4 Completion of Valuations

Let $|\cdot|$ be an arbitrary valuation on a field k. As we mention earlier, $|\cdot|$ is a norm on k, regarded k as a 1-dimensional vector space over k. Therefore, $(k, |\cdot|)$ is a normed linear space over k, some authors called it a normed field. In particular, it is a metric space in a natural way, i.e. for $a, b \in k$,

$$d(a,b) = |\, a - b \,|$$

is a distance function on k. Therefore, all analysis come in naturally. We quickly review some of them.

Let $|\cdot|$ be a valuation on k. We have the following definitions.

1. A sequence $\{a_n\}$ in k is said to converge to a if

   $$\lim_{n\to\infty} |\, a_n - a \,| = 0,$$

   i.e. for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

   $$|\, a_n - a \,| < \epsilon \text{ for all } n \geq N.$$

   Write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ *as* $n \to \infty$.

2. If $\lim_{n\to\infty} a_n = 0$, then $\{a_n\}$ is a null sequence.

3. $\{a_n\}$ is a Cauchy sequence in k if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

   $$|\, a_m - a_n \,| < \epsilon \text{ for all } m, n \geq N.$$

4. $\{a_n\}$ is bounded if there is a positive number M such that

   $$|\, a_n \,| \leq M \text{ for } n = 1, 2, \ldots.$$

We have the following facts whose proofs are the same as in metric space.

1. If $\{a_n\}$ converges, then its limit is unique.

2. A convergent sequences is also a Cauchy sequence.

3. A Cauchy sequence, in particular, a convergent sequence, is bounded.

4. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

$$\lim_{n\to\infty} a_n \pm b_n = a \pm b,$$

$$\lim_{n\to\infty} a_n b_n = ab,$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b} \text{ if } b \neq 0.$$

5. If $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence, then $\{a_n b_n\}$ is also a null sequence. In particular, if $\{a_n\}$ is a null sequence and $\{b_n\}$ is a Cauchy sequence, then $\{a_n b_n\}$ is a null sequence.

Now, we discuss the completion of a valuated field k. Let

$$\mathcal{C} = \{\{a_n\} |\ \{a_n\} \text{ is a Cauchy sequence in } k\}$$

and

$$\mathcal{N} = \{\{b_n\} |\ \{b_n\} \text{ is a null sequence in } k\}.$$

Define addition and multiplication on $\mathcal{C}$ as follows:
Given $\{a_n\}$ and $\{b_n\}$ in $\mathcal{C}$,

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

and

$$\{a_n\} \cdot \{b_n\} = \{a_n b_n\}.$$

Proposition 4.1 *With respect to the operations of addition and multiplication, $\mathcal{C}$ is a commutative ring with identity and $\mathcal{N}$ is a maximal ideal of $\mathcal{C}$. In particular, $\mathcal{C}/\mathcal{N}$ is a field.*

Proof. It suffices to prove the following facts:

(I) The addition and multiplication on $\mathcal{C}$ are well-defined, i.e., if $\{a_n\}, \{b_n\} \in \mathcal{C}$, then $\{a_n + b_n\}, \{a_n b_n\} \in \mathcal{C}$.

It follows from the following observations.

$$|\,(a_m + b_m) - (a_n + b_n)\,| \leq |\,a_m - a_n\,| + |\,a_n - b_n\,|$$

and

$$\begin{aligned} |\,a_m b_m - a_n b_n\,| &\leq |\,a_m b_m - a_m b_n + a_m b_n - a_n b_n\,| \\ &\leq |\,a_m\,|\,|\,b_m - b_n\,| + |\,b_n\,|\,|\,a_m - b_n\,| \\ &\leq M(|\,b_m - b_n\,| + |\,a_m - b_n\,|), \end{aligned}$$

where M is an upper bound for both $\{a_n\}$ and $\{b_n\}$.

(II) $(\mathcal{C}, +)$ is an abelian group.

(1) The addition on $\mathcal{C}$ is associative and commutative.

