Chapter 3 PERFORMANCE ANALYSIS OF INTERVAL FORECASTING

3.1 INTRODUCTION

In the time series analysis research, the most difficult work could be how to choose an appropriate model from a model base (a model family), which can honestly explain the trend of an underlying time series, such as exchange rate or index of stock volume. Two fundamental questions that often arise are: (1) Does there exist an appropriate statistical model that can account for this underlying process? (2) Does the dynamic model follow a liner or non-linear equation? (Do we need to use more than one equation, e.g. threshold model, to fit the time series?).

In the progress of the scientific research and analysis, the uncertainty in the statistical numerical data is the crux of the problem that the traditional mathematical model is hard to be established. Manski [20] has pointed out that the numerical data are over-demanded and over-explained. If we exploit this artificial accuracy to do causal analysis or measurement, it may lead to the deviation of the causal judgment, the misleading of the decision model, or the exaggerated difference between the forecast result and the actual data. Therefore, this chapter proposes to use the interval data to avoid such risks to happen.

Due to the uncertainty of the forecast points, intervals are used as the estimated forecast values. Nguyen and Wu [23] introduced fuzzy interval time series to forecast intervals. Taking stock market as an example, if it is desired to make a forecasting analysis to a certain stock, the daily highest and lowest prices of the stock are regarded as the boundary values of intervals. Then the future price intervals of the stock can be predicted by means of interval time series forecasting. Consequently, we can make comparatively objective decision by the predicted price interval rather than by the closing value or mean value.

When the interval calculation technology is applied to explore the model construction and the forecasting of interval time series, it is necessary to determine the validity of the forecasting method by means of the estimated errors between the forecast results and the actual intervals. Based on the four forecasting situations of the forecast interval, In order to perform the efficiency evaluation of interval forecasting, a criterion will be defined by means of the overlap parts and the non-overlap parts of the actual intervals and the forecast intervals. This chapter will define the mean squared error of interval and the mean relative interval error by combining the two factors of the center and the radius of interval. In addition, this chapter makes comparisons with the mean squared error of interval and the mean relative interval error.

While considering a good forecast interval, it is the most important whether the forecast interval does cover the actual interval. Moreover we define mean ratio of exclusive-*or* which is more sufficient to show the efficiency of interval forecasting. By proposing the forecasting performance evaluation for interval data, we will demonstrate the validation of the interval forecasting effect which will be helpful for the study and judgment on the choice of the interval forecasting models.

3.2 INTERVAL FORECASTING

3.2.1 Time Series Forecasting with Interval Data

In traditional analysis of time series, the data of time series is sampled from the values present at discrete points of time. However time is a continuous variable, the data variation between two consecutive samples cannot be known. Besides, the forecast result of a time series is merely a single value. Therefore, the forecasting by a set of discrete numerical data may be too subjective and restricted. In order to broaden more latitude of forecast result, the concept of interval time series is to represent the time series data in the form of interval. Then the centers and radii of interval time series are used to make analysis of forecasting. Thus, the result of interval time series forecasting is also in the form of interval obtained by the forecast center and radius.

While we consider the data to be of interval type, we must encounter the various problems of interval operations as well as the realistic meanings. Hayes [15] pointed out that the comparisons between intervals are more complicated than those of point-like numbers. Figure 3.1 shows 15 meaningful relations between intervals. For $\boldsymbol{X_1} = [\, a_1, b_1 \,]$ and $\boldsymbol{X_2} = [\, a_2, b_2 \,]$, the relations between two intervals is defined as follows : $(R_1, R_2, R_3, R_4) = (\#(a_1 - b_2), \#(a_1 - a_2), \#(b_1 - b_2), \#(b_1 - a_2))$, where $\#(d) = \begin{cases} < & , if\ d < 0 \\ > & , if\ d > 0 \\ = & , if\ d = 0 \end{cases}$. It is too ambiguous to name all these comparisons.

