國 立 政 治 大 學 應 用 數 學 系

碩 士 學 位 論 文

On Functional Equations in the Field of Meromorphic Functions

半純函數體中的函數方程

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Abstract

In this thesis, we use the theory of value distribution to study the existence of solution of the following functional equation:

$$\sum_{j=1}^{p} a_j(z) f_j(z)^{k_j} = 1,$$

where $a_1(z), \cdots, a_p(z)$ are meromorphic functions. For some special case, new and old examples of the solutions are given. For the general case, a necessary condition for the existence of solution is considered. Moreover, we obtain a result on the distribution of zeros and poles of a class of meromorphic functions.

中文摘要

在這篇論文中，我們將利用值分佈的理論來探討下列函數方程解的存在性與其性質：

$$\sum_{j=1}^{p} a_j(z) f_j(z)^{k_j} = 1,$$

其中 $a_1(z), \cdots, a_p(z)$ 爲半純函數。對某些特殊方程，除了文獻裡已知的結果外，我們亦提供其它的例子。一般而言，我們探討解存在的必要條件。另外，我們證明了某一類半純函數之零點與極點之分佈的結果。

1 Introduction

In this thesis, we will use the theory of value distribution to study the existence of solution of the following functional equation:

$$\sum_{j=1}^{p} a_j(z) f_j(z)^{k_j} = 1, \tag{1.1}$$

where $a_1(z), \cdots, a_p(z)$ are meromorphic functions.

First, we consider some special cases. In 1927, P. Montel [17] stated the following theorem.

Theorem 1.1 [17] *Let f, g be two transcendental entire functions. Then if m and n are integers ≥ 3, the functional equation*

$$f^n + g^m = 1$$

cannot hold.

Later in 1965, A.V. Jategaonkar [12] gave a complete proof of the theorem.

In 1966, Gross [4, 5] considered the functional equation

$$f^n + g^n \equiv 1 \tag{1.2}$$

and proved the following result.

Theorem 1.2 [5] *For $n > 2$, there do not exist two non-constant entire solutions that satisfy* (1.2).

Theorem 1.3 [4] *For $n > 3$, there do not exist two non-constant meromorphic functions f and g that satisfy* (1.2).

Theorem 1.4 [5] *For $n = 2$, all entire solutions of* (1.2) *for $n = 2$ are of the form*

$$f(z) = \cos(\eta(z)) \text{ and } g(z) = \sin(\eta(z)),$$

where $\eta(z)$ is an entire function.

Theorem 1.5 [4] *For $n = 2$, all meromorphic solutions of* (1.2) *for $n = 2$ are of the form*

$$f(z) = \frac{2\beta(z)}{1+\beta(z)^2}, \ g(z) = \frac{1-\beta(z)^2}{1+\beta(z)^2},$$

where β is a meromorphic function.

Moreover, Gross [5] considered an example of meromorphic solution of (1.2) for $n = 3$ as follow:

$$f = \frac{1}{2\wp}(1 + 3^{-1/2}\wp'),$$
$$g = \frac{-1}{2\wp}(1 - 3^{-1/2}\wp'),$$

where $\wp$ is the Weierstrass $\wp$-function.

In 1966, Baker [1] gave a general form of meromorphic solution of (1.2) for $n = 3$ as follows.

Theorem 1.6 [1] *If f and g are non-constant meromorphic functions satisfying* (1.2) *for $n = 3$, then f and g are of the form*

$$f(z) = F(\omega(z)),$$
$$g(z) = cG(\omega(z)) = cF(-\omega(z)) = F(-c^2\omega(z)),$$

where F and G are elliptic functions $\frac{1 + 3^{-1/2}\wp'(z)}{2\wp(z)}$ and $\frac{1 - 3^{-1/2}\wp'(z)}{2\wp(z)}$, respectively. $\omega(z)$ is a entire function and c is a cube-root of unity.

In 1970, Yang [22] considered a more general case of Theorem 1.1.

Theorem 1.7 [22] *The functional equation*

$$a(z)f^n(z) + b(z)g^m(z) = 1 \tag{1.3}$$

(a, b, f, g *are meromorphic functions and* m, n *are integers* ≥ 3) *cannot hold, if*

$$T(r, a) = o(T(r, f)), \quad T(r, b) = o(T(r, g)), \tag{1.4}$$

unless $m = n = 3$. *If* f, g *are entire and* (1.4) *holds, then* (1.3) *cannot hold even if* $m = n = 3$.

Another simple functional equation is

$$f^n + g^n + h^n \equiv 1 \tag{1.5}$$

In 1985, Hayman [9] proved a general theorem and, in 1979, D. J. Newman and M. Slater [19] gave a theorem which implies the following result.

Theorem 1.8 [9] *For* $n \geq 9$, *there do not exist three non-constant meromorphic functions* f, g, *and* h *that satisfies* (1.5).

Theorem 1.9 [19] *Let* f, g, *and* h *be solutions of* (1.5) *with* $n = 6$, *then* f, g, *and* h *can not be three non-constant polynomials.*

In 2002, Ishizaki [11] gave a new proof of the non-existence of transcendental meromorphic solution of (1.5) for $n \geq 9$.

For the functional equation (1.5), the solutions for $1 \leq n \leq 6$ can be given explicitly which will be discussed in section 2. However, for $n = 7, 8$, the existence of solution of (1.5) is still open.

For the general case of (1.1), Toda [21], K.-W. Yu and C.-C. Yang [25], and I. Lahiri and K.-W. Yu [13] gave a necessary condition for the existence of solution of (1.1) which will be discussed in section 4. Finally, we obtain a result on the distribution of zeros and poles of a class of meromorphic functions which will be proved in section 5.

2 Some Special Functional Equations

In this section, we collect some new and old examples of solutions of the functional equation (1.5):

$$f^n + g^n + h^n = 1$$

for $1 \leq n \leq 6$.

Example 2.1 (n=1) *Let* $f = e^z,\ g = e^{-z},\ h = -e^z - e^{-z} + 1$*, then*

$$f + g + h = 1.$$

In fact, given meromorphic functions g and h and set $f = 1 - g - h$.Then f, g and h satisfy (1.5).

Example 2.2 (n=2) *Let* $\alpha(z)$ *be a meromorphic function and defined*

$$f = \frac{\alpha^2 - 2}{\sqrt{3}},\ g = \frac{(\alpha^2 + 1)i}{\sqrt{3}},\ h = \sqrt{2}\alpha,$$

then

$$\begin{aligned} f^2 + g^2 + h^2 &= \frac{\alpha^4 - 4\alpha^2 + 4}{3} - \frac{\alpha^4 + 2\alpha^2 + 1}{3} + \frac{6\alpha^2}{3} \\ &= 1. \end{aligned}$$

Therefore, $f^2 + g^2 + h^2 = 1$ *has infinitely many solutions.*

Example 2.3 (n=3) *Lehmer* [14] *showed that* (1.5) *has infinitely many solutions. In fact, if* $\beta(z)$ *is a meromorphic function and let*

$$f(z) = 9\beta(z)^4,\ g(z) = -9\beta(z)^4 + 3\beta(z),\ h(z) = -9\beta(z)^3 + 1,$$

Then

$$\begin{aligned} f^3 + g^3 + h^3 &= (9\beta(z)^4)^3 + (-9\beta(z)^4 + 3\beta(z))^3 + (-9\beta(z)^3 + 1)^3 \\ &= 729\beta(z)^{12} + (-729\beta(z)^{12} + 729\beta(z)^9 - 243\beta(z)^6 + 27\beta(z)^3) + \\ &\quad (-729\beta(z)^9 + 243\beta(z)^6 - 27\beta(z)^3 + 1) \\ &= 1 \end{aligned}$$

Example 2.4 (n=4) *Gross [4] gave the following example:*

Let

$$
\begin{aligned}
f(z) &= 2^{1/4}(\sin^2 z - \cos^2 z + i \sin z \cos z),\\
g(z) &= (-1)^{1/4}(2i \sin z \cos z + \sin^2 z),\\
h(z) &= (-1)^{1/4}(2i \sin z \cos z - \cos^2 z).
\end{aligned}
$$

Then we have

$$f^4 + g^4 + h^4 = 1$$

Also Green [3] considered the following example: If $w(z)$ is a non-constant entire function, then the three transcednental entire functions

$$
\begin{aligned}
f =& 8^{-1/4}(e^{3w} + e^{-w}),\\
g =& (-8)^{-1/4}(e^{3w} - e^{-w}),\\
h =& (-1)^{1/4}e^{2w},
\end{aligned}
$$

satisfy

$$f^4 + g^4 + h^4 = 1$$

Example 2.5 (n=5) *Gundersen [8] gave the transcendental meromorphic solution as follows:*
Let $a_k = e^{2k\pi i/5}$ and $b_k = \dfrac{1}{a_k - 1}$, $1 \le k \le 4$, $c = \dfrac{b_3 b_4 - b_1 b_2}{(b_3 + b_4) - (b_1 + b_2)}$, $d = \sqrt{(c - b_1)(c - b_2)}$. Then $b_1 + b_2 \ne b_3 + b_4$, $c \ne 0$ and $d \ne 0$. Define

$$u(z) = 1 + \frac{1}{c + de^z},\ v(z) = 1 + \frac{1}{c + de^{-z}},\ w(z)^5 = \frac{u(z)^5 - 1}{v(z)^5 - 1}.$$

Let

$$f = u,\ g = e^{\frac{\pi}{5}i} vw,\ h = v.$$

Then

$$f^5 + g^5 + h^5 = 1.$$

Example 2.6 (n=6) *Gundersen* [7] *constructed three meromorphic functions* f, g *and* h *satisfying the functional equation* (1.5) *as* $n = 6$.

