorem 7.12 *Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series having sum x. Then every rearrangement of $\sum_{n=0}^{\infty} a_n$ also converges absolutely and has sum x.*

However, in the p-adic case, we have the following result.

Theorem 7.13 *Let $\sum_{n=0}^{\infty} a_n$ be a p-adic series. Then $\sum_{n=0}^{\infty} a_n$ converges with sum x if and only if every rearrangement of $\sum_{n=0}^{\infty} a_n$ converges with the same sum x.*

Proof. Suppose that $\sum_{n=0}^{\infty} a_n = x$ and $\sum_{n=0}^{\infty} a_n^{'}$ is an rearrangement of $\sum_{n=0}^{\infty} a_n$. By Theorem 7.1, $\sum_{n=0}^{\infty} a_n^{'}$ converges, so it suffices to show that $\sum_{n=0}^{\infty} a_n^{'} = x$. For any positive number ϵ, there is a positive integer N such that for all $n \geq N$,

$$|\, a_n \,|_p < \epsilon, \quad \left| a_n^{'} \right|_p < \epsilon, \text{ and } \left| \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N} a_n \right|_p < \epsilon.$$

Put $x_1 = \sum_{n=0}^{N} a_n$ and $x_1^{'} = \sum_{n=0}^{N} a_n^{'}$, and denote by x_2 and $x_2^{'}$, respectively, the sums of all terms in $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} a_n^{'}$ for which $|\, a_n \,|_p \geq \epsilon$ and $\left| a_n^{'} \right|_p \geq \epsilon$. It is clear that x_2 and $x_2^{'}$ have the same terms, hence $x_2 = x_2^{'}$. The sum x_1 differs from

x_2 by the terms satisfying $|\, a_n \,|_p < \epsilon$, and the sum $x_1^{'}$ differs from $x_2^{'}$ by the terms satisfying $\left|\, a_n^{'} \,\right|_p < \epsilon$. Therefore,

$$|\, x_1 - x_2 \,|_p < \epsilon \text{ and } \left|\, x_1^{'} - x_2^{'} \,\right|_p < \epsilon,$$

which implies that

$$\left|\, x_1 - x_1^{'} \,\right|_p < \epsilon.$$

Combining this with

$$\left|\sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N} a_n\right|_p < \epsilon,$$

we obtain

$$\left|\sum_{n=0}^{\infty} a_n - \sum_{n=0}^{N} a_n^{'}\right|_p < \epsilon.$$

Since N can be arbitrary large, we see that the series $\sum_{n=0}^{\infty} a_n^{'}$ converges and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n^{'} = x.$$

The other direction is obvious. ❑

Theorem 7.14 *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two p-adic series. Suppose that there exist positive constant* c *and* $N \in \mathbb{N}$ *such that* $|\, a_n \,|_p \le c \,|\, b_n \,|_p$ *for* $n \ge N$. *If* $\sum_{n=0}^{\infty} b_n$ *converges absolutely, then so is* $\sum_{n=0}^{\infty} a_n$.

Proof. By the comparison test, $\sum_{n=0}^{\infty} |\, a_n \,|_p$ converges. Hence, $\sum_{n=0}^{\infty} a_n$ converges absolutely. ❑

Theorem 7.15 *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two p-adic series, and suppose that* $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. *Then* $\sum_{n=0}^{\infty} a_n$ *absolutely converges if , and only if,* $\sum_{n=0}^{\infty} b_n$ *absolutely converges.*

Proof. There exists $N \in \mathbb{N}$ such that for all $n \ge N$, $\left| \frac{a_n}{b_n} - 1 \right|_p < \frac{1}{2}$. Therefore, for all $n \ge N$, we have

$$\frac{1}{2} < \left| \frac{a_n}{b_n} \right|_p < \frac{3}{2}.$$

Thus, for all $n \geq N$,

$$\frac{1}{2}\,|\,b_n\,|_p < |\,a_n\,|_p < \frac{3}{2}\,|\,b_n\,|_p\,,$$

and the theorem follows from Theorem 7.14. ❑

Remark. Theorem 7.15 also holds if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, provided that $c \neq 0$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, we can only conclude that absolute convergence of $\sum_{n=0}^{\infty} b_n$ implies the absolute convergence of $\sum_{n=0}^{\infty} a_n$.

In real analysis, we have the Dirichlet's test and Abel's test for series. We can generalize these two tests as follows.

Corollary 7.16 *Let $\sum_{n=0}^{\infty} a_n$ be a p-adic series whose partial sum form a bounded sequence and $\{b_n\}$ be a sequence in $\mathbb{Q}_p$ which converges to 0. Then $\sum_{n=1}^{\infty} a_n b_n$ converges to 0.*

***Proof*.** Let $A_n = a_1 + \cdots + a_n$ and assume that $|\,A_n\,|_p \leq M$ for all n. For all $n \in \mathbb{N}$,

$$\begin{aligned} |\,a_n b_n\,|_p &= |\,(A_n - A_{n-1})b_n\,|_p \\ &= |\,A_n - A_{n-1}\,|_p\,|\,b_n\,|_p \\ &\leq M\,|\,b_n\,|_p\,. \end{aligned}$$

Hence, $\lim_{n\to\infty} b_n = 0$ implies $\lim_{n\to\infty}(a_n b_n) = 0$. By Theorem 7.1, $\sum_{n=0}^{\infty} a_n b_n$ converges. ❑

Theorem 7.17 *The series $\sum_{n=1}^{\infty} a_n b_n$ converges to 0 if $\sum_{n=0}^{\infty} a_n$ converges and if $\{b_n\}$ is a sequence in $\mathbb{Q}_p$ which converges to 0.*

***Proof*.** Convergence of $\sum_{n=0}^{\infty} a_n$ implies $\{A_n\}$ is a bounded sequence, where $A_n = a_1 + \cdots + a_n$. Applying the preceding corollary, we complete the proof. ❑