3 Topology of Valuated Fields

From the definition of a valuation $|\cdot|$ on a field k, it is obvious that k with $|\cdot|$ becomes a normal linear space over k itself. Therefore, all the facts of normed topology can be applied to k. In this section, we will review some basic topological facts of a valuated field which can be found in [8,9]. Also, we will discuss some new phenomena, especially, in the non-Archimedean case.

Theorem 3.1 *Let $(k, |\cdot|)$ be a valuated field. Then the function*

$$d : k \times k \longrightarrow \mathbb{R}$$

defined by

$$d(x, y) = |\, x - y \,|$$

for all $x, y \in k$, is a metric (distance function) on k, that is, d satisfies:

(i) *For all $x, y \in k$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.*

(ii) *For all $x, y \in k$, $d(x, y) = d(y, x)$.*

(iii) *For all $x, y, z \in k$, $d(x, y) \leq d(x, z) + d(z, y)$.*

If, in addition, $|\cdot|$ is a non-Archimedean valuation on k, then d satisfies the strong triangle inequality: For all $x, y, z \in k$,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Moreover, if $d(x, z) \neq d(z, y)$, then

$$d(x, y) = \max\{d(x, z), d(z, y)\}.$$

Proof. It follows from the definition of valuation and Proposition 2.7. ❑

Now, a valuated field is automatically a metric space induced by its valuation.

Example 3.2 1. *The trivial valuation* $|\cdot|_0$ *on a field* k *induces the discrete metric space on* k.

2. *The ordinary absolute value* $|\cdot|_\infty$ *on* $\mathbb{R}$ *and* $\mathbb{C}$ *induces the Euclidean topology on* $\mathbb{R}$ *and* $\mathbb{C}$*, respectively.*

3. *The ordinary absolute value* $|\cdot|_\infty$ *on* $\mathbb{Q}$ *induces the induced topology of* $\mathbb{Q}$ *in* $\mathbb{R}$.

4. *The* p*-adic topology on* $\mathbb{Q}$ *is the topology induced by the* p*-adic valuation on* $\mathbb{Q}$.

Since every valuated field is a metric space, the following definitions are the same as in the case of metric space.

Definition 3.3 *Let* $(k, |\cdot|)$ *be a valuated field. Given* $a \in k$ *and* $r > 0$.

(i) *The set* $B(a;r) = \{x \in k \mid |x - a| < r\}$ *is called the open ball with center* a *and radius* r.

(ii) *The set* $\overline{B}(a;r) = \{x \in k \mid |x - a| \le r\}$ *is called the closed ball with center* a *and radius* r.

(iii) *A sequence* $\{a_n\}$ *in* k *is said to converge if there exists* $a \in k$ *such that* $\lim_{n\to\infty} |a_n - a| = 0$*. Moreover, if* $a = 0$*, then* $\{a_n\}$ *is called a null sequence.*

(iv) *A sequence* $\{a_n\}$ *in* k *is said to be Cauchy if, for all* $\epsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that, for all* $n, m \ge N$, $|a_n - a_m| < \epsilon$.

(v) *A sequence* $\{a_n\}$ *in* k *is said to be bounded if there exists* $M > 0$ *such that* $|a_n| \le M$ *for all* $n \in \mathbb{N}$.

As in elementary analysis, if a sequence converges, then its limit is unique, and is also a Cauchy sequence. Also, every Cauchy sequence is bounded. Obviously, a Cauchy sequence may not be convergent. Furthermore, if $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then $\{a_n b_n\}$ is also a null sequence. Of course, the general operations of convergent sequences hold in this case. In the classical case, if $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$, then $\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0$ and $\{a_n\}$ converges. However, if $\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0$, then $\{a_n\}$ may not converges. For example, let $a_n = \sqrt{n}$, $n = 1, 2, \ldots$. Then

$$\begin{aligned}
\lim_{n\to\infty} |\, a_{n+1} - a_n \,| &= \lim_{n\to\infty} \left| \sqrt{n+1} - \sqrt{n} \right| \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&= 0.
\end{aligned}$$

Clearly, $\{a_n\}$ diverges.

The above example shows that the sequence $\{a_n\}$ in $\mathbb{R}$ satisfying $\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0$ is not a Cauchy sequence in $\mathbb{R}$, hence it diverges. However, in the non-Archimedean case, the situation is quite different, in fact, we have the following theorem.

Theorem 3.4 *Let $\{a_n\}$ be a sequence in a non-Archimedean valuated field k. Then $\{a_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0$.*

Proof. Suppose that $\{a_n\}$ is a Cauchy sequence. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m, n \in \mathbb{N}$,

$$|\, a_m - a_n \,| < \epsilon.$$

In particular, for all $n \geq N$, $|\, a_{n+1} - a_n \,| < \epsilon$ which implies that

$$\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0.$$

To prove the converse, suppose that $\lim_{n\to\infty} |\, a_{n+1} - a_n \,| = 0$. Then, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$|\, a_{n+1} - a_n \,| < \epsilon.$$

Then, for all $m > n \geq N$,

$$\begin{aligned} |\, a_m - a_n \,| &= \left| \sum_{i=1}^{m-1} (a_{n+i} - a_{n+i-1}) \right| \\ &\leq \max_{1 \leq i \leq m-1} |\, a_{n+i} - a_{n+i-1} \,| \\ &< \epsilon. \end{aligned}$$

So $\{a_n\}$ is a Cauchy sequence in k. ❑

Example 3.5 *Consider the p-adic valuation on* $\mathbb{Q}$.

