2 General Theory of Valuations

In this section, we will define valuations on a field and study its basic algebraic and topological properties. Most of the materials can be found in [2,3,7,8,9]. Furthermore, a general procedure to produce valuations will be introduced and some important classes of examples will be given.

From now on, all rings in this thesis are commutative with identity 1.

Definition 2.1 *Let k be a field. A valuation on k is a function $|\cdot| : k \to \mathbb{R}$ satisfying the following conditions:*

(i) *For all $x \in k$,*

$$|x| \geq 0, \text{ and } |x| = 0 \text{ if and only if } x = 0.$$

(ii) *For all $x, y \in k$, $|xy| = |x||y|$.*

(iii) *$|\cdot|$ satisfies the triangle inequality: For all $x, y \in k$,*

$$|x+y| \leq |x| + |y|.$$

In this case, the pair $(k, |\cdot|)$ (or simply k) is called a valuated field. If, in addition, $|\cdot|$ satisfies the strong triangle inequality

$$|x+y| \leq \max\{|x|, |y|\}$$

for all $x, y \in k$, then $|\cdot|$ is called a non-Archimedean valuation on k and k is a non-Archimedean valuated field. Otherwise, $|\cdot|$ is called an Archimedean valuation and k is an Archimedean valuated field.

Note that the strong triangle inequality always implies the triangle inequality, but not converse as we will see.

Example 2.2 *Let k be a field and $|\cdot|_0 : k \longrightarrow \mathbb{R}$ be defined by*

$$|x|_0 = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then, obviously, $|\cdot|_0$ is a valuation on k, called the trivial valuation, which shows that every field admits at least one valuation.

Example 2.3 *Let $|\cdot|_\infty$ be the ordinary absolute value on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Then $|\cdot|_\infty$ is an Archimedean valuation on these fields, but not satisfies the strong triangle inequality, since $|1+1|_\infty = 2$ and $\max\{|1|_\infty, |1|_\infty\} = 1$.*

The following simple properties about valuations on a field k can be easily deduced from the definition.

Proposition 2.4 *Let $(k, |\cdot|)$ be a valued field. Then we have*

(i) $|1| = 1$.

(ii) $|-x| = |x|$ *for all* $x \in k$.

(iii) $|x^{-1}| = |x|^{-1}$ *for all* $x \in k$ *and* $x \neq 0$.

(iv) $||x| - |y|| \leq |x - y|$ *for all* $x, y \in k$.

Let $|\cdot|$ be a valuation on a field k and $k^* = k - \{0\}$. Denote the set $\{|x| \mid x \in k^*\}$ by $|k^*|$. Then, by definition of valuations, $|\cdot| : k^* \longrightarrow \mathbb{R}^*$ is a multiplicative group homomorphism, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. Therefore, the following proposition is obvious.

Proposition 2.5 *$|k^*|$ is a multiplicative subgroup of $\mathbb{R}^*$.*

Definition 2.6 *$|k^*|$ is called the value group of $|\cdot|$. Note that the image of $|\cdot|$ is $|k| = |k^*| \cup \{0\}$.*

For a non-Archimedean valuated field $(k, |\cdot|)$, we have some other important properties which are different from the Archimedean case.

Proposition 2.7 *Let $(k, |\cdot|)$ be a non-Archimedean valuated field. Then we have*

(i) $|\, n \cdot 1\,| \leq 1$ *for all* $n \in \mathbb{Z}$.

(ii) *Given* $x, y \in k$, *if* $|\,x\,| \neq |\,y\,|$, *then* $|\,x + y\,| = \max\{|\,x\,|, |\,y\,|\}$.

This is usually called the isosceles triangle property.

Proof. (i) follows from the strong triangle inequality of non-Archimedean valuations and induction. Now, to prove (ii), given $x, y \in k$ with $|\,x\,| \neq |\,y\,|$. We may assume that $|\,x\,| > |\,y\,|$. Then

$$\begin{aligned} |\,x\,| &= |\,x + y - y\,| \\ &\leq \max\{|\,x + y\,|, |-y|\} \\ &= \max\{|\,x + y\,|, |\,y\,|\} \text{ by (ii) of Proposition 2.4.} \\ &= |\,x + y\,| \text{ since } |\,y\,| < |\,x\,|, \end{aligned}$$

which proves (ii). ❑

Note that (ii) and its proof say that every triangle in k is isosceles and the length of base is less than or equal to the length of sides.

