5 Equivalence of Valuations

Given a field k and two valuations on k. When do they determine the same metric topology on k? In the literature, there are several ways to describe this relation. In this section, we will unify these equivalent conditions in Theorem 5.2 which are motivated by the following simple observation. Given a valuation $|\cdot|$ on k and $\lambda > 0$, define $|\cdot|_1 : k \longrightarrow \mathbb{R}$ by

$$|x|_1 = |x|^\lambda \text{ for all } x \in k.$$

Is $|\cdot|_1$ a valuation on k? In fact, it is almost the case.

Proposition 5.1 *As above, we have*

(i) *If $|\cdot|$ is Archimedean and $0 < \lambda \leq 1$, then $|\cdot|_1$ is an Archimedean valuation on k.*

(ii) *If $|\cdot|$ is non-Archimedean, then $|\cdot|_1$ is always a non-Archimedean valuation on k for every $\lambda > 0$.*

***Proof*.**

(i) Clearly, $|x|_1 \geq 0$ for all $x \in k$, and $|x|_1 = 0$ if and only if $x = 0$. Also, $|xy|_1 = |x|_1 |y|_1$ for all $x, y \in k$. To prove hat $|\cdot|$ satisfies the triangle inequality. Given $x, y \in k$, if $x = 0$ or $y = 0$, then it is obvious that

$$|x + y|_1 \leq |x|_1 + |y|_1 .$$

Now, assume that both x and y are nonzero, and $|\,x\,| \leq |\,y\,|$. We have

$$\begin{aligned}\left|1+\frac{x}{y}\right|_1 &= \left|1+\frac{x}{y}\right|^{\lambda} \\ &\leq (|\,1\,| + \left|\frac{x}{y}\right|)^{\lambda} \\ &= (1+\left|\frac{x}{y}\right|)^{\lambda} \\ &\leq (1+\left|\frac{x}{y}\right|) \\ &\leq (1+\left|\frac{x}{y}\right|^{\lambda}),\end{aligned}$$

which implies that

$$|\,x+y\,|_1 \leq |\,x\,|_1 + |\,y\,|_1 .$$

Therefore, $|\cdot|_1$ is a valuation on k, and it is Archimedean since $|\cdot|$ is Archimedean, $\mathbb{Z}$ is unbounded with respect to $|\cdot|$, so is with respect to $|\cdot|_1$.

(ii) In view of the proof of (i), it suffices to show that, for any $\lambda > 0$, $|\cdot|_1$ satisfies the strong triangle inequality. Again, we assume that both x and y are nonzero, otherwise, it is trivial. We have

$$\begin{aligned}|\,x+y\,|_1 &= |\,x+y\,|^{\lambda} \\ &\leq (\max\{|\,x\,|, |\,y\,|\})^{\lambda} \\ &= \max\{|\,x\,|^{\lambda}, |\,y\,|^{\lambda}\} \\ &= \max\{|\,x\,|_1, |\,y\,|_1\}.\end{aligned}$$

Therefore, $|\cdot|$ is a non-Archimedean valuation on k. ❑

Remark. In (i), if $\lambda > 1$, then $|\cdot|_1$ may not be a valuation on k. For example, consider ordinary absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ and $\lambda = 2$. Then

$$|\,1+1\,|_1 = |\,2\,|_1 = |\,2\,|_\infty^2 = 4,$$

but

$$|\,1\,|_1 + |\,1\,|_1 = |\,1\,|_\infty^2 + |\,1\,|_\infty^2 = 2,$$

so the triangle inequality fails for $|\cdot|_1$, and $|\cdot|_1$ is not a valuation on $\mathbb{Q}$.

From the above result, if $|\cdot|_1$ is also a valuation on k, then it is obvious that $|\cdot|_1$ and $|\cdot|$ induce the same metric topology on k and Cauchy sequences of k. In fact, we have the following Theorem.

Theorem 5.2 *Let $|\cdot|_1$ and $|\cdot|_2$ be two valuations on k. Then the following statements are equivalent:*

(i) *If $x \in k$, then $|x|_1 < 1$ if and only if $|x|_2 < 1$.*

(ii) *If $x \in k$, then $|x|_1 \leq 1$ if and only if $|x|_2 \leq 1$.*

(iii) *There exists $\lambda > 0$ such that $|\cdot|_2 = |\cdot|_1^\lambda$ on k.*

(iv) *$|\cdot|_1$ and $|\cdot|_2$ induce the same metric topology on k.*

(v) *$\{x_n\}$ is a Cauchy sequence with respect to $|\cdot|_1$ if and only if $\{x_n\}$ is a Cauchy sequence with respect to $|\cdot|_2$.*

(vi) *$\{x_n\}$ is a null sequence with respect to $|\cdot|_1$ if and only if $\{x_n\}$ is a null sequence with respect to $|\cdot|_2$.*

***Proof*.**

First, we consider the case that $|\cdot|_1$ is trivial.