1. $\{p^n\}$ *is a null sequence in* $\mathbb{Q}$ *with respect to* $|\cdot|_p$. *In fact,*

$$\begin{aligned} \lim_{n\to\infty} |\, p^n - 0 \,|_p &= \lim_{n\to\infty} |\, p^n \,|_p \\ &= \lim_{n\to\infty} p^{-n} \\ &= 0. \end{aligned}$$

   So $\lim_{n\to\infty} p^n = 0$.

2. *Let* $x_n = a_0 + a_1 p + \cdots + a_n p^n$, $n = 1, 2, \ldots$, *where* $0 \leq a_i \leq p-1$ *for all* $0 \leq i \leq n$. *Then* $\{x_n\}$ *is a Cauchy sequence in* $\mathbb{Q}$ *with respect to* $|\cdot|_p$. *In fact,*

$$\begin{aligned} \lim_{n\to\infty} |\, x_{n+1} - x_n \,|_p &= lim_{n\to\infty} \left|\, a_{n+1} p^{n+1} \,\right|_p \\ &= lim_{n\to\infty} \left|\, p^{n+1} \,\right|_p \\ &= lim_{n\to\infty} p^{-(n+1)} \\ &= 0. \end{aligned}$$

   By Theorem 3.4, $\{x_n\}$ *is a Cauchy sequence. We will see in section 6 that it converges in the p-adic number field* $\mathbb{Q}_p$.

3. *Let* $\{p_n\}$ *be the sequence of all prime integers, and define*

$$a_n = \prod_{i=1}^{n} p_i^n, \; n = 1, 2, \ldots$$

Then $\{a_n\}$ is a null sequence in $\mathbb{Q}$ with respect to $|\cdot|_p$. In fact, let $p = p_{n_0}$ and then $p^n | a_n$ for all $n \geq n_0$. So

$$\lim_{n\to\infty} |\, a_n - 0 \,|_p = \lim_{n\to\infty} |\, a_n \,|_p = \lim_{n\to\infty} p^{-n} = 0$$

which implies that $\{a_n\}$ is a null sequence in $\mathbb{Q}$.

In the remaining of this section, we will concentrate on the metric topology of the valuations defined in Example 2.17. Since their proofs are the same, we may only consider the p-adic valuation on $\mathbb{Q}$. The strange topological properties are due to the fact that the value group $|\,\mathbb{Q}^*\,|_p = \{p^n | n \in \mathbb{Z}\}$ is discrete.

Theorem 3.6 *Let $\{a_n\}$ be a sequence in $\mathbb{Q}$. Then we have*

(i) *If $\{a_n\}$ is a Cauchy sequence with respect to $|\cdot|_p$, then either $\lim_{n\to\infty} |\, a_n \,|_p = 0$ or there exists $n_0 \in \mathbb{N}$ such that $|\, a_n \,|_p = |\, a_{n_0} \,|$ for all $n \geq n_0$.*

(ii) *If $\lim_{n\to\infty} a_n = a$ with respect to $|\cdot|_p$, then either $a = 0$ or there exists $n_0 \in \mathbb{N}$ such that $|\, a_n \,|_p = |\, a \,|_p$ for all $n \geq n_0$.*

Proof. Suppose that $\{a_n\}_p$ is a Cauchy sequence with respect to $|\cdot|_p$. Then, by (iv) of Proposition 2.4, $\{|\, a_n \,|_p\}$ is a Cauchy sequence in $\mathbb{R}$, hence $\lim_{n\to\infty} |\, a_n \,|_p = c$. If $c = 0$, then we are done. Now, assume that $c > 0$. We have two cases:

Case 1. $c \in \{p^n |\, n \in \mathbb{Z}\}$

In this case, choose $\epsilon > 0$ small enough such that

$$(c - \epsilon, c + \epsilon) \cap \{p^n | n \in \mathbb{Z}\} = \{c\}.$$

Since $\lim_{n\to\infty} |\, a_n \,|_p = c$, there exists $n_0 \in \mathbb{N}$ such that $|\, a_n \,|_p \in (c - \epsilon, c + \epsilon)$ for all $n \geq n_0$. Since $|\, a_n \,|_p \in \{p^n | n \in \mathbb{Z}\}$ for all $n \in \mathbb{N}$, we conclude that $|\, a_n \,|_p = c$ for all $n \geq n_0$.