In order to derive some further properties of non-Archimedean field, we introduce some notation.

Let $(k, |\cdot|)$ be a non-Archimedean valuated field. Define

$$V = \{x \in k |\, |\,x\,| \leq 1\},$$

$$P = \{x \in k |\, |\,x\,| < 1\},$$

and

$$U = \{x \in k |\, |\,x\,| = 1\}.$$

Proposition 2.8 *As above, we have*

(i) *V is a subring of k, called the valuation ring of k. In particular, V is itself an integral domain.*

(ii) *P is the only maximal ideal in V. In particular, V is a local ring.*

(iii) *U is the group of units in V, called the group of units.*

***Proof*.**

(i) Given $x, y \in V$, we have

$$|\,x - y\,| \leq \max\{|\,x\,|, |\,y\,|\} \leq 1,$$

$$|\,xy\,| = |\,x\,|\,|\,y\,| \leq 1$$

and

$$|\,1\,| = 1,$$

which imply that V is a subring of k. Moreover, since $V \subseteq k$ and k is a field, V is an integral domain.

(ii) Given $x, y \in P$ and $a \in V$, we have

$$|\,x - y\,| \leq \max\{|\,x\,|, |\,y\,|\} < 1$$

and

$$|\,ax\,| = |\,a\,|\,|\,x\,| < 1$$

which imply that P is an ideal in V. Furthermore, if $x \in V$ and $x \notin P$, then $|\,x\,| = 1$. Therefore, $|\,x^{-1}\,| = |\,x\,|^{-1} = 1$ and $x^{-1} \in V$ which say that P is a maximal ideal and $V - P$ contains all units of V. In particular, P is the only maximal ideal in V.

(iii) As in the proof of (ii), U is the group of units in V.

$\square$

Now, given a valuation $|\cdot|$ on a field k, if $|\cdot|$ is non-Archimedean, then the set $\{n \cdot 1 | n \in \mathbb{Z}\}$ is bounded by (i) of Proposition 2.7, that is, $|\, n \cdot 1\,| \leq 1$ for all $n \in \mathbb{Z}$, and, for the ordinary absolute value $|\cdot|_\infty$ on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, the set $\{n \cdot 1 | n \in \mathbb{Z}\}$ is unbounded. In fact, this property characterizes valuations on a field k.

In order to make it clear, we may identify the ring of integers $\mathbb{Z}$ as a subring of k through the mapping, each $n \in \mathbb{Z}$ sends to $n \cdot 1 \in k$. Note that if k is of characteristic 0, then the mapping is injective and $\mathbb{Z}$ can be considered as a subring of k; if k is of characteristic p, where p is a prime integer, then the mapping is not injective and we may consider $\mathbb{F}_p$, the residue class field modulo p, as a subring of k.

Theorem 2.9 *Let $|\cdot|$ be a valuation on a field k. Then $|\cdot|$ is non-Archimedean if and only if $\mathbb{Z}$ is bounded with respect to $|\cdot|$. Equivalently, $|\cdot|$ is Archimedean if and only if $\mathbb{Z}$ is unbounded with respect to $|\cdot|$.*

***Proof*.** Suppose that $|\cdot|$ is non-Archimedean. Then $\mathbb{Z}$ is bounded by (i) of Proposition 2.7. To prove the converse, suppose that $\mathbb{Z}$ is bounded, say, $|\, n\,| \leq M$ for all $n \in \mathbb{Z}$. Given $x, y \in k$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
|\, x + y\,|^n &= |\,(x+y)^n\,| \\
&= \left| \sum_{i=0}^{n} \binom{n}{i} x^n y^{n-i} \right| \\
&\leq \sum_{i=0}^{n} \left| \binom{n}{i} \right| |\, x\,|^n\, |\, y\,|^{n-i} \\
&\leq M \sum_{i=0}^{n} |\, x\,|^n\, |\, y\,|^{n-i} \\
&\leq M \sum_{i=0}^{n} [\max\{|\, x\,|, |\, y\,|\}]^n \\
&= M(n+1)[\max\{|\, x\,|, |\, y\,|\}]^n,
\end{aligned}$$

which implies that, for all $n \in \mathbb{N}$,

$$|\,x+y\,| \le M^{\frac{1}{n}}(n+1)^{\frac{1}{n}} \max\{|\,x\,|, |\,y\,|\}.$$

Let $n \to \infty$ and use

$$\lim_{n\to\infty} M^{\frac{1}{n}} = 1 \text{ and } \lim_{n\to\infty}(n+1)^{\frac{1}{n}} = 1,$$