Given $\{a_n\}, \{b_n\}, \{c_n\} \in \mathcal{C}$, we have

$$(\{a_n\} + \{b_n\}) + \{c_n\} = \{a_n] + (\{b_n\} + \{c_n\})$$

and

$$\{a_n\} + \{b_n\} = \{b_n\} + \{a_n\}.$$

(2) There is a zero element $0 = \{0\}$ in $\mathcal{C}$, where $\{0\}$ is the sequence whose terms are 0.

(3) For each $\{a_n\} \in \mathcal{C}$, $\{-a_n\}$ is an additive inverse of $\{a_n\}$ in $\mathcal{C}$.

(III) For multiplication on $\mathcal{C}$, the associative law and distributive law hold

Given $\{a_n\}, \{b_n\}, \{c_n\} \in \mathcal{C}$, we have

$$(\{a_n\}\{b_n\})\{c_n\} = \{a_n\}(\{b_n\}\{c_n\})$$

and

$$\{a_n\}(\{b_n\} + \{c_n\}) = \{a_n\}\{b_n\} + \{a_n\}\{c_n\}.$$

(IV) $(\mathcal{C}, +, \cdot)$ is commutative ring with identity.

The multiplication is commutative since $\{a_n\}\{b_n\} = \{b_n\}\{a_n\}$. Moreover, there is a multiplicative identity $1 = \{1\}$ in $\mathcal{C}$, where $\{1\}$ is the constant sequence whose terms are 1.

(V) $\mathcal{N}$ is an ideal in $\mathcal{C}$.

Given $\{a_n\}, \{b_n\} \in \mathcal{N}$ and $\{c_n\} \in \mathcal{C}$, we have

$$\{a_n\} - \{b_n\} = \{a_n - b_n\} \in \mathcal{N}$$

and

$$\{c_n\}\{a_n\} = \{c_n a_n\} \in \mathcal{N}.$$

Therefore, $\mathcal{N}$ is an ideal in $\mathcal{C}$.

(VI) $\mathcal{N}$ is maximal.

Let I be an ideal in $\mathcal{C}$ such that $\mathcal{N} \subsetneq I \subseteq \mathcal{C}$. Then there exists a sequence $\{c_n\} \in I$ such that $\{c_n\} \notin \mathcal{N}$. Hence, there are some positive number λ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $|\, c_n \,| > \lambda$. Since, otherwise, for all $k \in \mathbb{N}$, there exists n_k such that $|\, c_{n_k} \,| < 1/k$. Assume that $\{n_k\}$ is increasing. Then

$$\lim_{k \to \infty} c_{n_k} = 0, \text{ i.e. } \{c_{n_k}\} \in \mathcal{N} \text{ is a null sequence,}$$

and so

$$\lim_{n \to \infty} c_n = 0, \text{ i.e. } \{c_n\} \text{ is a null sequence.}$$

It contradicts to that $\{c_n\} \notin \mathcal{N}$. Replace $\{c_n\}$ by $\{c_n'\} = \{1, \ldots, 1, c_{n_0}, c_{n_0+1}, \ldots, c_n, \ldots\}$ if necessary. Then $\{c_n'\}$ is a Cauchy sequence but not a null sequence. We may assume that $\{c_n'\} \in I$. (If some $c_i = 0$ for $1 \leq i \leq n_0 - 1$, then since $\mathcal{N} \subseteq I$ and $d = \{0, \ldots, 0, 1, 0, \ldots\} \in \mathcal{N}$. So $d + \{c_n\} \in I$. After at most $n_0 - 1$ steps, we get $\{c_n'\} \in I$.) For $m, n \geq n_0$,

$$\left|\, {c_m'}^{-1} - {c_n'}^{-1} \,\right| = \frac{|\, c_m' - c_n' \,|}{|\, c_m' c_n' \,|} \leq \frac{|\, c_m - c_n \,|}{\lambda^2}.$$

Then $\{(c_n')^{-1}\}$ is a Cauchy sequence, i.e., in $\mathcal{C}$. Since I is an ideal,

$$\{(c_n')^{-1}\}\{c_n'\} = 1 \in I$$

and then $I = \mathcal{C}$. So $\mathcal{N}$ is maximal.