Consider the differential equation

$$(F')^2 = K(F - A)(F - B)(F - C), \tag{2.1}$$

where $K \neq 0$ is a constant, and F is a meromorphic solution of (2.1). Then by a result of F. Rellich [20], F is an elliptic function. Set

$$w(z) = \alpha \frac{(4+2i)F(z) - (3+4i)A - (1-2i)B}{(4+2i)F(z) - 5A + (1-2i)B}.$$

Finally, define

$$f = \frac{w-b}{v},\ g = \frac{w+b}{v},\ h = i\frac{u}{v}.$$

Then

$$f^6 + g^6 + h^6 = 1.$$

3 Basic Theory of Value Distribution

In this section, we introduce and review some basic facts and notations in complex analysis and value distribution which will be used throughout the rest of the thesis. For the sake of brevity, proofs are omitted because they are standard and can be found in [2, 6, 10, 23, 24].

In Nevanlinna's value distribution theory, the following Poisson-Jensen's formula plays a very important role.

Theorem 3.1 (Poisson-Jensen's formula) *Let* $0 < R < \infty$ *and* f *be meromorphic in* $|z| < R$ *and* a_μ *and* b_ν *be the zeros and poles of* f *in* $|z| < R$, $1 \le \mu \le M$, $1 \le \nu \le N$, *respectively. If* $z = re^{i\theta}$, $0 \le r < R$, *and* $f(z) \ne 0, \infty$, *then we have*

$$\begin{aligned}\log|f(z)| =& \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi \\ &+ \sum_{\mu=1}^{M} \log\left|\frac{R(z-a_\mu)}{R^2 - \overline{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z-b_\nu)}{R^2 - \overline{b}_\nu z}\right|.\end{aligned}$$

By taking $z = 0$ in Theorem 3.1, we get the Jensen's formula.

Theorem 3.2 (Jensen's formula) *Under the assumption of Theorem 2.1, if* $f(0) \ne 0, \infty$, *then we have*

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\varphi - \sum_{\mu=1}^{M} \log\frac{R}{|a_\mu|} + \sum_{\nu=1}^{N} \log\frac{R}{|b_\nu|}.$$

The assumption $f(0) \ne 0, \infty$ in Theorem 3.1 can be eliminated. In fact, for $0 \le r < \infty$, let $n(r, f)$ denote the number of poles of f in $|z| \le r$ counting multiplicities. Consider the Laurent expansion of f at the origin

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots .$$

Note that $\lambda = n(0, \frac{1}{f}) - n(0, f)$. Consider the function

$$g(z) = \begin{cases} f(z)(\frac{R}{z})^\lambda & \text{if } z \neq 0 \\ c_\lambda R^\lambda & \text{if } z = 0, \end{cases}$$

then we have the generalized Jensen's formula.

Theorem 3.3 (generalized Jensen's formula) *Under the assumption of Theorem 3.1 without the condition $f(0) \neq 0, \infty$, then we have*

$$\log |\, c_\lambda \,| = \frac{1}{2\pi} \int_0^{2\pi} \log \big|\, f(Re^{i\varphi}) \,\big|\, d\varphi - \sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} - n(0, \frac{1}{f}) \log R$$
$$+ \sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} + n(0, f) \log R,$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

From now on, meromorphic function means meromorphic in the whole complex plane. First of all, we introduce the positive logarithmic function.

Definition 3.4 *For $x \geq 0$,*

$$\log^+ x = \max\{\log x, 0\} = \begin{cases} \log x & \textit{if } x \geq 1 \\ 0 & \textit{if } o \leq x < 1. \end{cases}$$

Obviously, $\log^+ x$ is a continuous non-negative increasing function on $[0, \infty)$ satisfying $\log x = \log^+ x - \log^+ \frac{1}{x}$ and $|\log x| = \log^+ x + \log^+ \frac{1}{x}$.

Let f be a meromorphic function, Nevanlinna [18] introduced the following notations.

Definition 3.5 *For $0 < r < \infty$,*

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

Definition 3.6 *For $0 < r < \infty$,*

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

where $n(t,f)$ denotes the number of poles of f in the disc $|\, z\,| \le t$ counting multiplicities. $N(r,f)$ is called the counting function of f.

For $0 \le r < \infty$, $n(r,f)$ denotes the number of poles of $f(z)$ in $|z| \le r$ counting multiplicities; $\overline{n}(r,f)$ denotes the number of poles of $f(z)$ in $|z| \le r$ ignoring multiplicities; $n_{k)}(r,1/f)$ (resp.$n_{(k}(r,1/f)$) denotes the number of zeros of $f(z)$ in $|z| \le r$ with order $\le k$(resp.$\ge k$) counting multiplicities; $\overline{n}_{k)}(r,1/f)$ (resp.$\overline{n}_{(k}(r,1/f)$) denotes the number of zeros of $f(z)$ in $|z| \le r$ with order $\le k$(resp.$\ge k$) ignoring multiplicities.

Definition 3.7 *For $0 < r < \infty$, the function $T(r,f)$ defined by*

$$T(r,f) = m(r,f) + N(r,f)$$

is called the (Nevanlinna) characteristic function of f.

It is clear that $T(r,f)$ is a non-negative increasing function and a convex function of $\log r$. Let f be given in Theorem 2.1. It follows from the integration by parts in Riemann-Stieltjes integral, we have

$$\sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} = \int_0^R \frac{n(t,\frac{1}{f}) - n(0,\frac{1}{f})}{t} dt$$

and

$$\sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} = \int_0^R \frac{n(t,f) - n(0,f)}{t} dt.$$

On the other hand, the generalized Jensen's formula can be rewritten as

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log^+ |\, f(Re^{i\varphi}) \,|\, d\varphi + \sum_{\nu=1}^{N} \log \frac{R}{|\, b_\nu \,|} + n(0,f)\log R \\
=&\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| \frac{1}{f(Re^{i\varphi})} \right| d\varphi + \sum_{\mu=1}^{M} \log \frac{R}{|\, a_\mu \,|} + n(0,\frac{1}{f})\log R + \log |\, c_\lambda \,|\,.
\end{aligned}$$

Therefore, we obtain

$$m(R,f)+N(R,f)=m(R,\frac{1}{f})+N(R,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

that is,

$$T(R,f)=T(R,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

which is another form of the generalized Jensen's formula and is also known as the Nevanlinna-Jensen's formula.

Theorem 3.8 (Nevanlinna-Jensen's formula) *Let f be a meromorphic function , then, for $r>0$,*

$$T(r,f)=T(r,\frac{1}{f})+\log|\,c_\lambda\,|\,,$$

where c_λ is the first non-zero coefficient of the Laurent expansion of f at 0.

By the Nevanlinna-Jensen's formula, we can get the Nevanlinna's first fundamental theorem.

Theorem 3.9 (Nevanlinna's First Fundamental Theorem) *Let f be a meromorphic function and a be a finite complex number. Then, for $r>0$, we have*

$$T(r,\frac{1}{f-a})=T(r,f)+\log|c_\lambda|+\varepsilon(a,r),$$

where c_λ is the first non-zero coefficient of the Laurent expansion of $\frac{1}{f-a}$ at 0, and

$$|\varepsilon(a,r)|\le\log^+|a|+\log 2.$$

Usually, Nevanlinna's first fundamental theorem is written as

$$T(r,\frac{1}{f-a})=T(r,f)+O(1).$$

Now, we come to the most important theorem in the theory of value distribution, namely, Nevanlinna's second fundamental theorem.

Theorem 3.10 (Nevanlinna's Second Fundamental Theorem) *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}$, $1 \leq j \leq q$, be q distinct finite values $(q \geq 2)$. Then*

$$m(r, f) + \sum_{j=1}^{q} m(r, \frac{1}{f - a_j}) \leq 2T(r, f) - N_1(r) + S(r, f),$$

where $N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'})$ and

$$S(r, f) = m(r, \frac{f'}{f}) + m(r, \sum_{j=1}^{q} \frac{f'}{f - a_j}) + O(1).$$

Given $a \in \mathbb{C}$, by Nevanlinna's first fundamental theorem,

$$m(r, \frac{1}{f - a}) = T(r, f) - N(r, \frac{1}{f - a}) + O(1).$$

Hence, Nevanlinna's second fundamental theorem can be rewritten as follows.

Theorem 3.11 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \leq j \leq q$, be q distinct values $(q \geq 3)$. Then*

$$(q - 2)T(r, f) < \sum_{j=1}^{q} N(r, \frac{1}{f - a_j}) - N_1(r) + S(r, f),$$

where $N_1(r)$ and $S(r, f)$ are given as in Theorem 3.10.

Note that, in Theorem 3.11, if some $a_j = \infty$, then $N(r, \frac{1}{f - a_j})$ should be read as $N(r, f)$.