We get

$$|\,x+y\,| \le \max\{|\,x\,|, |\,y\,|\}$$

which shows that $|\cdot|$ is non-Archimedean on k. ❑

Remark. From Theorem 2.9, if $|\cdot|$ is Archimedean on k, then, given $a \in k^*$ and $b \in k$, there exist $n \in \mathbb{Z}$ such that $|\,na\,| > |\,b\,|$. Apply it to the ordinary absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ or $\mathbb{R}$. We have the usual Archimedean property in analysis, namely, given two numbers $a > 0$ and b, there exists an integer n such that $na > b$.

Now, we develop a general procedure to produce non-Archimedean valuations on some well-known fields, for example the rational number field $\mathbb{Q}$ and the rational function field $k(x)$ over a field k, and then give three classes of important valuations.

Definition 2.10 *Let A be an integral domain. An order function on A is a function*

$$\text{ord} : A - \{0\} \longrightarrow \mathbb{Z}$$

satisfying:

(i) *For all $a, b \in A - \{0\}$, $\text{ord}(ab) = \text{ord}\, a + \text{ord}\, b$.*

(ii) *For all $a, b \in A - \{0\}$, $\text{ord}(a+b) \ge \min\{\text{ord}\, a, \text{ord}\, b\}$.*

In this case, $\text{ord}(a)$ *is called the order of a.*

For convenience, we define ord 0 to be $+\infty$ and $c^{\text{ord}\,0} = 0$ if $0 \le c < 1$ which will be used without mention. Note that under this convention, (i) and(ii), in the definition, still hold.

Let A be an integral domain and $k = \{\frac{a}{b} | a, b \in A, b \neq 0\}$ be its quotient field. For example, $\mathbb{Q}$ is the quotient field of $\mathbb{Z}$ and $k(x)$ is the quotient field of $k[x]$. The following theorem says that every order function on A induces a valuation on k.

Theorem 2.11 *Let* ord *be an order function in an integral domain A with quotient field k. Then it extends to an order function on k, still denoted by* ord*, in the following manner:*

$$\textit{Given } x \in k^*,\ \mathrm{ord}\, x = \mathrm{ord}\, a - \mathrm{ord}\, b \; if\ x = \frac{a}{b}.$$

Proof. First, we show that the definition is well-defined. Suppose that $x = \frac{a}{b}$ and $x = \frac{c}{d}$. Then $ad = bc$. By definition,

$$\mathrm{ord}\, a + \mathrm{ord}\, d = \mathrm{ord}\, b + \mathrm{ord}\, c$$

which implies that

$$\mathrm{ord}\, a - \mathrm{ord}\, b = \mathrm{ord}\, c - \mathrm{ord}\, d.$$

Therefore, ord x is well-defined.

Clearly, it extends the original order function on A. Finally, to show that *ord* is an order function on k. Given $x = \frac{a}{b}$ and $y = \frac{c}{d}$ in k^*, we have