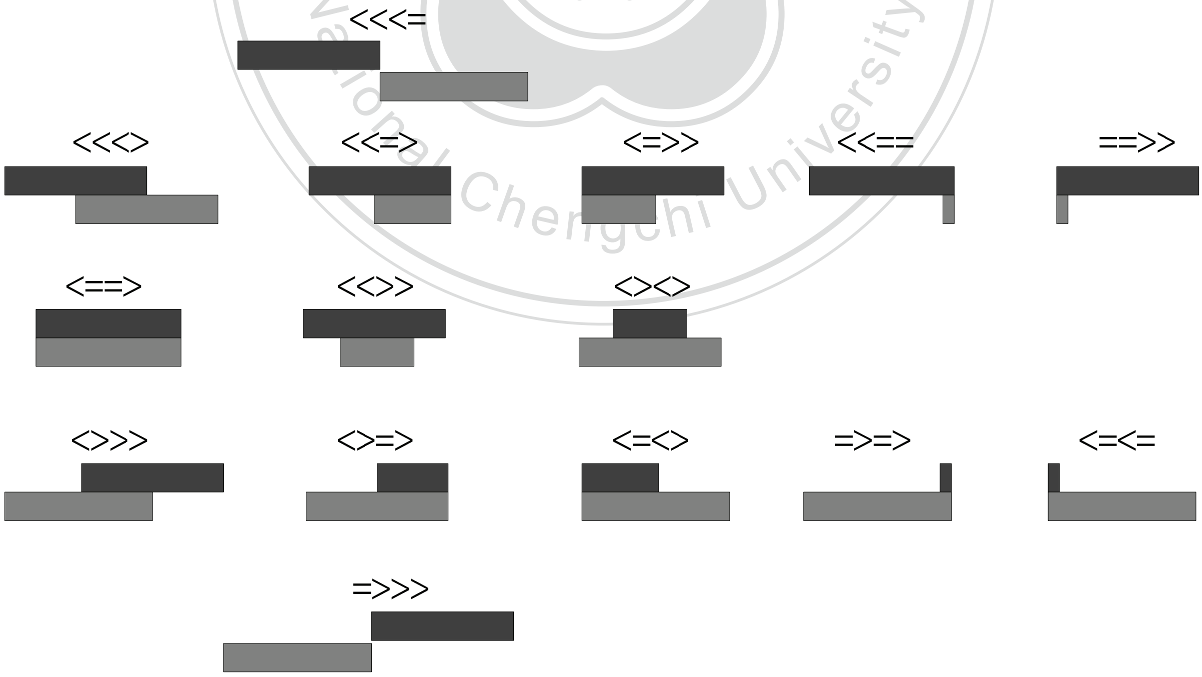

Figure 3.1 One encoding lists the relations of the four pairs of end points in a fixed sequence.

3.2.2 **Some Operations of Interval Data**

In this chapter, we concern the efficiency of interval forecasting mainly. And without loss of generality, the interval data which has the symmetric feature (*i.e.* $l = u$) is offered to perform the analysis of interval forecasting. Since several definitions relevant to interval time series were given in Section 2.2.2, therefore some interval operations should be defined for evaluating the performance interval forecasting as follows.

Definition 3.1 *Symmetric LR-type of interval data,* $\boldsymbol{X} = (c, r)_{LR}$

An interval data $\boldsymbol{X}$ *with the lower boundary a and the upper boundary b which is denoted as* $[a, b]$ *is of symmetric LR-type if the membership function is defined by*

$$\mu_X(x) = \begin{cases} L\left(\dfrac{c-x}{r}\right) & for\ x \le c \\ R\left(\dfrac{x-c}{r}\right) & for\ x \ge c \end{cases}$$

where L and R are defined as in Definition 2.2, the real number $c = (a + b) / 2$ *is the mean value of* $\boldsymbol{X}$*, and* $r = (b - a) / 2$*.*

Therefore, c is called as the center of $\boldsymbol{X}$*, r is called as the radius of* $\boldsymbol{X}$*, and* $\boldsymbol{X}$ *is denoted by* $\boldsymbol{X} = [a, b] = (c, r)_{LR}$.

The set difference $A - B$ is defined by $A - B = \left\{ x \mid x \in A \text{ and } x \notin B \right\}$ (Smith, [26]). While revising the definition of the set difference on the closed intervals, it should be a half-closed interval. But the closed intervals are used to the forecasting of interval time series and expressed by the boundary of the interval data in this chapter, so we make interval difference little diverse in the definition 3.1.

Definition 3.2 *The interval difference,* $\boldsymbol{X}_1 - \boldsymbol{X}_2$

Let $\boldsymbol{X}_1 = [a_1, b_1] = (c_1, r_1)_{LR}$ *and* $\boldsymbol{X}_2 = [a_2, b_2] = (c_2, r_2)_{LR}$ *be two interval data and* $a_1 \le a_2 \le b_1 \le b_2$*. Then the interval difference* $\boldsymbol{X}_1 - \boldsymbol{X}_2$ *is defined as*

$$\begin{aligned} \boldsymbol{X}_1 - \boldsymbol{X}_2 &= [a_1, b_1] - [a_2, b_2] \\ &= \left(\bar{c} - \bar{r}, (\partial c - \partial r)/2\right)_{LR}, \end{aligned}$$

where $\bar{c} = (c_1 + c_2)/2$, $\bar{r} = (r_1 + r_2)/2$, $\partial c = c_2 - c_1$, *and* $\partial r = r_2 - r_1$.