(VII) $\mathcal{C}/\mathcal{N}$ is a field. It follows from the general theory of commutative ring.

❑

In the following, if we say that k is a valuated field, then its valuation is denoted by $|\cdot|_k$.

Definition 4.2 *A valuated field K is called an extension of the valuated field k if K is a field extension of k and $|a|_K = |a|_k$ for all $a \in k$, i.e., $|\cdot|_K$ is an extension of $|\cdot|_k$. In this case, we refer to $|\cdot|_k$ as the restriction of $|\cdot|_K$ to k and to $|\cdot|_K$ as a prolongation of $|\cdot|_k$ to K.*

Definition 4.3 *Let K and L be valuated fields. An isometry $\sigma : K \longrightarrow L$ is called analytic if $|a|_K = |\sigma(a)|_L$ for all $a \in K$ i.e. σ is an isometry with respect to $|\cdot|_K$ and $|\cdot|_L$.*

We know that $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the ordinary absolute value $|\cdot|$ on $\mathbb{Q}$. Hence, we have the following definition.

Definition 4.4 *A valuated field k is said to be complete if every Cauchy sequence in k converges.*

Example 4.5 $\mathbb{R}$ *with ordinary absolute value is complete, but* $\mathbb{Q}$ *is not.*

There are non-Archimedean valuated fields which are not complete, such as $\mathbb{Q}$ with p-adic valuation. Like the case of non-complete metric space, we can always make a completion. We wish to investigate the possibility of embedding a valuated field in a complete valuated field which is the smallest one in some sense. As we shall see, this can be done by mimicking the process by which the field $\mathbb{Q}$ with ordinary absolute value is embedded in the field of real numbers $\mathbb{R}$ with ordinary absolute value.

Definition 4.6 *Let k be a valuated field. A valuated field K is called a completion of k if*

(i) *K is a field extension of k.*

(ii) *K is complete.*

(iii) *$\overline{k} = K$ i.e. k is dense in K, i.e. every element of K is the limit of some sequence in k.*

We will construct the completion of a valuated field in the same way as we do in metric space.

Theorem 4.7 *Let k be a valuated field. Then*

(a) *There is a completion of k;*

(b) *If K is a completion of k and L is a complete valuated field which is an extension of k, then there exists an analytic k-isometry from K onto some subfield of L.*

(c) *Any two completions K and L of k are analytically k-isometric. i.e. there exists an analytic k-isometry from K onto L.*

Proof. We know that

$$K \equiv \mathcal{C}/\mathcal{N}$$

is a field. Define

$$\varphi : k \longrightarrow K$$

by, for all $a \in k$,

$$\varphi(a) = \{a\} + \mathcal{N},$$

where $\{a\} = \{a, a, \ldots\}$ is the constant sequence whose terms are a. Since $\lim_{n\to\infty} a = a \neq 0$, if $a \neq 0$, then $\{a\}$ is not a null sequence. φ is an injective homomorphism. Identifying a with $\{a\} + \mathcal{N}$. We may consider k as a subfield of K. So K is a field extension of k.

Define $|\cdot|_K : K \longrightarrow \mathbb{R}$ by

$$|\{a_n\} + \mathcal{N}|_K = \lim_{n\to\infty} |a_n|_k \text{ for all } \{a_n\} + \mathcal{N} \in K.$$

The whole proof of Theorem 4.7 follows from the following claims:

(i) $|\cdot|_K$ is a well-defined valuation on K. If $\{a_n\} + \mathcal{N} = \{b_n\} + \mathcal{N}$, then $\{a_n\} - \{b_n\} \in \mathcal{N}$. i.e. $\{a_n - b_n\}$ is a null sequence in K and then

$$\lim_{n\to\infty} |a_n - b_n|_k = 0.$$

On the other hand, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in k so $\{|a_n|_k\}$ and $\{|b_n|_k\}$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, both $\lim_{n\to\infty} |a_n|_k$ and $\lim_{n\to\infty} |b_n|_k$ exist and

$$\lim_{n\to\infty} |a_n|_k = \lim_{n\to\infty} |b_n|_k .$$

Therefore, $|\cdot|_K$ is well-defined on K.