$$\begin{aligned}
\mathrm{ord}(xy) &= \mathrm{ord}(\frac{ac}{bd}) \\
&= \mathrm{ord}(ac) - \mathrm{ord}(bd) \\
&= \mathrm{ord}\, a + \mathrm{ord}\, c - \mathrm{ord}\, b - \mathrm{ord}\, d \\
&= \mathrm{ord}\, a - \mathrm{ord}\, b + \mathrm{ord}\, c - \mathrm{ord}\, d \\
&= \mathrm{ord}(\frac{a}{b}) + \mathrm{ord}(\frac{c}{d}) \\
&= \mathrm{ord}\, x + \mathrm{ord}\, y,
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{ord}(x+y) &= \operatorname{ord}(\frac{a}{b}+\frac{c}{d}) \\
&= \operatorname{ord}(\frac{ad+bc}{bd}) \\
&= \operatorname{ord}(ad+bc) - \operatorname{ord}(bd) \\
&\geq \min\{\operatorname{ord}(ad), \operatorname{ord}(bc)\} - \operatorname{ord} b - \operatorname{ord} d \\
&= \min\{\operatorname{ord} a + \operatorname{ord} d, \operatorname{ord} b + \operatorname{ord} c\} - \operatorname{ord} b - \operatorname{ord} d \\
&= \min\{\operatorname{ord} a - \operatorname{ord} b, \operatorname{ord} c - \operatorname{ord} d\} \\
&= \min\{\operatorname{ord}(\frac{a}{b}), \operatorname{ord}(\frac{c}{d})\} \\
&= \min\{\operatorname{ord} x, \operatorname{ord} y\}.
\end{aligned}$$

Therefore, by definition, ord is an order function on k. ❑

Again, we take the convention that $\operatorname{ord} 0 = +\infty$.

Theorem 2.12 *As in Theorem 2.11, given $c \geq 1$, the function*

$$|\cdot| : k \longrightarrow \mathbb{R}$$

defined by

$$|x| = \begin{cases} c^{-\operatorname{ord} x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a non-Archimedean valuation on k.

Proof. If $c = 1$, then $|\cdot| = |\cdot|_0$ is the trivial valuation on k. Now, we assume that $c > 1$. To show that $|\cdot|$ is a non-Archimedean valuation on k.

(i) Clearly, $|x| \geq 0$ for all $x \in k$, and $|x| = 0$ if and only if $x = 0$.

(ii) Given $x, y \in k$,

$$\begin{aligned} |\,xy\,| &= c^{-\mathrm{ord}(xy)} \\ &= c^{-\mathrm{ord}\,x - \mathrm{ord}\,y} \text{ by Theorem 2.11} \\ &= c^{-\mathrm{ord}\,x} \cdot c^{-\mathrm{ord}\,y} \\ &= |\,x\,|\,|\,y\,|. \end{aligned}$$

(iii) Given $x, y \in k$,

$$\begin{aligned} |\,x+y\,| &= c^{-\mathrm{ord}(x+y)} \\ &\le c^{-\min\{\mathrm{ord}\,x, \mathrm{ord}\,y\}} \text{ by Theorem 2.11} \\ &= \max\{c^{-\mathrm{ord}\,x}, c^{-\mathrm{ord}\,y}\} \\ &= \max\{|\,x\,|, |\,y\,|\}. \end{aligned}$$

Hence, $|\cdot|$ is a non-Archimedean valuation on k. ❑

Note that if we emphasize the valuation in Theorem 2.12, then we will denote it by $|\cdot|_c$.

Corollary 2.13 *As in Theorem 2.12, if $c > 1$ and $d > 1$, then there exists $\lambda > 0$ such that, for all $x \in k$,*

$$|\,x\,|_c = |\,x\,|_d^{\lambda}.$$

Proof. Given $x \in k^*$, we have

$$|\,x\,|_c = c^{-\mathrm{ord}\,x} \text{ and } |\,x\,|_d = d^{-\mathrm{ord}\,x},$$

so

$$\log|\,x\,|_c = -(\mathrm{ord}\,x)\log c$$

and

$$\log|\,x\,|_d = -(\mathrm{ord}\,x)\log d.$$

Therefore,

$$\frac{\log|\,x\,|_c}{\log c} = \frac{\log|\,x\,|_d}{\log d},$$

that is,

$$\log|\,x\,|_c = \log|\,x\,|_d \cdot \frac{\log c}{\log d}.$$

Set

$$\lambda = \frac{\log c}{\log d}.$$

Then $\lambda > 0$ and

$$\log|\,x\,|_c = \log|\,x\,|_d^{\lambda}.$$

Hence,

$$|\,x\,|_c = |\,x\,|_d^{\lambda}$$

which is true for all $x \in k$. ❑

Remark. The relation $|\cdot|_c = |\cdot|_d^{\lambda}$ in Corollary 2.13 will be clear in section 4. In fact, $|\cdot|_c$ and $|\cdot|_d$ are equivalent. Therefore, the choice of $c > 1$ is not important and, usually, we will take $c = e$ in most abstract cases.