Let $n_1(t) = 2n(t, f) - n(t, f') + n(t, \frac{1}{f'})$ and let $\overline{n}(t, f)$ denote the number of distinct poles of f in $|z| \leq t$. Define

$$\overline{N}(r, f) = \int_0^r \frac{\overline{n}(t, f) - \overline{n}(0, f)}{t} dt + \overline{n}(0, f) \log r,$$

which is called the reduced counting function of f. Note that, if z_0 is a pole of f of order k in $|z| \leq t$, then z_0 is counted $k - 1$ times by $n_1(r)$. Similarly, for a finite value a, if z_0 is a zero of $f-a$ of order k in $|z| \le t$, then z_0 is also counted $k-1$ times by $n_1(r)$. Hence,

$$\sum_{j=1}^{q} N(r, \frac{1}{f-a_j}) - N_1(r) \le \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}).$$

Therefore, we have the third form of Nevanlinna's second fundamental theorem.

Theorem 3.12 *Let f be a non-constant meromorphic function and $a_j \in \mathbb{C}_\infty$, $1 \le j \le q$, be q distinct values ($q \ge 3$). Then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \overline{N}(r, \frac{1}{f-a_j}) + S(r,f),$$

where $S(r,f)$ is given as in Theorem 3.10.

Definition 3.13 *Let f be a non-constant meromorphic function and $a \in \mathbb{C}_\infty$. The deficiency of a with respect to f is defined by*

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,\frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

In Nevanlinna's second fundamental theorem, the remainder term $S(r,f)$ is a complicated object which can be estimated by using the method of logarithmic derivative. It turns out that $S(r,f)$ is small comparing to $T(r,f)$. In order to make it clear, we need the concept of the growth of meromorphic function.

Classically, we use the maximum modulus to measure the growth of an entire function.

Definition 3.14 *Let f be a meromorphic function. The order λ of f is defined to be*

$$\lambda = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$$

and the lower order μ of f is defined to be

$$\mu = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Definition 3.15 *Let $f(z)$ and $a(z)$ be meromorphic functions. If $T(r,a) = S(r,f)$, then $a(z)$ is called a small function of $f(z)$.*

Let f be an entire function. Define, for $r \geq 0$,

$$M(r,f) = \max_{|z \leq r|} |f(z)|.$$

Then the relation between $M(r,f)$ and $T(r,f)$ is given as follows.

Theorem 3.16 *Let $0 \leq r < R < \infty$ and f be an entire function, we have*

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

In particular,

$$T(r,f) \leq \log^+ M(r,f) \leq 3T(2r,f).$$

By Theorem 3.16, the order and lower order of an entire function are unambiguous. Now, we can state the properties of $S(r,f)$.

Lemma 3.17 *Let f be a non-constant meromorphic function. If f is of finite order, then*

$$m(r, \frac{f'}{f}) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$m(r, \frac{f'}{f}) = O(\log(rT(r,f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

Theorem 3.18 *Let f be a non-constant meromorphic function and $S(r,f)$ be defined in Theorem 3.10. If f is of finite order, then*

$$S(r,f) = O(\log r), \quad (r \to \infty).$$

If f is of infinite order, then

$$S(r,f) = O(\log(rT(r,f))), \quad (r \to \infty, r \notin E),$$

where E is a set of finite measure.

In the thesis, we will denote by $S(r,f)$ any quantity satisfy $S(r,f)=o(T(r,f))$ as $r\to\infty$ if f is of finite order, and $S(r,f)=o(T(r,f))$ as $r\to\infty, r\notin E$ if f is of infinite order, where E is a set of finite measure.

By Lemma 3.17, $m(r,\frac{f'}{f})=S(r,f)$. Moreover, Milloux [16] proved the following.

Theorem 3.19 *Let f be a non-constant meromorphic function and k be a positive integer and let*

$$\Psi(z)=\sum_{i=1}^{k}a_i(z)f^{(i)}(z),$$

where $a_1(z),a_2(z),\ldots,a_k(z)$ are small functions of f. Then

$$m(r,\frac{\Psi}{f})=S(r,f).$$

Now we record some well-known results on two meromorphic functions sharing four or five small functions as follows.

We have the generalization of second fundamental theorem for three small functions.

Theorem 3.20 [23] *Let f be a non-constant meromorphic function and $a_1(z)$, $a_2(z)$ and $a_3(z)$ are three distinct small function. Then*

$$T(r,f)<\sum_{j=1}^{3}\overline{N}(r,\frac{1}{f-a_j})+S(r,f),$$

Finally, we state a Nevanlinna theorem about the linear combination of meromorphic functions which will be needed in section 4.

Theorem 3.21 [23] *Suppose $f_1,\cdots,f_n$ are linearly independent meromorphic functions satisfying the following identity*

$$\sum_{j=1}^{n}f_j=1$$

Then for $1 \leq j \leq n$, we have

$$T(r, F_1) \leq \sum_{j=1}^{n} N(r, \frac{1}{f_n}) - \sum_{j=1}^{n} N(r, f_n) + N(r, D) - N(r, \frac{1}{D}) + o(T(r))$$

where D is the Wronskian of $f_1, \cdots, f_n$, and

$$T(r) = \max_{1 \leq k \leq n} \{T(r, f_k)\},$$

E is a set with finite linear measure.

4 Some Necessary Conditions for the Existence of Solution of Functional Equations

In this section, we study the necessary condition for the existence of solution of the functional equation (1.1) considered by Toda [21], K.-W. Yu and C.-C. Yang [25], and I. Lahiri and K.-W. Yu [13]. For completeness, we include their proofs.

Toda [21] first considered the necessary condition for the existence of entire solutions. Before stating the theorem, we need a lemma.

Lemma 4.1 [21] *Let $g_0, \cdots, g_p$ (≥ 1) be $p+1$ non-constant meromorphic functions in $|z| < \infty$ satisfying*

$$\sum_{i=0}^{p} \alpha_i g_i = 1, \ \alpha_0, \cdots, \alpha_p \neq 0, \tag{4.1}$$

with constant coefficients and $\delta(\infty, g_i) = 1$ $(i = 0, \cdots, p)$. Then , we have

$$\sum_{i=0}^{p} \theta_p(0, g_i) \leq p.$$

Here,

$$\theta_p(0, g_i) = 1 - \limsup_{r \to \infty} \frac{N_p(r, 0, g_i)}{T(r, g_i)}$$

and

$$N_p(r, 0, g_i) = \sum_{a_k \neq 0} \log^+ \frac{r}{|a_k|} + \min(\rho_0, p) \log r$$

where the summation is taken over the different zeros $a_k(\neq 0)$ *of* g_i *counted* $\min(\rho_k, p)$ *times at* a_k, ρ_k*(resp.* ρ_0*) being the order of multiplicity of zero of* g_i *at* a_k*(resp. 0).*

Proof. Suppose $g_0, \cdots, g_p$ are linearly independent, then the Wronskian $\Delta = W(g_0, \cdots, g_p) \neq 0$. By differentiating both sides of (4.1), we have

$$\sum_{i=0}^{p} (\alpha_i g_i)^{(\mu)} = 0, \ \mu = 1, \ldots, p \tag{4.2}$$

Now let $\widetilde{\Delta} = \Delta / g_0, \cdots, g_p$ and $\widetilde{\Delta_i} = W(g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_p) g_i / g_0, \cdots, g_p$. By (4.2)

$$\alpha_0, \cdots, \alpha_p \widetilde{\Delta} = \frac{1}{g_0, \cdots, g_p} \begin{vmatrix} \alpha_0 g_0 & \cdots & \alpha_p g_p \\ \vdots & & \vdots \\ (\alpha_0 g_0)^{(p)} & \cdots & (\alpha_p g_p)^{(p)} \end{vmatrix}$$

$$= \frac{1}{g_0, \cdots, g_p} \begin{vmatrix} \alpha_0 g_0 & \cdots & \alpha_{i-1} g_{i-1} & 1 & \alpha_{i+1} g_{i+1} & \cdots & \alpha_p g_p \\ \vdots & \cdots & (\alpha_{i-1} g_{i-1})^{(1)} & 0 & (\alpha_{i+1} g_{i+1})^{(1)} & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ (\alpha_0 g_0)^{(p)} & \cdots & (\alpha_{i-1} g_{i-1})^{(p)} & 0 & (\alpha_{i+1} g_{i+1})^{(p)} & \cdots & (\alpha_p g_p)^{(p)} \end{vmatrix}$$

$$= \frac{\alpha_0, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_p}{g_0, \cdots, g_p} \begin{vmatrix} g_0 & \cdots & g_{i-1} & 1 & g_{i+1} & \cdots & g_p \\ \vdots & \cdots & \vdots & 0 & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ g_0^{(p)} & \cdots & g_{i-1}^{(p)} & 0 & g_{i+1}^{(p)} & \cdots & g_p^{(p)} \end{vmatrix}$$

$$= \frac{\alpha_0, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_p}{g_i} \widetilde{\Delta}_i$$

which implies

$$g_i = \frac{\widetilde{\Delta}_i}{\alpha_i \widetilde{\Delta}}.$$

Consider

$$
\begin{aligned}
m(r,g_i) &= m(r,\frac{\widetilde{\Delta}_i}{\alpha_i\widetilde{\Delta}}) \le m(r,\widetilde{\Delta}_i) + m(r,\frac{1}{\widetilde{\Delta}}) + m(r,\alpha_i) \\
&\le m(r,\widetilde{\Delta}_i) + m(r,\frac{1}{\widetilde{\Delta}}) + N(r,\frac{1}{\widetilde{\Delta}}) + O(1) \\
&\le m(r,\widetilde{\Delta}_i) + T(r,\frac{1}{\widetilde{\Delta}}) + O(1) \le m(r,\widetilde{\Delta}_i) + T(r,\widetilde{\Delta}) + O(1) \\
&\le m(r,\widetilde{\Delta}_i) + m(r,\widetilde{\Delta}) + N(r,\widetilde{\Delta}) + O(1) \\
&\le N(r,\widetilde{\Delta}) + S(r) \\
&\le \sum_{j=0}^{p} N_p(r,0,g_j) + p\sum_{j=0}^{p} \overline{N}(r,g_j) + S(r) \\
&\le \sum_{j=0}^{p} N_p(r,0,g_j) + p\sum_{j=0}^{p} N(r,g_j) + S(r),
\end{aligned}
$$

where

$$
S(r) = \begin{cases} O(1) & ,\text{if } g_j \text{ are rational} \\ o\left(\sum_{j=0}^{p} T(r,g_j)\right) & ,r \notin E \end{cases} .
$$