Definition 3.3 *Exclusive-OR* (*XOR*), $\boldsymbol{X}_1 \Delta \boldsymbol{X}_2$

Let $\boldsymbol{X}_1 = [a_1, b_1] = (c_1, r_1)_{LR}$ *and* $\boldsymbol{X}_2 = [a_2, b_2] = (c_2, r_2)_{LR}$ *be two interval data and* $a_1 \le a_2 \le b_1 \le b_2$. *Then the exclusive-or denoted* $\boldsymbol{X}_1 \Delta \boldsymbol{X}_2$, *is defined by*

$$\begin{aligned} \boldsymbol{X}_1 \Delta \boldsymbol{X}_2 &= (\boldsymbol{X}_1 - \boldsymbol{X}_2) \cup (\boldsymbol{X}_2 - \boldsymbol{X}_1) \\ &= \left(\bar{c} - \bar{r}, (\partial c - \partial r)/2\right)_{LR} \cup \left(\bar{c} + \bar{r}, (\partial c + \partial r)/2\right)_{LR}. \end{aligned}$$

Example 3.1 Let $\boldsymbol{X_1} = [1, 3] = (2, 1)_{LR}$, $\boldsymbol{X_2} = [2, 6] = (4, 2)_{LR}$, then

$$\boldsymbol{X}_1 \Delta \boldsymbol{X}_2 = [1, 2] \cup [3, 6] = (1.5, 0.5)_{LR} \cup (4.5, 1.5)_{LR}.$$ ♦

3.2.3 **Properties of Interval Time Series**

Let $\{\boldsymbol{X}_t = [a_t, b_t] = (c_t, r_t)_{LR}\}$ be a sequence of interval data and $\hat{\boldsymbol{X}}_t = [\hat{a}_t, \hat{b}_t] = (\hat{c}_t, \hat{r}_t)_{LR}$ be the forecast interval with respect to $\boldsymbol{X}_t = [a_t, b_t] = (c_t, r_t)_{LR}$. In the analysis and forecasting of interval time series, there are four forecasting situations:

(1) If $\hat{a}_t \le a_t \le b_t \le \hat{b}_t$, then the forecast interval is too wide, and denoted by *FIW*.

(2) If $a_t \le \hat{a}_t \le \hat{b}_t \le b_t$, then the forecast interval is too narrow, and referred to as *FIN*.

(3) If $a_t \le \hat{a}_t \le b_t \le \hat{b}_t$, then the forecast interval inclines to the right, and indicated as *FIR*.

(4) If $\hat{a}_t \le a_t \le \hat{b}_t \le b_t$, then the forecast interval inclines to the left, expressed as *FIL*.

It is difficult to know which forecasting situation is better than the others. By calculating the length of the exclusive-*or*, it can help us to find out which forecast interval is better for forecasting. So we will introduce some properties for forecasting situations.

Property 3.1 The interval length of *XOR* for *FIW*

If $\hat{\boldsymbol{X}}$ is *FIW*, then the interval length of *XOR* is $\left\|\boldsymbol{X} \Delta \hat{\boldsymbol{X}}\right\| = -2\partial r$, where $\partial r = r - \hat{r}$.

Proof: Since $\hat{\boldsymbol{X}}$ is *FIW*, $\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}} = (\boldsymbol{X}-\hat{\boldsymbol{X}})\cup(\boldsymbol{X}-\hat{\boldsymbol{X}}) = [\hat{a}, a]\cup[b,\hat{b}]$. By $a = c - r$, $\hat{a} = \hat{c} - \hat{r}$, $b = c + r$, and $\hat{b} = \hat{c} + \hat{r}$, we have $a - \hat{a} = (c-\hat{c}) - (r-\hat{r})$ and $\hat{b} - b = (\hat{c} - c) + (\hat{r} - r)$. Therefore,

$$\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = \|[\hat{a},\ a]\| + \left\|[b,\ \hat{b}]\right\| = (a-\hat{a}) + (\hat{b}-b) = -2(r-\hat{r}) = -2\partial r. \quad ■$$

Property 3.2 The interval length of *XOR* for *FIN*

If $\hat{\boldsymbol{X}}$ is *FIN*, then the interval length of *XOR* is $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = 2\partial r$, where $\partial r = r - \hat{r}$.