Now, the question becomes the following: What kind of integral domain admitting an order function?

It is well-known in basic abstract algebra [4,5] that every Euclidean domain is a principal ideal domain, and every principal ideal domain is a unique factorization domain. Moreover, the ring of integers $\mathbb{Z}$ with absolute value and the polynomial ring $k[x]$ with degree function are known to be Euclidean domains.

Given a unique factorization domain A and fixed an irreducible element $p \in A$, define

$$\mathrm{ord}_p : A - \{0\} \longrightarrow \mathbb{Z}$$

in the following manner: Given $x \in A - \{0\}$, let $n \in \mathbb{Z}$ be the largest integer such that $p^n | x$, that is, $x = p^n x'$, where p and x' are relatively prime in A. Then

$$\mathrm{ord}_p\, x = n.$$

Equivalently, $x = p^{\text{ord}_p\, x} \cdot x'$, $(p, x') = 1$.

Theorem 2.14 *Let A be a unique factorization domain. Then the function* ord_p *defined above is an order function on A.*

Proof. Clearly, $\text{ord}_p : A - \{0\} \longrightarrow \mathbb{Z}$. Given $x, y \in A - \{0\}$, write

$$x = p^{\text{ord}_p\, x} \cdot x' \textit{ and } y = p^{\text{ord}_p\, y} \cdot y',$$

where $(p, x') = 1$ and $(p, y') = 1$. We have

$$xy = p^{\text{ord}_p\, x + \text{ord}_p\, y} \cdot (x'y')$$

and $(p, x'y') = 1$. So

$$\text{ord}_p(xy) = \text{ord}_p\, x + \text{ord}_p\, y.$$

Finally, we assume that $\text{ord}_p\, x \leq \text{ord}_p\, y$. Then

$$\begin{aligned} x + y &= p^{\text{ord}_p\, x} x' + p^{\text{ord}_p\, y} y' \\ &= p^{\text{ord}_p\, x} \{x' + p^{\text{ord}_p\, y - \text{ord}_p\, x} y'\} \end{aligned}$$

which implies that

$$\text{ord}_p(x + y) \geq \text{ord}_p\, x = \min\{\text{ord}_p\, x, \text{ord}_p\, y\}.$$

Therefore, ord_p is an order function on A. ❑

Corollary 2.15 *Let A be a unique factorization domain with quotient field k. If* ord_p *is defined as above, then it extends to an order function on k.*

Proof. It follows from Theorem 2.11. ❑

Corollary 2.16 *As in Corollary 2.15, given $c \geq 1$, the function*

$$|\cdot|_p : k \longrightarrow \mathbb{R}$$

defined by

$$|\,x\,|_p = \begin{cases} c^{-\mathrm{ord}_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

defines a non-Archimedean valuation on k.

Proof. It follows from Theorem 2.12. ❑

Now, we can exhibit the main examples in the theory of valuations.

Example 2.17 *Consider the ring of integer $\mathbb{Z}$ which is a unique factorization domain. Given a prime integer p, we have the order function ord_p on $\mathbb{Z}$ which can be extended to an order function on its quotient field $\mathbb{Q}$ to obtain a non-Archimedean valuation $|\cdot|_p$ on $\mathbb{Q}$. By the remark to Corollary 2.13, we may take $c = p$ in this case. More precisely,*

$$|\,x\,|_p = \begin{cases} p^{-\mathrm{ord}_p x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

which is a non-trivial non-Archimedean valuation on $\mathbb{Q}$.