Therefore,

$$
\begin{aligned}
T(r) &\equiv \max_{0\le i\le p}\{m(r,g_i) + N(r,g_i)\} \\
&\le \max_{0\le i\le p}\{\sum_{j=0}^{p} N_p(r,0,g_j) + p\sum_{j=0}^{p} N(r,g_j) + N(r,g_i) + S(r)\} \\
&\le \sum_{j=0}^{p} N_p(r,0,g_j) + (p+1)\sum_{j=0}^{p} N(r,g_j) + S(r) \qquad (4.3)
\end{aligned}
$$

By definition of $\theta_p(0,g_i)$ and $\delta(\infty,g_i)$, for arbitary $\varepsilon > 0$, there exists a positive number r_0 such that for any given $r \ge r_0$, we have

$$
N_p(r,0,g_i) < (1-\theta_p(0,g_i)+\varepsilon)T(r,g_i) \qquad (4.4)
$$

and

$$
N(r,g_i) < (1-\delta(\infty,g_i)+\varepsilon)T(r,g_i). \qquad (4.5)
$$

Since $\delta(\infty, g_i) = 1$, for all $i = 0, \cdots p$, by (4.3), (4.4), (4.5), we have

$$
\begin{aligned}
T(r) \leq & \sum_{j=0}^{p}(1-\theta_p(0,g_j)+\varepsilon)T(r,g_j) + (p+1)\sum_{j=0}^{p}(1-\delta(\infty,g_j)+\varepsilon)T(r,g_j) \\
\leq & \sum_{j=0}^{p}(1-\theta_p(0,g_j)+\varepsilon)T(r) + (p+1)\sum_{j=0}^{p}\varepsilon T(r)
\end{aligned}
$$

which imply

$$
\begin{aligned}
0 \leq & \sum_{j=0}^{p}(1-\theta_p(0,g_j)+\varepsilon)T(r) + (p+1)\sum_{j=0}^{p}\varepsilon T(r) - T(r) \\
= & \left(\sum_{j=0}^{p}(1-\theta_p(0,g_j)+\varepsilon) + (p+1)\sum_{j=0}^{p}\varepsilon - 1\right)T(r) \\
= & \left(p+1-\sum_{j=0}^{p}\theta_p(0,g_j) + (p+1)\varepsilon + (p+1)^2\varepsilon - 1\right)T(r) \\
= & \left(p-\sum_{j=0}^{p}\theta_p(0,g_j) + (p+1)\varepsilon + (p+1)^2\varepsilon\right)T(r).
\end{aligned}
$$

Therefore, we obtain

$$
p-\sum_{j=0}^{p}\theta_p(0,g_j) + (p+1)\varepsilon + (p+1)^2\varepsilon \geq 0,
$$

which is true for all $\varepsilon > 0$. Hence,

$$
\sum_{j=0}^{p}\theta_p(0,g_j) \leq p.
$$

In the case when $g_0, \cdots, g_p$ are linearly dependent. The original functional equation can be reduce to $\sum_{i=0}^{s}\gamma_i g_i = 1$, where $g_0, \cdots, g_s$ are linearly independent with $s < p$ as follows. By renumbering, we may assume that $\{g_0, \cdots, g_s\}$ is the maximal linear independent subset of $\{g_0, \cdots, g_p\}$. Then

$$
g_k = \sum_{j=0}^{s}\beta_{k,j}g_j, \ s+1 \leq k \leq p,
$$

where $\beta_{k,j}$ are constants. Hence,

$$\begin{aligned} 1 &= \sum_{i=0}^{p} \alpha_i g_i \\ &= \sum_{j=0}^{s} \alpha_j g_j + \sum_{k=s+1}^{p} \alpha_k \sum_{j=0}^{s} \beta_{k,j} g_j \\ &= \sum_{j=0}^{s} \left(\alpha_j + \sum_{k=s+1}^{p} \alpha_k \beta_{k,j} \right) g_j \\ &= \sum_{j=0}^{s} \gamma_j g_j, \end{aligned}$$

where $\gamma_j = \left(\alpha_j + \sum_{k=s+1}^{p} \alpha_k \beta_{k,j} \right)$.

Clearly, we may assume that $\gamma_j \neq 0,\ 0 \leq j \leq s$. Thus, we have

$$\sum_{j=0}^{s} \gamma_j g_j = 1,\ \gamma_j \neq 0$$

By the result of the first part, we have

$$\sum_{j=0}^{s} \theta_s(0, g_j) \leq s$$

Obviously, for $s < p$, we have $\theta_p(0, g_i) \leq \theta_s(0, g_i) \leq 1$. Hence

$$\begin{aligned} \sum_{j=0}^{p} \theta_p(0, g_j) &= \sum_{j=0}^{s} \theta_p(0, g_j) + \sum_{j=s+1}^{p} \theta_p(0, g_j) \\ &\leq \sum_{j=0}^{s} \theta_s(0, g_j) + \sum_{j=s+1}^{p} \theta_p(0, g_j) \\ &\leq s + (p - s) = p \end{aligned}$$

❑

Next, we state the result of Toda [21].

Theorem 4.2 [21] *Let* $f_0, \cdots, f_p$ $(p \geq 1)$ *be* $p+1$ *non-constant entire functions and let* $a_0, \cdots, a_p$ *be* $p+1$ *meromorphic functions* $(\neq 0)$ *in* $|z| < \infty$ *such that*

$$T(r, a_i) = o(T(r, f_i)),\ (i = 0, \cdots, p)$$

Then, if the following functonal equation

$$\sum_{i=0}^{p} a_i f_i^{n_i} = 1$$

holds for some integers $n_0, \cdots, n_p (\geq 1)$*, it must be*

$$\sum_{i=0}^{p} \frac{1}{n_i} \geq \frac{1}{p}.$$

Proof. We know that

$$T(r, f^n) \sim nT(r, f),\ (r \to \infty)$$

Since

$$\begin{aligned} T(r, a_i f_i^{n_i}) &\leq T(r, a_i) + n_i T(r, f_i) = n_i T(r, f_i) + S(r, f_i) \\ &\leq n_i T(r, f_i) + T(r, a_i) + S(r, f_i) \\ &\leq T(r, a_i f_i^{n_i}) + S(r, f_i) \end{aligned}$$

Thus, we get

$$T(r, a_i f_i^{n_i}) \sim n_i T(r, f_i),\ (r \to \infty)$$

By definition of $N_p(r, \frac{1}{g_i})$, we have

$$N_p(r, \frac{1}{a_i f_i^{n_i}}) \leq N_p(r, \frac{1}{a_i}) + p\overline{N}(r, \frac{1}{f_i^{n_i}}) = N_p(r, \frac{1}{a_i}) + p\overline{N}(r, \frac{1}{f_i})$$

$$\frac{N_p(r, \frac{1}{a_i f_i^{n_i}})}{T(r, a_i f_i^{n_i})} \leq \frac{N_p(r, \frac{1}{a_i})}{T(r, a_i f_i^{n_i})} + \frac{p\overline{N}(r, \frac{1}{f_i})}{T(r, a_i f_i^{n_i})}$$

Therefore,

$$\limsup_{r\to\infty} \frac{N_p(r, \frac{1}{a_i f_i^{n_i}})}{T(r, a_i f_i^{n_i})} \leq \limsup_{r\to\infty} \frac{p\overline{N}(r, \frac{1}{f_i})}{T(r, a_i f_i^{n_i})} = \frac{p}{n_i} \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f_i})}{T(r, a_i f_i)} \leq \frac{p}{n_i}$$

By definition of θ_p,

$$1 - \limsup_{r\to\infty} \frac{N_p(r, \frac{1}{a_i f_i^{n_i}})}{T(r, a_i f_i^{n_i})} = \theta_p(0, a_i f_i^{n_i}) \geq 1 - \frac{p}{n_i}.$$

Since each $a_i(z)$ is a small function, by Lemma 4.1, we have

$$\sum_{i=0}^{p}\left(1-\frac{p}{n_i}\right) \leq \sum_{i=0}^{p}\theta_p(0, a_i f_i^{n_i}) \leq p$$

which implies that

$$\sum_{i=0}^{p}\frac{1}{n_i} \geq \frac{1}{p}.$$

❑

For the case $p = 3$ in (1.1), since $\frac{3}{n} \geq \frac{1}{2}$ if $n \leq 6$. It follows from Theorem 4.2, the functional equation admits no entire solution if $n > 6$. In general, we have the following consequence.