Case 2. $c \notin \{p^n |\ n \in \mathbb{Z}\}$

In this case, choose $\epsilon > 0$ small enough such that

$$(c - \epsilon, c + \epsilon) \cap \{p^n | n \in \mathbb{Z}\} = \phi.$$

By $\lim_{n\to\infty} |\, a_n \,|_p = c$, there exists $n_0 \in \mathbb{N}$ such that $|\, a_n \,|_p \in (c - \epsilon, c + \epsilon)$ for all $n \geq n_0$ which is impossible.

Therefore, (i) is proved, and (ii) follows from (i) and $\lim_{n\to\infty} |\, a_n \,|_p = |\, a \,|_p$ by (iv) of Proposition 2.4. ❑

Remark. The properties stated in Theorem 3.6 are different from the classical case. For example, $\lim_{n\to\infty} \frac{n}{n+1} = 1$, but $\left|\, \frac{n}{n+1} \,\right|_\infty \neq 1$ for all $n \in \mathbb{N}$.

Another strange properties say that all balls are both open and closed, and every point of a ball is the center of the ball.

Theorem 3.7 *In $\mathbb{Q}$ with the p-adic topology. We have*

(i) *Every open ball $B(a; r)$ is also closed.*

(ii) *For all $b \in B(a; r)$, $B(b; r) = B(a; r)$.*

(iii) *Every closed ball $\overline{B}(a; r)$ is also open.*

(iv) *For all $b \in \overline{B}(a; r)$, $\overline{B}(b; r) = \overline{B}(a; r)$.*

(v) *The boundary $\partial B(a; r) = \{x \in \mathbb{Q} |\, |\, x - a \,|_p = r\}$ is both open and closed in $\mathbb{Q}$.*

Proof. Note that, in arbitrary metric space, open balls are open, and closed balls are closed.

(i) If $r = p^m$ for some $m \in \mathbb{Z}$, then choose $\epsilon > 0$ small enough such that

$$s = p^m - \epsilon \notin \{p^n | n \in \mathbb{Z}\}$$

and $s > 0$. We have

$$B(a;r) = \overline{B}(a;s)$$

So $B(a;r)$ is closed.

(ii) Given $b \in B(a;r)$, for $x \in B(a;r)$,

$$\begin{aligned} |x-b|_p &= |x-a+a-b|_p \\ &\le \max\{|x-a|_p, |a-b|_p\} \\ &< r, \end{aligned}$$

which implies that $x \in B(b;r)$. Therefore,

$$B(a;r) \subseteq B(b;r).$$

Similarly, $B(b;r) \subseteq B(a;r)$. Hence, $B(a;r) = B(b;r)$.

(iii) and (iv) can be proved as in (i) and (ii), and (v) follows from $\partial B(a;r) = \overline{B}(a;r) - B(a;r)$. ❑

Remark. In (v) of Theorem 3.7, $\partial B(a;r)$ may be empty. In fact, if $r \notin \{p^n | \, n \in \mathbb{Z}\}$, then $\partial B(a;r) = \phi$. Moreover, if $r = p^n$ for some $n \in \mathbb{Z}$, then $\partial B(a;p^n) = \{x \in \mathbb{Q} | \, |x-a|_p = p^n\}$.

Corollary 3.8 *Let $B(a;r)$ and $B(b;s)$ be two open balls in $\mathbb{Q}$. Then either $B(a;r) \cap B(b;s) = \phi$ or one is contained in the other. similarly for closed balls.*

Proof. Suppose that $c \in B(a;r) \cap B(b;s)$. Then, by (ii) of Theorem 3.7, $B(c;r) = B(a;r)$ and $B(c;s) = B(b;s)$. If $r < s$, then

$$\begin{aligned} B(a;r) &= B(c;r) \\ &\subseteq B(c;s) \\ &= B(b;s). \end{aligned}$$

Similarly, if $r > s$, then $B(b;s) \subseteq B(a;r)$. ❑

Corollary 3.9 *$\mathbb{Q}$ can be expressed as countable disjoint union of open(closed) balls.*

***Proof*.** The collection $\mathcal{B} = \{B(a; p^n)|\ a \in \mathbb{Q}, n \in \mathbb{Z}\}$ of open balls is countable. According to Theorem 3.7, $\mathbb{Q}$ can be expressed as a union of a sub-collection of $\mathcal{B}$ which consists of disjoint open balls. ❑

Theorem 3.10 *The p-adic topology on $\mathbb{Q}$ is totally disconnected, i.e. every connected component in $\mathbb{Q}$ is a singleton.*

***Proof*.** It suffices to show that if $A \subseteq \mathbb{Q}$ contains at least two points, then A is disconnected.

Let $x, y \in A$ and $x \neq y$. Then $r = |\,x - y\,|_p > 0$, $y \notin B(x; r)$. and

$$A = [B(a; r) \cap A] \cup [(\mathbb{Q} - B(a; r)) \cap A].$$

Since $B(a; r)$ is also closed by (i) of Theorem 3.7, A can be expressed as two non-empty disjoint open subsets of A. Therefore, A is disconnected. ❑