(ii) For all $\{a_n\} + \mathcal{N} \in K$, $|\{a_n\} + \mathcal{N}|_K \geq 0$ and $|\{a_n\} + \mathcal{N}|_K = 0$ if and only if $\{a_n\} + \mathcal{N} = 0$.

(iii) For all $\{a_n\} + \mathcal{N}, \{b_n\} + \mathcal{N} \in K$,

$$\begin{aligned} |(\{a_n\} + \mathcal{N})(\{b_n\} + \mathcal{N})|_K &= |\{a_n b_n\} + \mathcal{N}|_K \\ &= \lim_{n\to\infty} |a_n b_n|_k \\ &= \lim_{n\to\infty} |a_n|_k \cdot \lim_{n\to\infty} |b_n|_k \\ &= |\{a_n\} + \mathcal{N}|_K \, |\{b_n\} + \mathcal{N}|_K . \end{aligned}$$

(iv) For all $\{a_n\}+\mathcal{N}, \{b_n\}+\mathcal{N} \in K$,

$$\begin{aligned}
|\,(\{a_n\}+\mathcal{N})+(\{b_n\}+\mathcal{N})\,|_K &= |\,\{a_n+b_n\}+\mathcal{N}\,|_K \\
&= \lim_{n\to\infty} |\,a_n+b_n\,|_k \\
&\le \lim_{n\to\infty} |\,a_n\,|_k + \lim_{n\to\infty} |\,b_n\,|_k \\
&= |\,\{a_n\}+\mathcal{N}\,|_K + |\,\{b_n\}+\mathcal{N}\,|_K\,.
\end{aligned}$$

Hence, $|\cdot|_K$ is a valuation on K by (ii), (iii) and (iv).

(v) For all $a \in k$,

$$|\,\{a\}+\mathcal{N}\,|_K\,(=|\,\varphi(a)\,|_K) = \lim_{n\to\infty} |\,a\,|_k = |\,a\,|_k\,.$$

So $|\cdot|_K$ is an extension of $|\cdot|_k$.

(vi) K is complete, and every element of K is the limit of some sequence in k. i.e. $\overline{k} = K$.

Let $A_i = \{a_n^i\}+\mathcal{N}$, $i=1,2,\ldots$, and $\{A_i\}$ be a Cauchy sequence in K. Then for all $\epsilon > 0$, there is $i_0 \in \mathbb{N}$ such that, for all $i, j \ge i_0$,

$$\big|\,(\{a_n^i\}+\mathcal{N}) - (\{a_n^j\}+\mathcal{N})\,\big|_K < \epsilon$$

i.e.

$$\lim_{n\to\infty} \big|\,a_n^i - a_n^j\,\big|_k < \epsilon.$$

Since, for each $i \in \mathbb{N}$, $\{a_n^i\}_{n=1}^{\infty} \in \mathcal{C}$ is a Cauchy sequence in k, there exists $N_i \in \mathbb{N}$ such that, for all $m, n \ge N_i$,

$$\big|\,a_n^i - a_m^i\,\big|_k < \frac{1}{i}.$$

In particular, for each $n \ge N_i$,

$$\big|\,a_n^i - a_{N_i}^i\,\big| < \frac{1}{i},\ i=1,2,\ldots.$$

Let $B_i = \{a_{N_i}^i\}+\mathcal{N}$, where $\{a_{N_i}^i\}$ is the constant sequence whose terms are $a_{N_i}^i$. Then we have