Corollary 4.3 *Let f_i and a_i be defined as in Theorem 4.5. If*

$$\sum_{i=0}^{p}\frac{1}{n_i} < \frac{1}{p},$$

then $\sum_{i=0}^{p} a_i(z) f_i^{n_i}(z) = 1$ *cannot hold.*

Before we prove the result of K.-W. Yu and C.-C. Yang, we need the following lemma.

Lemma 4.4 [25] *Let $F_1, \cdots, F_p$ be p (≥ 2) linearly independent non-constant meromorphic functions. If*

$$\sum_{j=1}^{p} F_j = 1,$$

then,

$$T(r) \leq 2(p-1)\left[\sum_{j=1}^{p}\overline{N}(r, F_j) + \sum_{j=1}^{p}\overline{N}(r, \frac{1}{F_j})\right] + S(r),$$

where $T(r) = \max_{1\leq j\leq p} T(r, F_j)$ *and* $S(r) = o\left(\sum_{j=1}^{p} T(r, F_j)\right)$*, with a possible exceptional set of r of finite linear measure.*

Proof. Let $D = W(F_1, F_2, \cdots, F_p)$. As in proof of Lemma 4.1, we have

$$F_j = \frac{\Delta_j}{\Delta},$$

where

$$\Delta = \frac{D}{F_1 \cdots F_p} \text{ and } \Delta_j = \frac{F_j D_j}{F_1 \cdots F_p}$$

with D_j are the minor determinants of elements F_j, $1 \le j \le p$. By the first fundamental theorem,

$$\begin{aligned}
m(r, F_j) &\le m(r, \Delta_j) + m(r, \frac{1}{\Delta}) + O(1) \\
&\le m(r, \Delta_j) + m(r, \frac{1}{\Delta}) + N(r, \frac{1}{\Delta}) - N(r, \frac{1}{\Delta}) + O(1) \\
&\le m(r, \Delta_j) + T(r, \frac{1}{\Delta}) - N(r, \frac{1}{\Delta}) + O(1) \\
&\le m(r, \Delta_j) + T(r, \Delta) - N(r, \frac{1}{\Delta}) + O(1) \\
&\le m(r, \Delta_j) + m(r, \Delta) + N(r, \Delta) - N(r, \frac{1}{\Delta}) + O(1) \\
&\le N(r, \Delta) - N(r, \frac{1}{\Delta}) + S(r)
\end{aligned} \tag{4.6}$$

We know that if z_0 is not a pole and a zero of $F_j, 1 \le j \le p$, then z_0 is not a pole of Δ. Thus, the poles of Δ come from the zeros and poles of F_j, $1 \le j \le p$. But some of them may be canceled in the expression of Δ, that is, some zeros or poles of $F_1, \cdots, F_j$ may not be a pole of Δ. Suppose that z_0 is a pole or a zero of F_1 and is not a pole and a zero of F_j, for all $j = 2 \cdots p$. Then $\frac{F_1^{(n)}}{F_1}$ has a pole of order at most $p - 1$ at z_0. By these simple argument we have

$$N(r, \Delta) \le (p-1) \left[\sum_{j=1}^{p} \overline{N}(r, F_j) + \sum_{j=1}^{p} \overline{N}(r, \frac{1}{F_j}) \right]$$

On the other hand, by the first fundamental theorem and (4.6),

$$
\begin{aligned}
T(r) &= \max_{1\le j\le p} T(r,F_j) \\
&= \max_{1\le j\le p}[m(r,F_j)+N(r,F_j)] \\
&\le N(r,\Delta) - N(r,\frac{1}{\Delta}) + \max_{1\le j\le p} N(r,F_j) + S(r) \\
&\le (p-1)\left[\sum_{j=1}^{p}\overline{N}(r,F_j) + \sum_{j=1}^{p}\overline{N}(r,\frac{1}{F_j})\right] \\
&\quad + \max_{1\le j\le p}\left[N(r,F_j) - N(r,\frac{1}{\Delta})\right] + S(r)
\end{aligned}
\tag{4.7}
$$

Since

$$N(r,F_j) = N(r,\frac{F_j\Delta}{\Delta}) \le N(r,F_j\Delta) + N(r,\frac{1}{\Delta})$$

and

$$
F_j\Delta = \begin{vmatrix}
F_1^{(1)}/F_1 & \cdots & F_{j-1}^{(1)}/F_{j-1} & F_{j+1}^{(1)}/F_{j+1} & \cdots & F_p^{(1)}/F_p \\
F_1^{(2)}/F_1 & \cdots & F_{j-1}^{(2)}/F_{j-1} & F_{j+1}^{(2)}/F_{j+1} & \cdots & F_p^{(2)}/F_p \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
F_1^{(p-1)}/F_1 & \cdots & F_{j-1}^{(p-1)}/F_{j-1} & F_{j+1}^{(p-1)}/F_{j+1} & \cdots & F_p^{(p-1)}/F_p
\end{vmatrix},
$$

we have

$$
\begin{aligned}
N(r,F_j) - N(r,\frac{1}{\Delta}) &\le (p-1)\left[\sum_{i\ne j}\overline{N}(r,F_i) + \sum_{i\ne j}\overline{N}(r,\frac{1}{F_i})\right] \\
&\le (p-1)\left[\sum_{j=1}^{p}\overline{N}(r,F_j) + \sum_{j=1}^{p}\overline{N}(r,\frac{1}{F_j})\right]
\end{aligned}
\tag{4.8}
$$

Combine (4.7) and (4.8), we have

$$T(r) \le 2(p-1)\left[\sum_{j=1}^{p}\overline{N}(r,F_j) + \sum_{j=1}^{p}\overline{N}(r,\frac{1}{F_j})\right] + S(r)$$

❑

Remark. If $\delta(\infty,F_j)=1,\ 1\le j\le p$ in Lemma 4.4, then $\overline{N}(r,F_j)=S(r)$ and

$N(r, F_j) = S(r)$. In this case, (4.6) and (4.7) can be reduced as follows.

$$\begin{aligned} m(r, F_j) &\leq m(r, \Delta_j) + m(r, \frac{1}{\Delta}) + O(1) \\ &\leq m(r, \Delta_j) + T(r, \frac{1}{\Delta}) + O(1) \\ &\leq m(r, \Delta_j) + T(r, \Delta) + O(1) \\ &\leq m(r, \Delta_j) + m(r, \Delta) + N(r, \Delta) + O(1) \\ &\leq N(r, \Delta) + S(r) \end{aligned}$$

and

$$\begin{aligned} T(r) &= \max_{1\leq j\leq p} T(r, F_j) \\ &= \max_{1\leq j\leq p} [m(r, F_j) + N(r, F_j)] \\ &\leq N(r, \Delta) + S(r) \\ &\leq (p-1)\sum_{j=1}^{p} \overline{N}(r, \frac{1}{F_j}) + S(r) \\ &\leq (p-1)\sum_{j=1}^{p} \frac{1}{k_j} N(r, \frac{1}{f_j^{k_j}}) + S(r) \\ &\leq (p-1)\sum_{j=1}^{p} \frac{1}{k_j} T(r, \frac{1}{f_j^{k_j}}) + S(r) \\ &\leq (p-1)\sum_{j=1}^{p} \frac{1}{k_j} T(r) + S(r) \end{aligned}$$

Then it reduced to Theorem 4.2.

Now, we come to a result of K.-W. Yu and C.-C. Yang.

Theorem 4.5 [25] *Let $f_1, \ldots, f_p$ be p (≥ 2) non-constant meromorphic functions and let $a_1, \ldots, a_p$ be p meromorphic functions $(\not\equiv 0)$ such that*

$$T(r, a_j) = o(T(r, f_j)), \ 1 \leq j \leq p.$$

Then, if the f_j and a_j $(1 \leq j \leq p)$ satisfy equation

$$\sum_{j=1}^{p} a_j(z) f_j^{k_j}(z) = 1,$$

for some integer $k_1, \ldots, k_p$*, then*

$$\sum_{j=1}^{p} \frac{1}{k_j} \geq \frac{1}{4(p-1)}.$$

Proof. By Lemma 4.4, we need to consider $\overline{N}(r, \frac{1}{a_j f_j^{k_j}})$ and $\overline{N}(r, a_j f_j^{k_j})$ as follows. With a simple calculation, we have

$$\begin{aligned}
\overline{N}(r, \frac{1}{a_j f_j^{k_j}}) &\leq \overline{N}(r, \frac{1}{a_j}) + \frac{1}{k_j} N(r, \frac{1}{f_j^{kj}}) \\
&\leq T(r, \frac{1}{a_j}) + \frac{1}{k_j} T(r, \frac{1}{f_j^{k_j}}) \\
&\leq T(r, a_j) + \frac{1}{k_j} T(r, f_j^{k_j}) + O(1) \\
&\leq \frac{1}{k_j} T(r) + S(r) \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
\overline{N}(r, a_j f_j^{k_j}) &\leq \overline{N}(r, a_j) + \frac{1}{k_j} N(r, f_j^{kj}) \\
&\leq T(r, a_j) + \frac{1}{k_j} T(r, f_j^{k_j}) + O(1) \\
&\leq \frac{1}{k_j} T(r) + S(r) \qquad (4.10)
\end{aligned}$$

By (4.9), (4.10) and Lemma 4.4, we obtain

$$T(r) \leq 4(p-1) \left(\sum_{j=1}^{p} \frac{1}{k_j} \right) T(r) + S(r)$$

Thus, we have $\sum_{j=1}^{p} \frac{1}{k_j} \geq \frac{1}{4(p-1)}$ and the proof is finished. ❑

Finally, we consider the following result of I. Lahiri and K.-W. Yu. Before we start, we need some lemmas.