Suppose that (i) holds. Then it is easy to see that $|x|_1 > 1$ if and only if $|x|_2 > 1$. Hence, $|x|_1 = 1$ if and only if $|x|_2 = 1$. So (ii) holds also.

Suppose that (ii) holds. We will show that $|\cdot|_2$ is also trivial on k. Hence, (i) holds. If there exists $x \in k$ and $x \neq 0$ such that $|x|_2 \neq 1$, then we may assume that $|x|_2 > 1$, otherwise consider x^{-1}. By assumption, $|x|_1 > 1$ which is impossible. So $|\cdot|_2$ is trivial on k.

Suppose that (i) holds. By assumption, $|\cdot|_2$ is trivial on k. Choose arbitrary $\lambda > 0$, (iii) holds.

Suppose that (iii) holds. Then both $|\cdot|_1$ and $|\cdot|_2$ define the discrete metric on k. So they induce the same metric topology on k, namely, the discrete topology on k.

Suppose that (iv) holds. Then $|\cdot|_1$ and $|\cdot|_2$ are trivial on k. So (v) holds automatically.

Suppose that (v) holds. Given $x \in k$ with $|x|_1 < 1$, obviously, $|x|_2 < 1$. Conversely, if $|x|_2 < 1$, then $\{x^n\}$ converges to zero, with respect to $|\cdot|_2$, so it is a Cauchy sequence with respect to $|\cdot|_2$. By assumption, $\{x^n\}$ is a Cauchy sequence with respect to $|\cdot|_1$. Since $|\cdot|_1$ is trivial, we conclude that $x = 0$, hence $|x|_1 < 1$. So (i) holds.

Suppose that (iii) holds. Then (vi) is trivial.

Suppose that (vi) holds. If $|x|_1 < 1$, then obviously, $|x|_2 < 1$. Conversely, if $|x|_2 < 1$, then $\{x^n\}$ is a null sequence with respect to $|\cdot|_2$. By assumption, $\{x^n\}$ is a null sequence with respect to $|\cdot|_1$. As $|\cdot|_1$ is trivial, we must have $x = 0$. So $|x|_1 < 1$. Hence, (i) holds.

Therefore, all statements are equivalent if one of $|\cdot|_1$ or $|\cdot|_2$ is trivial on k. In fact, we have proved that $|\cdot|_1$ and $|\cdot|_2$ are both trivial on k.

Now, we assume that both $|\cdot|_1$ and $|\cdot|_2$ are non-trivial valuations on k.

Suppose that (i) holds. Then (i) implies that $|x|_1 > 1$ if and only if $|x|_2 > 1$ and $|x|_1 = 1$ if and only if $|x|_2 = 1$. So (ii) holds.

Suppose that (ii) holds. Then (ii) implies that $|x|_1 > 1$ if and only if $|x|_2 > 1$. Now, if $|x|_1 < 1$, then $\left|\frac{1}{x}\right|_1 > 1$. So $\left|\frac{1}{x}\right|_2 > 1$. Therefore, $|x|_2 < 1$. Similarly, $|x|_2 < 1$ if and only if $|x|_1 < 1$. So (i) holds.

Suppose that (i) holds. Since $|\cdot|_1$ is nontrivial, there exists $c \in k$ and $c \neq 0$ such that $|c|_1 \neq 1$. Given $x \in k$, $x \neq 0$ and $n, m \in \mathbb{N}$, we want to compare

$$|c^n x^m|_1 \text{ and } |c^n x^m|_2.$$

Take log of both numbers, we obtain

$$\log|c^n x^m|_1 = n\log|c|_1 + m\log|x|_1$$

and

$$\log|c^n x^m|_2 = n\log|c|_2 + m\log|x|_2.$$

Graph both lines,

$$n\log|c|_1 + m\log|x|_1 = 0$$

and

$$n\log|c|_2 + m\log|x|_2 = 0$$

in the nm-plane. Suppose that they are not the same; say the first one eventually lies above the second one in the first quadrant. Then the region between them satisfies the equations

$$n\log|c|_1 + m\log|x|_1 < 0$$

and

$$n\log|c|_2 + m\log|x|_2 > 0.$$

It is clear that we can find integer point (n, m) in this region. For this point, $\log|c^n x^m|_1 < 0$, but $\log|c^n x^m|_2 > 0$, i.e. $|c^n x^m|_1 < 1$, but $|c^n x^m|_2 > 1$ which contradicts to (ii). Therefore, they are the same line, hence have the same slope. i.e.