Proof: For $\hat{\boldsymbol{X}}$ is *FIN*, $\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}} = [a,\hat{a}]\cup[\hat{b},b]$. From $\hat{a} - a = (\hat{c}-c) - (\hat{r}-r)$ and $b - \hat{b} = (c-\hat{c}) + (r-\hat{r})$, $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = \|[a,\ \hat{a}]\| + \left\|[\hat{b},\ b]\right\| = (\hat{a}-a) + (b-\hat{b}) = 2\partial r$. ■

Property 3.3 The interval length of *XOR* for *FIR*

If $\hat{\boldsymbol{X}}$ is *FIR*, then the interval length of *XOR* is $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = -2\partial c$, where $\partial c = c - \hat{c}$.

Proof: If $\hat{\boldsymbol{X}}$ is *FIR*, $\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}} = [a,\hat{a}]\cup[b,\hat{b}]$ By $\hat{a} - a = (\hat{c}-c) - (\hat{r}-r)$ and $\hat{b} - b = (\hat{c}-c) + (\hat{r}-r)$, $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = \|[a,\ \hat{a}]\| + \left\|[b,\ \hat{b}]\right\| = (\hat{a}-a) + (\hat{b}-b) = -2\partial c$. ■

Property 3.4 The interval length of *XOR* for *FIL*

If $\hat{\boldsymbol{X}}$ is *FIL*, then the interval length of *XOR* is $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = 2\partial c$, where $\partial c = c - \hat{c}$.

Proof: When $\hat{\boldsymbol{X}}$ is *FIL*, $\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}} = [\hat{a},a]\cup[\hat{b},b]$. By $a - \hat{a} = (c-\hat{c}) - (r-\hat{r})$ and $b - \hat{b} = (c-\hat{c}) + (r-\hat{r})$, $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = \|[\hat{a},\ a]\| + \left\|[\hat{b},\ b]\right\| = (a-\hat{a}) + (b-\hat{b}) = 2\partial c$. ■

Example 3.2 Let $\boldsymbol{X} = [1, 3] = (2, 1)_{LR}$, $\hat{\boldsymbol{X}} = [2, 6] = (4, 2)_{LR}$. Then $\hat{\boldsymbol{X}}$ is *FIR*.

From Property 3.3, we get $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = 2(4-2) = 4$. ♦

Example 3.3 Let $\boldsymbol{X} = [3, 5] = (4, 1)$, $\hat{\boldsymbol{X}} = [2, 6] = (4, 2)$. Then $\hat{\boldsymbol{X}}$ is *FIW*.

From Property 3.1, we obtain $\left\|\boldsymbol{X}\,\Delta\,\hat{\boldsymbol{X}}\right\| = 2(2-1) = 2$. ♦

3.3 EFFICIENCY EVALUATION FOR INTERVAL FORECASTING

How to evaluate the forecasting performance for interval data is an important issue. In Section 2.3.2, we propose two techniques of the efficiency analysis for interval forecasting. They are the mean squared error of interval, denoted as *MSEI*, and the mean relative interval error, denoted as *MRIE*. The following will make a comparison between *MESI* and *MRIE*.

3.3.1 The Comparison of *MESI* and *MRIE*

Definition 3.4 *Mean squared error of interval* (*MSEI*)

Let $\{\boldsymbol{X}_t = (c_t, r_t)_{LR}\}$ *be an interval time series and* $\hat{\boldsymbol{X}}_t = (\hat{c}_t, \hat{r}_t)_{LR}$ *be the forecast interval, the mean squared error of interval (MSEI) is defined by*

$$\begin{aligned} MSEI &= \frac{1}{s}\sum_{t=1}^{s} D^2\left(\boldsymbol{X}_{n+t}, \hat{\boldsymbol{X}}_{n+t}\right) \\ &= \frac{1}{s}\sum_{t=1}^{s}\left((c_{n+t}-\hat{c}_{n+t})^2 + ((c_{n+t}-r_{n+t})-(\hat{c}_{n+t}-\hat{r}_{n+t}))^2 + ((c_{n+t}+r_{n+t})-(\hat{c}_{n+t}+\hat{r}_{n+t}))^2\right) \\ &= \frac{1}{s}\sum_{t=1}^{s}\left(3(c_{n+t}-\hat{c}_{n+t})^2 + 2(r_{n+t}-\hat{r}_{n+t})^2\right) \end{aligned}$$

where n denotes the current time, s is the number of the preceding intervals, and $\hat{c}_t$ *and* $\hat{r}_t$ *are the estimations of* c_t *and* r_t *respectively.*

Definition 3.5 *Mean relative interval error* (*MRIE*)