Lemma 4.6 *Let* f *be a meromorphic function and* p *be a positive integer. Then we have*

$$N_p(r, 0, f) \leq p\overline{N}(r, 0, f).$$

Proof. Clearly, It follows from the definition of $N_p(r, 0, f)$. ❑

Lemma 4.7 [15] *Let $p \geq 2$ and $F_1, \cdots, F_p$ be linearly independent non-constant meromorphic functions. If*

$$\sum_{j=1}^{p} F_j(z) = 1,$$

then we have

$$T(r, F_1) < \sum_{j=1}^{p} N_{p-1}(r, \frac{1}{F_j}) + A_p \sum_{j=1}^{p} \overline{N}(r, F_j) + o(T(r)),$$

where $T(r) = \max_{1 \leq j \leq p} T(r, F_j)$ *with* $r \notin E$ *and* $A_p = \begin{cases} \frac{1}{2} & \text{if } p = 2 \\ \frac{2p-3}{3} & \text{if } p = 3, 4, 5 \\ \frac{2p+1-2\sqrt{2p}}{2} & \text{if } p \geq 6 \end{cases}$.

Proof. By Theorem 3.21, we have

$$T(r, F_1) < \sum_{j=1}^{p} N(r, \frac{1}{F_j}) - \sum_{j=2}^{p} N(r, F_j) + N(r, D) - N(r, \frac{1}{D}) + o(T(r)))$$

where D is the Wronskian of $F_1, \cdots, F_p$. Let

$$N(r) = \sum_{j=1}^{p} N(r, \frac{1}{F_j}) - \sum_{j=1}^{p} N(r, F_j) + N(r, D) - N(r, \frac{1}{D})$$

and

$$N^*(r) = \sum_{j=1}^{p} N_{p-1}(r, \frac{1}{F_j}) + A_p \sum_{j=1}^{p} \overline{N}(r, F_j).$$

It suffice to show that

$$N(r) \leq N^*(r) \tag{4.11}$$

We say that z is a b-point of f if $f(z) - b = 0$. Then we define the following functions:

$$\mu_f^b(z) = \begin{cases} m, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \geq 1, \\ 0, & z \text{ is not a } b\text{-point of } f, \end{cases}$$

$$\overline{\mu}_f^b(z) = \begin{cases} 1, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \geq 1, \\ 0, & z \text{ is not a } b\text{-point of } f, \end{cases}$$

and

$$\nu_f^b(z) = \begin{cases} m, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \leq n-1, \\ n-1, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m > n-1, \\ 0, & z \text{ is not a } b\text{-point of } f. \end{cases}$$

Let

$$\mu = \sum_{j=1}^{p} \mu_{F_i}^0 - \sum_{j=2}^{p} \mu_{F_i}^\infty + \mu_D^\infty - \mu_D^0 \text{ and } \mu^* = \sum_{j=1}^{p} \nu_{F_i}^0 + A_p \sum_{j=1}^{p} \overline{\mu}_{F_i}^\infty.$$

In order to prove (4.11), it is reduced to show that $\mu(z) \leq \mu^*(z)$. Now, consider the following two cases.

Case 1. z is not a pole of F_j, $1 \leq j \leq p$.

In this case, z is also not a pole of D. This implies that

$$\mu_D^\infty(z) = 0, \ \sum_{j=2}^{p} \mu_{F_j}^\infty = 0, \text{ and } \sum_{j=1}^{p} \overline{\mu}_{F_j}^\infty = 0. \tag{4.12}$$

Since z is a zero of $F_i^{(p-1)}$ with multiplicity at least $\mu_{F_j}^0(z) - \nu_{F_j}^0(z)$, the number of zeros of D with multiplicity is at least $\sum_{j=1}^{p} \left(\mu_{F_j}^0(z) - \nu_{F_j}^0(z)\right)$. Hence we have

$$\mu_D^0(z) \geq \sum_{j=1}^{p} \left(\mu_{F_j}^0(z) - \nu_{F_j}^0(z)\right). \tag{4.13}$$

By (4.12), (4.13) and the definitions of μ and μ^*, we obtain that

$$\mu(z) \leq \mu^*(z).$$

Case 2. z is a pole of some F_j.

We consider two subcases:

Subcase 1. z is not a pole of F_1. Without loss of generality, we may assume that z is a zero of F_i with multicity m_i, $i = 1, \cdots, k$, $1 \leq k \leq n-2$ and a pole of F_i

with multicity m_i, $i = k+1, \cdots, n$. Since $F_1+\cdots+F_n = 1$, we have $D = (-1)^{n+1}D_1$, where

$$D_1 = \begin{vmatrix} F_1' & \cdots & F_{n-1}' \\ \vdots & \ddots & \vdots \\ F_1^{(n-1)} & \cdots & F_{n-1}^{(n-1)} \end{vmatrix}$$

Let

$$q = \sum_{i=1}^{k}(m_i - \nu_{F_i}^0(z)) - \sum_{i=k+1}^{n-1} m_i - \frac{(n+k)(n-k-1)}{2}$$

If $q \geq 0$, then z is a zero of D with multiplicity at least q. Otherwise, z is a pole of D with multiplicity as least $-q$. Thus, we have

$$\mu_D^0(z) - \mu_D^\infty(z) \geq q$$

This implies that

$$\begin{aligned} \mu_D^0(z) - \mu_D^\infty(z) &\geq \sum_{i=1}^{k}(m_i - \nu_{F_i}^0(z)) - \sum_{i=k+1}^{n-1} m_i - \frac{(n+k)(n-k-1)}{2} \\ \Rightarrow \sum_{i=1}^{k} \nu_{F_i}^0 + \frac{(n+k)(n-k-1)}{2} &\geq \mu_D^\infty(z) - \mu_D^0(z) + \sum_{i=1}^{k} \mu_{F_i}^0 - \sum_{i=k+1}^{n-1} \mu_{F_i}^\infty \\ \Rightarrow \sum_{i=1}^{k} \nu_{F_i}^0 + \frac{(n+k)(n-k-1)}{2} &\geq \mu(z) + \mu_{F_n}^\infty \end{aligned}$$

Since $\mu_{F_n}^\infty \geq 1$, we have

$$\mu(z) \leq \sum_{i=1}^{k} \nu_{F_i}^0 + \frac{(n+k)(n-k-1)}{2} - 1,$$

and

$$\mu^*(z) = \sum_{i=1}^{k} \nu_{F_i}^0 + A_n(n-k).$$

It is clearly that $\frac{(n+k)(n-k-1)}{2} - 1 \leq A_n(n-k)$. Thus, we have

$$\mu(z) \leq \mu^*(z).$$

Subcase 2. z is a pole of F_1. Without loss of generality, we may assume that z is a pole of $F_1, F_{k+1}, \cdots, F_n$ with multicity m_i respectively, and a pole of F_i with multicity m_i, $i = 2, \cdots, k,\ 2 \le k \le n-1$. Since $F_1 + \cdots + F_n = 1$, we have $D = D_2$, where

$$D_2 = \begin{vmatrix} F_2' & \cdots & F_n' \\ \vdots & \ddots & \vdots \\ F_2^{(n-1)} & \cdots & F_n^{(n-1)} \end{vmatrix}$$

Let

$$q = \sum_{i=2}^{k}(m_i - \nu_{F_i}^0(z)) - \sum_{i=k+1}^{n} m_i - \frac{(n-k)(n+k-1)}{2}$$

If $q \ge 0$, then z is a zero of D with multiplicity at least q. Otherwise, z is a pole of D with multiplicity as least $-q$. Thus, we have

$$\mu_D^0(z) - \mu_D^\infty(z) \ge q$$

This implies that

$$\mu_D^0(z) - \mu_D^\infty(z) \ge \sum_{i=2}^{k}(m_i - \nu_{F_i}^0(z)) - \sum_{i=k+1}^{n} m_i - \frac{(n-k)(n+k-1)}{2}$$

$$\Rightarrow \sum_{i=2}^{k} \nu_{F_i}^0 + \frac{(n-k)(n+k-1)}{2} \ge \mu_D^\infty(z) - \mu_D^0(z) + \sum_{i=1}^{k} \mu_{F_i}^0 - \sum_{i=k+1}^{n} \mu_{F_i}^\infty$$

Then we have

$$\sum_{i=1}^{k} \nu_{F_i}^0 + \frac{(n-k)(n+k-1)}{2} \ge \mu(z).$$

and

$$\mu^*(z) = \sum_{i=2}^{k} \nu_{F_i}^0 + A_n(n-k+1).$$

Since $(n+k-1)(n-k)/2 \le A_n(n-k+1)$ for $1 \le k \le n-1$. Hence,

$$\mu(z) \le \mu^*(z).$$

which completes the proof. ❑

Combine Lemma 4.6 and Lemma 4.7, we have

Lemma 4.8 [13] *Let p be a positive integer and $p \ge 2$. Let $F_1, \cdots, F_p$ be p linearly independent non-constant meromorphic functions. If*

$$\sum_{j=1}^{p} F_j(z) = 1,$$

then we have

$$T(r) \leq (p-1)\sum_{j=1}^{p} \overline{N}(r, \frac{1}{F_j}) + A_p \sum_{j=1}^{p} \overline{N}(r, F_j) + o(T(r)),$$

where $T(r)$ and A_p is defined as in Lemma 4.7.