$$|\,A_i - B_i\,|_K = \lim_{n\to\infty} \big|\,a_n^i - a_{N_i}^i\,\big|_k \le \frac{1}{i},$$

for all $i = 1, 2, \ldots$, so

$$\lim_{i\to\infty} |\, A_i - B_i \,|_K = 0.$$

Since $\{A_i\}$ is a Cauchy sequence in K, so is $\{B_i\}$. Let $\{z_n\}$ be the sequence defined by

$$z_i = a^i_{N_i},\ i = 1, 2, \ldots.$$

Then, for all $i, j \in \mathbb{N}$,

$$\begin{aligned} |\, z_i - z_j \,|_k &= \left|\, a^i_{N_i} - a^j_{N_j} \,\right|_k \\ &= \lim_{n\to\infty} \left|\, a^i_{N_i} - a^j_{N_j} \,\right|_k \\ &= |\, B_i - B_j \,|_K \,. \end{aligned}$$

Since $\{B_i\}$ is a Cauchy sequence in K,

$$\lim_{i,j\to\infty} |\, z_i - z_j \,|_k = \lim_{i,j\to\infty} |\, B_i - B_j \,|_K = 0.$$

Therefore, $\{z_n\} \in \mathcal{C}$ i.e., a Cauchy sequence in k. Also,it implies that

$$\begin{aligned} |\, B_i - (\{z_n\} + \mathcal{N}) \,|_K &= \lim_{n\to\infty} \left|\, a^i_{N_i} - z_n \,\right|_k \\ &= \lim_{n\to\infty} \left|\, a^i_{N_i} - a^n_{N_n} \,\right|. \end{aligned}$$

Given $\epsilon > 0$, there is a positive integer N such that for all $i, j \geq N$,

$$|\, z_i - z_j \,|_k < \frac{\epsilon}{2},$$

i.e. for all $i, j \geq N$,

$$\left|\, a^i_{N_i} - a^j_{N_j} \,\right|_k < \frac{\epsilon}{2}.$$

So, for each $i \geq N$,

$$|\, B_i - (\{z_n\} + \mathcal{N}) \,|_K \leq \frac{\epsilon}{2} < \epsilon,$$

i.e. B_i converges to $\{z_n\} + \mathcal{N}$ in K. Finally, by

$$\lim_{i\to\infty} |\, A_i - B_i \,|_K = 0,$$

we conclude that $\{A_i\}$ converges to $\{z_n\} + \mathcal{N}$ in K. This proves that K is complete.

Next, to prove that $\overline{k} = K$. Here, we identify $a \in k$ with $\{a\} + \mathcal{N}$ in K. Given $A = \{a_n\} + \mathcal{N} \in K$, since $\{a_n\}$ is a Cauchy sequence in k, for all $\epsilon > 0$, there exists N such that

$$|\, a_n - a_m \,|_k < \frac{\epsilon}{2}$$

for all $n, m \geq N$. Let $X_n = \{a_n\} + \mathcal{N} \in k$, where $\{a_n\}$ is a constant sequence whose terms are a_n. Then, for all $n \geq N$,

$$\begin{aligned} |\, a_n - A \,|_K &= |\, X_n - A \,|_K \\ &= \lim_{m \to \infty} |\, a_n - a_m \,|_k \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

i.e., $\{X_n\}$ converges to A in K and $X_n \in k$ for all k. So $\overline{k} = K$. Therefore, K is a completion of k.

(vii) Let L be a complete valuated field which is an extension of k. We must find an analytic k-isometry of K onto some subfield of L.
Since $A \in K$, we may choose a sequence $\{a_n\}$ in k such that $\lim_{n \to \infty} a_n = A$ in A by (vi). Define

$$\sigma(A) = \lim_{n \to \infty} a_n \in L.$$

This defines an analytic k-isometry of K onto the subfield $\sigma(K)$ of L.

(viii) If K and L are both completion of k, then the analytic k-isometry defined in (vii) is an analytic k-isometry from K onto L.

❑