Let $\{\boldsymbol{X}_t = (c_t, r_t)_{LR}\}$ *be an interval time series and* $\hat{\boldsymbol{X}}_t = (\hat{c}_t, \hat{r}_t)_{LR}$ *be the forecast interval, the mean relative interval error (MRIE) is given by*

$$MRIE = \frac{1}{s}\sum_{t=l}^{s}\frac{\left|c_{n+t}-\hat{c}_{n+t}\right|}{\left\|\boldsymbol{X}_{n+t}\Theta\hat{\boldsymbol{X}}_{n+t}\right\|_{(*)}} = \frac{1}{s}\sum_{t=l}^{s}\frac{\left|c_{n+t}-\hat{c}_{n+t}\right|}{r_t+\hat{r}_t} \quad ,$$

where n denotes the current time, s is the number of the preceding intervals, and $\hat{c}_t$ *is the estimation of* c_t.

The mean squared error of interval is a statistic often used to calculate the efficiency of the statistical estimated value. It is convenient for investigating the interval forecasting. But at some forecasting situations, it will obtain an incorrect outcome if we use the mean squared error of interval to analyze the efficiency of the forecasting interval.

Consider the interval $\boldsymbol{X}$ = [4, 7] = (5.5, 1.5 $)_{LR}$, and the forecast intervals $\hat{\boldsymbol{X}}_1$ = [1, 8] = (4.5, 3.5$)_{LR}$ and $\hat{\boldsymbol{X}}_2$ = [6, 8] = (7, 1$)_{LR}$ obtained by two different forecasting methods respectively. The *MSEI* of $\hat{\boldsymbol{X}}_1$, denoted as $MSEI_1$, is 11. The *MSEI* of $\hat{\boldsymbol{X}}_2$, denoted as $MSEI_2$, is 7.25. Then $\hat{\boldsymbol{X}}_2$ is a better forecast interval than $\hat{\boldsymbol{X}}_1$ by comparing $MSEI_1$ and $MSEI_2$. Actually, it is not true. While considering the efficiency of interval forecasting, it is the most important whether the forecast interval does cover the actual interval. Explicitly speaking, a forecast result is better if the center $\hat{c}$ is closer to the center c and their interval overlap is larger.

When the mean relative interval error is employed to evaluate the efficiency of the forecast interval, the *MRIE* of $\hat{\boldsymbol{X}}_1$ is $MRIE_1$ = 0.2 and the *MRIE* of $\hat{\boldsymbol{X}}_2$ is $MRIE_2$ = 0.6. Although the radius of $\hat{\boldsymbol{X}}_1$ is larger than that of $\hat{\boldsymbol{X}}_2$, the central point of $\hat{\boldsymbol{X}}_1$ is closer to the central point of $\boldsymbol{X}$. Since the range of $\hat{\boldsymbol{X}}_1$ covers the range of the actual interval $\boldsymbol{X}$ is more than the range of $\hat{\boldsymbol{X}}_2$ does. As a result, we still regard $\hat{\boldsymbol{X}}_1$ as the better forecast interval than $\hat{\boldsymbol{X}}_2$.

3.3.2 **The Feasibility Analysis of *MSEI* and *MRIE***

As described in Section 2.3.2, the error between the forecast interval $\hat{\boldsymbol{X}}_t$ and the actual interval X_t contains two parts, the position error and the length error. The former is the distance between the central points of two intervals, while the latter is the difference between the radii of two intervals. If the mean squared errors of the position and the length are always summed up, it will be hard to discern the efficiencies of the forecasting methods between the position and the length.

It is preferred to use *MRIE* since it seems to be superior to *MSEI*. But there are some questionable problems in the four forecasting situations. For instance, the interval $\boldsymbol{X} = [4, 7] = (5.5, 1.5)_{LR}$, and the forecast intervals $\hat{\boldsymbol{X}}_1 = [1, 8] = (4.5, 3.5)_{LR}$ and $\hat{\boldsymbol{X}}_2=[0,10]=(5,5)_{LR}$ are obtained by two different forecasting methods. Then the *MRIE* of $\hat{\boldsymbol{X}}_1$, denoted as $MRIE_1$, is 0.1. The *MRIE* of $\hat{\boldsymbol{X}}_2$, denoted as $MRIE_2$, is 0.08. Intuitively, $\hat{\boldsymbol{X}}_2$ looks like better than $\hat{\boldsymbol{X}}_1$ by evaluating $MRIE_1$ and $MRIE_2$. Is it right? Since $\hat{\boldsymbol{X}}_1$ and $\hat{\boldsymbol{X}}_2$ are *FIW*s, the forecast radius is longer, the *MRIE* will be smaller. Hence the *MRIE* is not an ideal method especially when the forecast interval is too wide.