Proof. Without loss of generality, we may assume that $T(r) = \max T(r, F_i) = T(r, F_1)$. Therefore, by Lemma 4.7,

$$T(r) \leq (p-1)\sum_{j=1}^{p} N_{p-1}(r, \frac{1}{F_j}) + A_p \sum_{j=1}^{p} \overline{N}(r, F_j) + o(T(r)).$$

Furthermore, by Lemma 4.6,

$$T(r) \leq (p-1)\sum_{j=1}^{p} \overline{N}(r, \frac{1}{F_j}) + A_p \sum_{j=1}^{p} \overline{N}(r, F_j) + o(T(r)).$$

❑

Now, we can use Lemma 4.6, Lemma 4.7 and Lemma 4.8 to prove the result of I. Lahiri and K.-W. Yu.

Theorem 4.9 [13] *Let p be a positive integer and $p \geq 2$. Let $f_1, \cdots, f_p$ be p non-constant meromorphic functions and $a_1, \cdots, a_p$ be p non-constant meromorphic functions such that*

$$T(r, a_j) = o(T(r, f_j)), \quad 1 \leq j \leq p,$$

as $r \to +\infty$ and $r \notin E$. If f_j and a_j, $1 \leq j \leq p$, satisfy

$$\sum_{j=1}^{p} a_j(z) f_j^{k_j}(z) = 1$$

for some positive integers $k_1, \cdots, k_p$, then

$$\sum_{j=1}^{p} \frac{1}{k_j} \geq \frac{1}{p-1+A_p}$$

where A_p is defined as in Lemma 4.7.

Proof. Suppose that $F_j = a_j f_j^{k_j}, 1 \le j \le p$ are linearly independent. Let $T(r) = \max_{1\le j\le p} T(r, F_j)$ and $S(r) = o\left(\sum_{j=1}^{p} T(r, f_j)\right)$. As in Theorem 4.5, we have the following inequality.

$$\begin{aligned}\overline{N}(r, \frac{1}{a_j f_j^{k_j}}) &\le \overline{N}(r, \frac{1}{a_j}) + \frac{1}{k_j} N(r, \frac{1}{f_j^{k_j}}) \\ &\le T(r, \frac{1}{a_j}) + \frac{1}{k_j} T(r, \frac{1}{f_j^{k_j}}) \\ &\le T(r, a_j) + \frac{1}{k_j} T(r, f_j^{k_j}) + O(1) \\ &\le \frac{1}{k_j} T(r) + S(r) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\overline{N}(r, a_j f_j^{k_j}) &\le \overline{N}(r, a_j) + \frac{1}{k_j} N(r, f_j^{k_j}) \\ &\le T(r, a_j) + \frac{1}{k_j} T(r, f_j^{k_j}) \\ &\le \frac{1}{k_j} T(r) + S(r) \end{aligned} \tag{4.15}$$

By (4.14), (4.15) and Lemma 4.8, we obtain that

$$T(r) \le (p - 1 + A_p) \left(\sum_{j=1}^{p} \frac{1}{k_j}\right) T(r) + S(r).$$

Thus, we have $\sum_{j=1}^{p} \frac{1}{k_j} \ge \frac{1}{p-1+A_p}$.

In the case $F_j = a_j f_j^{k_j}$, $1 \le j \le p$ are linearly dependent, we may find a maximal linearly independent subset $\{F_{j_1}, \cdots, F_{j_q}\}$ of $\{F_j\}$ such that

$$b_1(z) F_{j_1}(z) + \cdots + b_q(z) F_{j_q}(z) = 1$$

where $b_l(z)$ are constant multiply of $a_j(z)$ such that $b_l(z) = o(T(r, F_{j_l}))$. By the case above and A_p is increasing with p, we have

$$\sum_{j=1}^{p} \frac{1}{k_j} \ge \sum_{l=1}^{q} \frac{1}{k_{j,l}} \ge \frac{1}{q-1+A_q} \ge \frac{1}{p-1+A_p}.$$

Thus, we finish the proof. ❏

5 The Distribution of Zeros and Poles of a Class of Meromorphic Functions

In this section, we will prove our main result of this thesis. Namely, theorem 5.3 below.

Now, we consider the following problem:

Given meromorphic functions $f_1, \cdots, f_p$ and positive integers $k_1, \cdots, k_p$. Set

$$F(z) = \sum_{j=1}^{p} R_j(z) f_j^{k_j}(z),$$

where $R_1(z), \cdots, R_p(z)$ are rational functions. What is the value distribution of $F(z)$?

In fact, we will prove that, under some conditions, $F(z)$ has infinitely many zeros or poles or some of partial sums (one of which may be F) are equal to zero identically.

The first result on this problem was proved by Toda. In fact, Toda proved the following result.

Theorem 5.1 [21] *Let $f_1, \cdots, f_p$ be $p\ (\geq 2)$ non-constant entire functions and let $k_1, \cdots, k_p$ be p integers not less than one such that at least one of $f_i^{k_i}/f_j^{k_j}\ (i \neq j)$ is transcendental and*

$$\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{p-1}.$$

Then, for rational functions $R_j(z)(\neq 0)(j = 1, \cdots, p)$

$$F(z) \equiv \sum_{j=1}^{p} R_j(z) f_j^{k_j}(z)$$

has infinitely many zeros or some of partial sums (one of which may be F) are equal to zero identically.

Later, K.-W. Yu and C.-C. Yang [25] considered the above problem for meromorphic functions and state the following result without proof. For completeness, we provide a proof.

Theorem 5.2 [25] *Let $f_1, \cdots, f_p$ be p (≥ 2) non-constant transcendental meromorphic functions and let $k_1, \cdots, k_p$ be p integers not less than one and at least one of $\dfrac{f_i^{k_i}}{f_j^{k_j}}$ is transcendental such that*

$$\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{4(p-1)}.$$

Then we have

$$F(z) = \sum_{j=1}^{p} R_j(z) f_j^{k_j},$$

where $R_j(z)$ $(\neq 0)$, $1 \leq j \leq p$ are rational functions, has infinitely many zeros or poles, or some of the partial sums of F are equal to zero identically.

Proof. Suppose that the assertion is false. This means that $F(z)$ has finitely many zeros and poles and has no partial sum is equal to zero. In particular, $F(z) \neq 0$. We may assume that $F(z)$ has zeros $\{a_1, \cdots, a_n\}$ and poles $\{b_1, \cdots, b_m\}$, counting multicity. Therefore, $\prod_{s=1}^{m}(z-b_s)F(z)/\prod_{t=1}^{n}(z-a_t)$ has no zeros and poles. Then we can write $F(z)$ as

$$F(z) = R(z)e^{f(z)},$$

where $R(z) = \prod_{t=1}^{n}(z-a_t)/\prod_{s=1}^{m}(z-b_s)$ is rational and $f(z)$ is entire. Let

$$g_j(z) = -\frac{f(z)}{k_j},\ 1 \leq j \leq p.$$

It is obvious that $g_j(z)$ is entire, $1 \leq j \leq p$. From $F(z) = \sum_{j=1}^{p} R_j(z) f_j^{k_j}$, we get

$$1 = \sum_{j=1}^{p} R^{-1} R_j (f_j e^{g_j})^{k_j}$$

Since R_jR^{-1} is rational which implies that

$$T(r, R_iR^{-1}) = O(\log r) \text{ as } r \to \infty,\ j = 1, \cdots, p.$$

By assumption, $\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{4(p-1)}$, some of $f_je^{g_j}$ must be rational. Otherwise, all $f_je^{g_j}$ are transcendental meromorphic function, we have

$$T(r, R_jR^{-1}) = o(T(r, f_je^{g_j})), \text{ for all } 1 \leq j \leq p.$$

By Theorem 4.5, $\sum_{j=1}^{p} \frac{1}{k_j} \geq \frac{1}{4(p-1)}$ which is impossible. Actually, the number s of j such that $f_je^{g_j}$ is rational is less than $p-2$. In fact, if $s \geq p-1$, it forces that all of $f_ie^{g_i}$ are rational. In this case

$$\frac{(f_ie^{g_i})^{k_i}}{(f_je^{g_j})^{k_j}} = \frac{f_i^{k_i}}{f_j^{k_j}}$$

is rational, which is a contradiction. Thus,

$$1 \leq s \leq p-2.$$

Suppose that $f_1e^{g_1}, \cdots, f_se^{g_s}$ are rational, so that $\sum_{j=1}^{s} R^{-1}R_j(f_je^{g_j})^{k_j}$ is rational and is not equal to 1 identically since $\sum_{j=s+1}^{p} R_jf_j^{k_j}$ is not equal to zero identically. Therefore,

$$\begin{aligned}\sum_{j=s+1}^{p} R_jR^{-1}(f_je^{g_j})^{k_j} &= 1 - \sum_{j=1}^{s} R_jR^{-1}(f_je^{g_j})^{k_j} \\ &\equiv B(z), \qquad (5.1)\end{aligned}$$

where $B(z)$ is a nonzero rational function.