$$\frac{\log|x|_1}{\log|x|_2} = \frac{\log|c|_1}{\log|c|_2} \text{ for all } x \neq 0$$

which implies that

$$|x|_2 = (e^{\log|x|_1})^{\frac{\log|c|_2}{\log|c|_1}} = |x|_1^{\alpha},$$

where $\alpha = \frac{\log|c|_2}{\log|c|_1} > 0$ since $|c|_1 < 1$ if and only if $|c|_2 < 1$ by (i). Therefore, $|x|_1 = |x|_2^{\lambda}$, where $\lambda = \frac{1}{\alpha} > 0$. This proves (iii).

Suppose that (iii) holds. Then (iv) follows from the following observation of open balls with respect to $|\cdot|_1$ and $|\cdot|_2$:

$$\begin{aligned} B_2(a;r) &= \{x \in k \,|\, |x-a|_2 < r\} \\ &= \{x \in k \,|\, |x-a|_1^{\lambda} < r\} \\ &= \{x \in k \,|\, |x-a|_1 < r^{\frac{1}{\lambda}}\} \\ &= B_1(a; r^{\frac{1}{\lambda}}). \end{aligned}$$

and

$$\begin{aligned} B_1(a;r) &= \{x \in k \,|\, |x-a|_1 < r\} \\ &= \{x \in k \,|\, |x-a|_2^{\frac{1}{\lambda}} < r\} \\ &= \{x \in k \,|\, |x-a|_2 < r^{\lambda}\} \\ &= B_2(a; r^{\lambda}), \end{aligned}$$

where $B_i(a;r)$ is the open ball with center a and radius $r > 0$ with respect to $|\cdot|_i$, $i = 1, 2$.

Suppose that (iv) holds, i.e. $|\cdot|_1$ and $|\cdot|_2$ induce the same metric topology on k. Let $\{x_n\}$ be a Cauchy sequence in k with respect to $|\cdot|_1$. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $B_1(0;\delta) \subseteq B_2(0;\epsilon)$. Choose $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$|x_m - x_n|_1 < \delta, \text{ i.e., } x_m - x_n \in B_1(0;\delta).$$

Hence, for all $m, n \geq N$,

$$x_m - x_n \in B_2(0;\epsilon), \text{ i.e., } |x_m - x_n| < \epsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in k with respect to $|\cdot|_2$. So (v) holds. Similarly, for the other direction.

Suppose (v) holds and $|\,x\,|_1 < 1$. Then $\{x^n\}$ converges to 0 with respect to $|\cdot|_1$, in particular, $\{x^n\}$ is a Cauchy sequence with respect to $|\cdot|_1$, by assumption, $\{x^n\}$ is a Cauchy sequence with respect to $|\cdot|_2$. Since $|\,x\,|_1 < 1,\ x \neq 1$. Let $\epsilon = |\,x-1\,|_2$. Then $\epsilon > 0$. By definition, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|\,x^m - x^n\,|_2 < \epsilon.$$

In particular, for all $n \geq N$,

$$\left|\,x^{n+1} - x^n\,\right|_2 < \epsilon, \text{ i.e., } |\,x\,|_2^n\,|\,x-1\,|_2 < \epsilon = |\,x-1\,|_2\,.$$

which implies that $|\,x\,|_2^n < 1$, i.e. $|\,x\,|_2 < 1$. Similarly, for the other direction, and we have (i). We have shown that (i),(ii),(iii),(iv),(v) are equivalent.

Suppose that (iii) holds. Then (vi) is trivial.

Suppose (vi) holds and $|\,x\,|_1 < 1$. Then $\{x^n\}$ is a null sequence with respect to $|\cdot|_1$. By assumption, $\{x^n\}$ is a null sequence with respect to $|\cdot|_2$. It is easy to see that $|\,x\,|_2 < 1$. Similarly for the other direction, and (i) holds.