How do we know which one is better interval forecasting in the four forecasting situations? For example, the actual interval is $\boldsymbol{X} = [\ 4, 7\ ] = (\ 5.5, 1.5\ )_{LR}$, and the forecast intervals are $\hat{\boldsymbol{X}}_1 = [\ 2.2, 8.4\ ] = (\ 5.3, 3.1\ )_{LR}$ and $\hat{\boldsymbol{X}}_2 = [\ 4.2, 6\ ] = (\ 5.1, 0.9\ )_{LR}$. The *MSEI* of $\hat{\boldsymbol{X}}_1$ is 5.24 and the *MSEI* of $\hat{\boldsymbol{X}}_2$ is 1.2. The *MRIE* of $\hat{\boldsymbol{X}}_1$ is 0.04 and the *MRIE* of $\hat{\boldsymbol{X}}_2$ is 0.17. Is $\hat{\boldsymbol{X}}_1$ better than $\hat{\boldsymbol{X}}_2$ by observing their *MRIE*s? Or is $\hat{\boldsymbol{X}}_2$ better than $\hat{\boldsymbol{X}}_1$ by examining their *MSEI*s? It is very difficult to describe which one is superior between them.

$\hat{\boldsymbol{X}}_1$ is *FIW* that means it can cover all range of the actual interval $\boldsymbol{x}$, whereas $\hat{\boldsymbol{X}}_2$ is *FIN* which is enclosed by the actual interval $\boldsymbol{X}$. If the forecast interval is too wide, it could be forced to include some 'noisy information'. In consequence it will disturb our decision. On the contrary, while the forecast interval is too narrow such as $\hat{\boldsymbol{X}}_2$, it maybe lose some 'important information'. Thus, it will mislead the executive's judgment. The similar question always happens when the forecast interval is *FIL* or *FIR*. They conclude some noisy information and lose some important information at the same time. It is unfair to compare the forecasting efficiency with the different forecasting situations. Sometimes it depends on policymaker's requirement. If we try to clarify how better in the same forecasting situation, the *XOR* can offer a good explanation in the forecasting efficiency. We will present another technique for forecasting efficiency analysis.

3.4 THE MEAN RATIO OF EXCLUSIVE-*OR*

3.4.1 The Mean Ratio of *XOR*

Generally speaking, if the center and radius of the forecast interval are almost matched the center and radius of the actual interval respectively, then it is a better interval forecasting. Therefore, when the length of *XOR* showing non-overlap of the actual interval and the forecast interval is small, it appears the forecast interval covers more the actual interval. Using the character of *XOR*, we offer another technique of the efficiency analysis for the interval time series forecasting.

Definition 3.6 *Mean ratio of exclusive-or* (*MRXOR*)

Let $\{\boldsymbol{X}_t = (c_t, r_t)_{LR}\}$ *be an interval time series and* $\hat{\boldsymbol{X}}_t = (\hat{c}_t, \hat{r}_t)_{LR}$ *be the forecast interval, the mean ratio of exclusive-or (MRXOR) is given as follows:*

$$MRXOR = \frac{1}{s}\sum_{t=1}^{s}\frac{\left\|\boldsymbol{X}_{n+t}\,\Delta\,\hat{\boldsymbol{X}}_{n+t}\right\|}{\left\|\boldsymbol{X}_{n+t}\right\|},$$

where n denotes the current time, and s is the number of the preceding intervals.

Definition 3.7 *The efficiency of MRXOR*

Let $\{\boldsymbol{X}_t = (c_t, r_t)_{LR}\}$ *be an interval time series, and let the forecast interval time series* $\left\{\hat{\boldsymbol{X}}_{1t} = (\hat{c}_{1t}, \hat{r}_{1t})\right\}$ *and* $\left\{\hat{\boldsymbol{X}}_{2t} = (\hat{c}_{2t}, \hat{r}_{2t})\right\}$ *be obtained by two different forecasting methods. If* *the MRXOR of* $\{\hat{X}_{1t}\}$*, denoted as MRXOR*$_1$*, is smaller than the MRXOR of* $\{\hat{X}_{2t}\}$*, denoted as MRXOR*$_2$*, then we say the forecast interval* $\{\hat{X}_{1t}\}$ *is more efficient as compared to the forecast interval* $\{\hat{X}_{2t}\}$*. i.e.* $\{\hat{X}_{1t}\}$ *is more efficient than* $\{\hat{X}_{2t}\}$*, if MRIXOR*$_1$ *< MRIXOR*$_2$*.*

Example 3.4 Let the interval samples be $\boldsymbol{X}_1$ = [4, 6] = (5, 1)$_{LR}$ and $\boldsymbol{X}_2$ = [5, 8] = (6.5, 1.5)$_{LR}$, the forecast intervals are $\hat{\boldsymbol{X}}_1$ = [2.8, 5.4] = (4.1, 1.3)$_{LR}$, and $\hat{\boldsymbol{X}}_2$ = [3.8, 7.8] = (5.8, 2)$_{LR}$.