We can write (5.1) as follows

$$\sum_{j=s+1}^{p} P_j(f_je^{g_j})^{k_j} = 1$$

where $P_j = \frac{R_jR^{-1}}{B}$ is rational, $s+1 \leq j \leq p$.

Since $f_j e^{g_j}$ are transcendental and $P_j(\neq 0)$ are rational for all $j = s+1, \cdots, p$. It follows that

$$T(r, P_j) = o(T(r, f_j e^{g_j})),\ j = s+1, \cdots, p.$$

By Theorem 4.2 we have,

$$\sum_{j=1}^{p} \frac{1}{k_j} > \sum_{j=s+1}^{p} \frac{1}{k_j} \geq \frac{1}{4(p-s-1)} > \frac{1}{4(p-1)}$$

which contracts to $\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{4(p-1)}$. Thus, we obtain the theorem. ❑

Finally, we state and prove our main theorem as follows.

Main Theorem *Let $f_1, \cdots, f_p$ be p (≥ 2) non-constant transcendental meromorphic functions and let $k_1, \cdots, k_p$ be p integers not less than one and at least one of $\frac{f_i^{k_i}}{f_j^{k_j}}$ is transcendental such that*

$$\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{p-1+A_p}.$$

Then we have

$$F(z) = \sum_{j=1}^{p} R_j(z) f_j^{k_j},$$

where $R_j(z)$ $(\neq 0)$, $1 \leq j \leq p$ are rational functions, has infinitely many zeros or poles, or some of the partial sums of F are equal to zero identically.

Proof. Assume that the statement is false. This means that $F(z)$ has only finitely many zeros and poles and has all partial sums of F are not identically zero. This implies that $F(z) \neq 0$. We may assume that $F(z)$ has zeros $\{a_1, \cdots, a_n\}$ and poles $\{b_1, \cdots, b_m\}$, counting multiplicity. Therefore, $\prod_{s=1}^{m}(z-b_s)F(z)/\prod_{t=1}^{n}(z-a_t)$ has no zeros and poles. Then we can write $F(z)$ as

$$F(z) = R(z)e^{f(z)},$$

where $R(z) = \prod_{t=1}^{n}(z-a_t)/\prod_{s=1}^{m}(z-b_s)$ is rational and $f(z)$ is entire.

Let

$$g_j(z) = -\frac{f(z)}{k_j},\ 1 \le j \le p.$$

It is obvious that $g_j(z)$ is entire, $1 \le j \le p$. From $F(z) = \sum_{j=1}^{p} R_j(z) f_j^{k_j}$, we get

$$1 = \sum_{j=1}^{p} R^{-1} R_j (f_j e^{g_j})^{k_j}$$

By assumption, $R_j R^{-1}$ is rational which implies that

$$T(r, R_j R^{-1}) = O(\log r) \text{ as } r \to \infty,\ j = 1, \cdots, p.$$

By $\sum_{j=1}^{p} \frac{1}{k_j} < \frac{1}{p-1+A_p}$ and Theorem 4.9, some of $f_j e^{g_j}$ must be rational. Otherwise, all $f_j e^{g_j}$ are transcendental, we have

$$T(r, R_j R^{-1}) = S(r, f_j e^{g_j}),\ 1 \le j \le p.$$

By Theorem 4.9, $\sum_{j=1}^{p} \frac{1}{k_j} \ge \frac{1}{p-1+A_p}$ which is impossible. Also, the number s of j such that $f_j e^{g_j}$ is rational is less than $p-2$. If $s \ge p-1$, it force that all of $f_i e^{g_i}$ are rational. In this case

$$\frac{(f_j e^{g_j})^{k_j}}{(f_j e^{g_j})^{n_j}} = \frac{f_i^{n_i}}{f_j^{n_j}}$$

is rational, which is a contradiction. Thus,

$$1 \le s \le p-2.$$

Suppose that $f_1 e^{g_1}, \cdots, f_s e^{g_s}$ are rational, so that $\sum_{j=1}^{s} R^{-1} R_j (f_j e^{g_j})^{k_j}$ is rational and is not equal to 1 identically since $\sum_{j=s+1}^{p} R_j f_j^{k_j}$ is not equal to zero identically. Therefore,

$$\sum_{j=s+1}^{p} R_j R^{-1} (f_j e^{g_j})^{k_j} = 1 - \sum_{j=1}^{s} R_j R^{-1} (f_j e^{g_j})^{k_j}$$
$$\equiv B(z),$$

where $B(z)$ is a nonzero rational function.

We have

$$\sum_{j=s+1}^{p} P_j(f_j e^{g_j})^{k_j} = 1$$

with $P_j = \dfrac{R_j R^{-1}}{B}$ is rational, $s+1 \leq j \leq p$.

Since $f_j e^{g_j}$ are transcendental and P_j are rational, for $j = s+1, \cdots, p$. It shows that

$$T(r, P_j) = S(r, f_j e^{g_j}),\ j = s+1, \cdots, p.$$

By Theorem 4.9, we have

$$\sum_{j=s+1}^{p} \frac{1}{k_j} \geq \frac{1}{(p-s)-1+A_{p-s}}.$$

So

$$\begin{aligned}\sum_{j=1}^{p} \frac{1}{k_j} &> \sum_{j=s+1}^{p} \frac{1}{k_j} \\ &> \frac{1}{p-s-1+A_{p-s}} \\ &> \frac{1}{p-1+A_p},\end{aligned}$$

where A_p is increasing in p which is a contradiction. ❑

References

[1] I. N. Baker, *On a class of meromorphic functions*, Proc. Amer. Math. Soc., 17 (1966), pp. 819–822.

[2] C.-T. Chuang and C.-C. Yang, *Fix-points and factorization of meromorphic functions*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1990. Translated from the Chinese.

[3] M. L. Green, *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math., 97 (1975), pp. 43–75.

[4] F. Gross, *On the equation* $f^n + g^n = 1$, Bull. Amer. Math. Soc., 72 (1966), pp. 86–88.

[5] F. Gross, *On the functional equation* $f^n + g^n = h^n$, Amer. Math. Monthly, 73 (1966), pp. 1093–1096.

[6] F. Gross, *Factorization of meromorphic functions*, Mathematics Research Center, Naval Research Laboratory, Washington, D. C., 1972.

[7] G. G. Gundersen, *Meromorphic solutions of* $f^6 + g^6 + h^6 \equiv 1$, Analysis (Munich), 18 (1998), pp. 285–290.

[8] G. G. Gundersen, *Meromorphic solutions of* $f^5 + g^5 + h^5 \equiv 1$, Complex Variables Theory Appl., 43 (2001), pp. 293–298. The Chuang special issue.

[9] W. Hayman, *Warings Problem für analytische Funktionen*, Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber., (1984), pp. 1–13 (1985).

[10] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[11] K. Ishizaki, *A note on the functional equation* $f^n+g^n+h^n = 1$ *and some complex differential equations*, Comput. Methods Funct. Theory, 2 (2002), pp. 67–85.

[12] A. V. Jategaonkar, *Elementary proof of a theorem of P. Montel on entire functions*, J. London Math. Soc., 40 (1965), pp. 166–170.

[13] I. Lahiri and K.-W. Yu, *On generalized Fermat type functional equations*, Comput. Methods Funct. Theory, 7 (2007), pp. 141–149.

[14] D. H. Lehmer, *On the diophantine equation* $x^3 + y^3 + z^3 = 1$, Journal of the London Mathematical Society, 31 (1956), pp. 275–280.

[15] P. Li and C.-C. Yang, *Some further results on the unique range sets of meromorphic functions*, Kodai Math. J., 18 (1995), pp. 437–450.

[16] H. Milloux, *Les fonctions méromorphes et leurs dérivées. Extensions d'un théorème de M. R. Nevanlinna. Applications*, Actualités Sci. Ind., no. 888, Hermann et Cie., Paris, 1940.

[17] P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthiers-Villars, Paris, 1927.

[18] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthiers-Villars, Paris, 1929.

[19] D. J. Newman and M. Slater, *Waring's problem for the ring of polynomials*, Journal of Number Theory, 11 (1979), pp. 477–487.

[20] F. Rellich, *Elliptische Funktionen und die ganzen Lösungen von* $y'' = f(y)$, Math, 47 (1940), pp. 153–160.

[21] N. Toda, *On the functional equation* $\sum_{i=0}^{p} a_i f_i^{n_i} = 1$, Tôhoku Mathematical Journal, 23 (1971), pp. 289–299.

[22] C.-C. Yang, *A generalization of a theorem of p. montel on entire functions*, Proceedings of the American Mathematical Society, 26 (1970), pp. 332–334.

[23] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, vol. 557 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 2003.

[24] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993. Translated and revised from the 1982 Chinese original.

[25] K.-W. Yu and C.-C. Yang, *A note for waring's type of equations for a ring of meromorphic functions*, Indian Journal of Pure and Applied Mathematics, 33 (2002), pp. 1495–1502.