Definition 2.18 *Given a prime integer p, $|\cdot|_p$ is called the p-adic valuation on $\mathbb{Q}$.*

Example 2.19 *The following table shows some p-adic values of rational number with respect to different prime integers.*

$p\backslash\mathbb{Q}$	-1	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{10}{11}$	15	66	77
2	1	2^{-1}	2^{-2}	2^{-1}	2^{-1}	1	2^{-1}	1
3	1	3^1	1	3^{-1}	1	3^{-1}	3^{-1}	1
5	1	1	5^1	1	5^{-1}	5^{-1}	1	1
7	1	1	1	7^1	1	1	1	7^{-1}
11	1	1	1	1	11^1	1	11^{-1}	11^{-1}

It follows from the definition, the valuation ring of $|\cdot|_p$ on $\mathbb{Q}$ is

$$V = \{x \in \mathbb{Q} |\ x = \frac{a}{b},\ a, b \in \mathbb{Z},\ b \neq 0,\ (a, b) = 1 \text{ and } p \nmid b\}.$$

which contains $\mathbb{Z}$. Also, the maximal ideal in $\mathbb{Z}_p$ is

$$P = \{x \in \mathbb{Z}_p |\ x = \frac{a}{b},\ (a, b) = 1 \text{ and } p|a\},$$

which contains $\mathbb{Z}$, the group of units in $\mathbb{Z}_p$ is

$$U = \{x \in \mathbb{Z}_p |\ x = \frac{a}{b},\ (a, b) = 1 \text{ and } p \nmid ab\},$$

and the value group is

$$|\,\mathbb{Q}^*\,|_p = \{p^n |\ n \in \mathbb{Z}\}.$$

Example 2.20 *Consider the polynomial ring $k[x]$ over a field k which is a unique factorization domain. Given an irreducible polynomial $p(x) \in k[x]$, denote it by $p = p(x)$. We have the order function* ord_p *on $k[x]$ which can be extended to an order function on its quotient field $k(x)$, the field of rational functions over k, to obtain a non-Archimedean valuation $|\cdot|_p$ on $k(x)$. In this case, we may take $c = e$. More precisely,*

$$|\,f(x)\,|_p = \begin{cases} e^{-\text{ord}_p f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

which is a nontrivial non-Archimedean valuation on $k(x)$.

Example 2.21 *Consider the polynomial ring $\mathbb{C}[z]$ over the complex number field. Since $\mathbb{C}$ is algebraically closed, every irreducible polynomial is nothing but a linear polynomial. Therefore, given $a \in \mathbb{C}$, we write* ord_a *for* $\text{ord}_{(z-a)}$. *For all $f(z) \in \mathbb{C}[z] - \{0\}$,*

$$f(z) = (z - a)^{\text{ord}_a f(z)}\, h(z),$$

where $h(z) \in \mathbb{C}[z]$ and $(z - a)$ are relatively prime which means that $h(a) \neq 0$. Hence, $\text{ord}_a f(z)$ *is just the multiplicity of the zero of $f(z)$ at $z = a$. If $f(z) \in \mathbb{C}(z)$,* *then* $\operatorname{ord}_a f(z)$ *is the multiplicity of the zero (respectively, pole) of* $f(z)$ *at* $z = a$ *if* $\operatorname{ord}_a f(z) > 0$ *(respectively,* $\operatorname{ord}_a f(z) < 0$ *).*

Therefore, as the examples above, the function

$$|\,f(z)\,|_a = \begin{cases} e^{-\operatorname{ord}_a f(z)} & \text{if } f(z) \neq 0 \\ 0 & \text{if } f(z) = 0 \end{cases}$$

defines a nontrivial non-Archimedean valuation on $\mathbb{C}(z)$.

For $a = \infty$*, we define* $\operatorname{ord}_\infty$ *on* $\mathbb{C}[z] - \{0\}$ *by*

$$\operatorname{ord}_\infty f(z) = -\deg f(z),$$

where $\deg f(z)$ *is the degree of* $f(z)$*. It is easy to see that it also defines an order function on* $\mathbb{C}[z]$*, and then it extends to an order function on* $\mathbb{C}(z)$*. Therefore, the function*

$$|\,f(z)\,|_\infty = \begin{cases} e^{-\operatorname{ord}_\infty f(z)} & \text{if } f(z) \neq 0 \\ 0 & \text{if } f(z) = 0 \end{cases}$$

defines a nontrivial non-Archimedean valuation on $\mathbb{C}(z)$.

Remark. As in Example 2.19, we can also find the valuation ring, maximal ideal, the group of units and the value group in Example 2.20 and 2.21.