Since $\hat{\boldsymbol{X}}_1$ and $\hat{\boldsymbol{X}}_2$ are *FIL*s and from Property 3.4, $\left\|\boldsymbol{X}_1 \Delta \hat{\boldsymbol{X}}_1\right\|$ = 2 (5 − 4.1) = 1.8 and $\left\|\boldsymbol{X}_2 \Delta \hat{\boldsymbol{X}}_2\right\|$ = 2 (6.5 − 5.8) = 1.4. Thus the mean ratio of exclusive-*or* is given by

$$MRXOR = \frac{1}{2}\left(\frac{1.8}{2} + \frac{1.4}{3}\right) = 0.68. \qquad \blacklozenge$$

Example 3.5 Let the interval sample be $\boldsymbol{X}$ = [4, 7] = (5.5, 1.5)$_{LR}$, the forecast intervals be $\hat{\boldsymbol{X}}_1$ = [2.2, 8.4] = (5.3, 3.1)$_{LR}$ and $\hat{\boldsymbol{X}}_2$ = [4.2, 6] = (5.1, 0.9)$_{LR}$. Since $\hat{\boldsymbol{X}}_1$ is *FIW*, we have $\left\|\boldsymbol{X} \Delta \hat{\boldsymbol{X}}_1\right\|$ = 2(3.1−1.5) = 3.2. By Property 3.1, $MRXOR_1 = \frac{3.2}{3} = 1.07$. Similarly, $\hat{\boldsymbol{X}}_2$ is *FIN,* then we have $\left\|\boldsymbol{X} \Delta \hat{\boldsymbol{X}}_2\right\|$ = 2(1.5−0.9) = 1.2 by Property 3.2. Therefore, $MRXOR_2 = \frac{1.2}{3} = 0.40$. Because *MRXOR*$_1$ > *MRXOR*$_2$, $\hat{\boldsymbol{X}}_2$ is more efficient than $\hat{\boldsymbol{X}}_1$. ♦

3.4.2 **Discussion of MRXOR in Different Forecasting Situations**

If we consider two sets of forecast intervals having the same forecasting situation, *MRXOR* will be an excellent method of efficiency analysis. What information can be revealed by *MRXOR* in the forecast solutions? Assume $\hat{\boldsymbol{X}}_1 = (\hat{c}_1, \hat{r}_1)_{LR}$ and $\hat{\boldsymbol{X}}_2 = (\hat{c}_2, \hat{r}_2)_{LR}$ attained by different forecasting methods are the forecast solutions of the actual interval. Their mean ratios of exclusive-*or* are *MRXOR*$_1$ and *MRXOR*$_2$ respectively. The effect of *MRXOR* is discussed according to four forecasting situations as follows.

Case 1: When $\hat{X}_1$ and $\hat{X}_2$ are FIWs.

If $\|\hat{X}_1\| < \|\hat{X}_2\|$, then $MRXOR_1 < MRXOR_2$. It means that $\hat{X}_2$ has more noisy information than $\hat{X}_1$. Therefore, $\hat{X}_1$ is more efficient than $\hat{X}_2$.

When the forecast interval time series $\left\{\hat{X}_t = (\hat{c}_t, \hat{r}_t)_{LR} \mid t = 1, 2, \ldots, s\right\}$ are all *FIW*s*,* what should we do for this state? Because the interval radius influences the length of *XOR* from Property 3.1*,* we should correct the forecasting method of the interval radius first.

Case 2: When $\hat{X}_1$ and $\hat{X}_2$ are FINs.

If $\|\hat{X}_1\| < \|\hat{X}_2\|$, then $MRXOR_1 > MRXOR_2$. It means that $\hat{X}_1$ lose more information than $\hat{X}_2$. Therefore, $\hat{X}_2$ is more efficient than $\hat{X}_1$.

Considering the forecast interval time series $\left\{\hat{X}_t = (\hat{c}_t, \hat{r}_t)_{LR} \mid t = 1, 2, \ldots, s\right\}$ are all *FIN*s. From Property 3.2, the interval radius dominates the length of *XOR*. Then the forecasting method of the interval radius should be properly corrected.