Suppose that (i) holds and (iii) has established. Therefore, (i), (iii), (vi) are equivalent, and the theorem is proved. ❑

Definition 5.3 *Two valuations $|\cdot|_1$ and $|\cdot|_2$ on a field k are said to be equivalent if they satisfy one of the conditions in Theorem 5.2. In this case, we write $|\cdot|_1 \sim |\cdot|_2$.*

Remark.

(i) Clearly, $\sim$ is an equivalence relation among all valuations on k by Theorem 5.2.

(ii) Trivial valuations can not be equivalent to non trivial relations by Theorem 5.2.

Corollary 5.4 *Let $|\cdot|_1$ and $|\cdot|_2$ be valuations on a field k. If $|\cdot|_1 \sim |\cdot|_2$ on k, then both are Archimedean or non-Archimedean.*

Proof. It follows from Theorem 5.2 and Theorem 2.9. ❑

Corollary 5.5 *Let p and q be two distinct prime integers. Then $|\cdot|_p$ and $|\cdot|_q$ are not equivalent on $\mathbb{Q}$. Similarly, if $p(x)$ and $q(x)$ are distinct irreducible polynomial in $k[x]$, then $|\cdot|_{p(x)}$ and $|\cdot|_{q(x)}$ are not equivalent on $k(x)$.*

Proof. It is easy to see that $\{p^n\}$ is a null sequence on $\mathbb{Q}$ with respect to $|\cdot|_p$, but not for $|\cdot|_q$, and $\{q^n\}$ is a null sequence on $\mathbb{Q}$ with respect to $|\cdot|_q$, but not for $|\cdot|_p$. Therefore, by Theorem 5.2, $|\cdot|_p$ and $|\cdot|_q$ are not equivalent. Similarly, if $p(x)$ and $q(x)$ are two distinct irreducible polynomials on $k[x]$, then $|\cdot|_{p(x)}$ and $|\cdot|_{q(x)}$ defined in Example 2.20 are not equivalent valuations on $k(x)$. ❑

Example 5.6 *Given a prime integer p and $c > 1$. Define on $\mathbb{Q}$,*

$$|x| = \begin{cases} c^{-\operatorname{ord}_p x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$$

Then $|\cdot|$ is a valuation on $\mathbb{Q}$ and $|\cdot| = |\cdot|_p^{\lambda}$ on $\mathbb{Q}$, where $\lambda = \frac{\log c}{\log p}$. In particular, $|\cdot| \sim |\cdot|_p$ on $\mathbb{Q}$.

In fact, for all $x \neq 0$ in $\mathbb{Q}$, we have

$$|x| = c^{-\operatorname{ord}_p x}.$$

Hence,

$$\begin{aligned} \log|x| &= -\operatorname{ord}_p x \cdot \log c \\ &= -\operatorname{ord}_p x \cdot \log p \cdot \frac{\log c}{\log p} \\ &= \log p^{(\operatorname{ord}_p x)\lambda} \end{aligned}$$

which implies that

$$|\,x\,| = p^{-(\mathrm{ord}_p\, x)\lambda} = |\,x\,|_p^{\lambda}.$$

Therefore, $|\cdot| \sim |\cdot|_p$ *on* $\mathbb{Q}$ *by Theorem 5.2. Similarly, given an irreducible* $p(x)$ *in* $k[x]$ *and* $c, d > 1$*. Then as in Example 2.20, the valuations on* $k(x)$ *defined by*

$$|\,f(x)\,|_p = \begin{cases} c^{-\mathrm{ord}_p\, f(x)}, & \text{if } f(x) \neq 0; \\ 0, & \text{if } f(x) = 0. \end{cases}$$

and

$$|\,f(x)\,|_1 = \begin{cases} d^{-\mathrm{ord}_p\, f(x)}, & \text{if } f(x) \neq 0; \\ 0, & \text{if } f(x) = 0. \end{cases}$$

are equivalent on $k(x)$*.*

In general, we have the following well-known result whose proof can be found in [2,3,7,8,9].

Theorem 5.7 *(Ostrowski) Every valuation on* $\mathbb{Q}$ *is equivalent to either the trivial valuation* $|\cdot|_0$*, p-adic valuation* $|\cdot|_p$ *or the ordinary absolute value* $|\cdot|_\infty$ *on* $\mathbb{Q}$*.*