Case 3: When $\hat{X}_1$ and $\hat{X}_2$ are FIRs.

The interval center can manipulate the *XOR* through Property 3.3. When the center of $\hat{X}_1$ is closer to the center of $\boldsymbol{X}$ than that of $\hat{X}_2$, it presents $\hat{X}_1$ covers more vital information and contains less boisterous information than $\hat{X}_2$. That is, if $c < \hat{c}_1 < \hat{c}_2$, then $MRXOR_1 < MRXOR_2$. Therefore, $\hat{X}_1$ is more efficient than $\hat{X}_2$.

When the proceeding forecast interval time series $\left\{\hat{X}_t = (\hat{c}_t, \hat{r}_t)_{LR} \mid t = 1, 2, \ldots, s\right\}$ are all *FIR*s*,* the forecasting method of the interval center should be modified.

Case 4: When $\hat{X}_1$ and $\hat{X}_2$ are FILs.

As the same argument in Case 3, *XOR* can be operated by the interval center through Property 3.4. When the center of $\hat{X}_1$ is closer to the center of $\boldsymbol{x}$ than that of $\hat{X}_2$, $\hat{X}_1$ encloses more essential information and has fewer confusing information than $\hat{X}_2$ does. That is, if $c > \hat{c}_1 > \hat{c}_2$, then $MRXOR_1 < MRXOR_2$. Therefore, $\hat{X}_1$ is more efficient than $\hat{X}_2$.

Once the proceeding forecast interval time series $\left\{ \hat{X}_t = (\hat{c}_t, \hat{r}_t)_{LR} \,|\, t = 1, 2, \ldots, s \right\}$ are all *FIL*s, we should modify the forecasting method of the interval center.

3.5 EMPIRICAL STUDIES

In this section, we use two examples to illustrate the efficiency analysis of forecasting techniques. One case is to contrast among the efficiency of the three forecasting situations. The other case is to demonstrate the forecasting performance of the temperature.

Table 3.1 The actual intervals and the three sets of forecasting intervals.

Actual interval		**Case 1**		**Case 2**		**Case 3**	
$[a_t, b_t]$	(c_t, r_t)	$[\hat{a}_{1t}, \hat{b}_{1t}]$	$(\hat{c}_{1t}, \hat{r}_{1t})$	$[\hat{a}_{2t}, \hat{b}_{2t}]$	$(\hat{c}_{2t}, \hat{r}_{2t})$	$[\hat{a}_{3t}, \hat{b}_{3t}]$	$(\hat{c}_{3t}, \hat{r}_{3t})$
[32.25,34.95]	(33.60,1.35)	[31.10,35.55]	(33.33,2.22)	[33.69,36.35]	(35.02,1.33)	[32.88,34.20]	(33.54,0.66)
[28.10,32.00]	(30.05,1.95)	[27.75,34.25]	(31.00,3.25)	[29.34,33.10]	(31.22,1.88)	[29.94,31.66]	(30.80,0.86)
[26.85,31.75]	(29.30,2.45)	[26.00,33.40]	(29.70,3.70)	[27.95,32.73]	(30.34,2.39)	[27.85,29.95]	(28.90,1.05)
[27.10,30.00]	(28.55,1.45)	[25.95,31.35]	(28.65,2.70)	[28.20,31.28]	(29.74,1.54)	[28.04,29.44]	(28.74,0.70)
[26.00,27.35]	(26.68,0.68)	[24.85,31.20]	(28.02,3.17)	[26.44,27.76]	(27.10,0.66)	[26.80,27.30]	(27.05,0.25)
[26.20,28.85]	(27.52,1.33)	[24.50,30.60]	(27.55,3.05)	[26.45,29.05]	(27.75,1.30)	[26.35,27.15]	(26.75,0.40)

3.5.1 **The Efficiency Analysis of Three Forecasting Situations**

Table 3.1 lists the actual intervals and three cases which are forecast values obtained respectively by three simulated forecasting methods. Figure 3.2 illustrates the actual intervals and the forecast intervals in Case 1. The dark solid line represents the actual interval and the gray dash line symbolizes the forecast interval. It demonstrates the forecast intervals are *FIW*s. The forecast intervals of Case 2 are *FIR*s in Figure 3.3. In Figure 3.4, the forecast intervals attained in Case 3 are *FIN*s. Table 3.2 demonstrates their *MSEI*, *MRIE* and *MRXOR*.

Table 3.2 The comparison of evaluating forecasting performance for simulated interval data.

	MSEI	MRIE	MRXOR
Case 1	6.39	0.12*	1.28
Case 2	3.06	0.31	0.62
Case 3	2.49*	0.23	0.58*

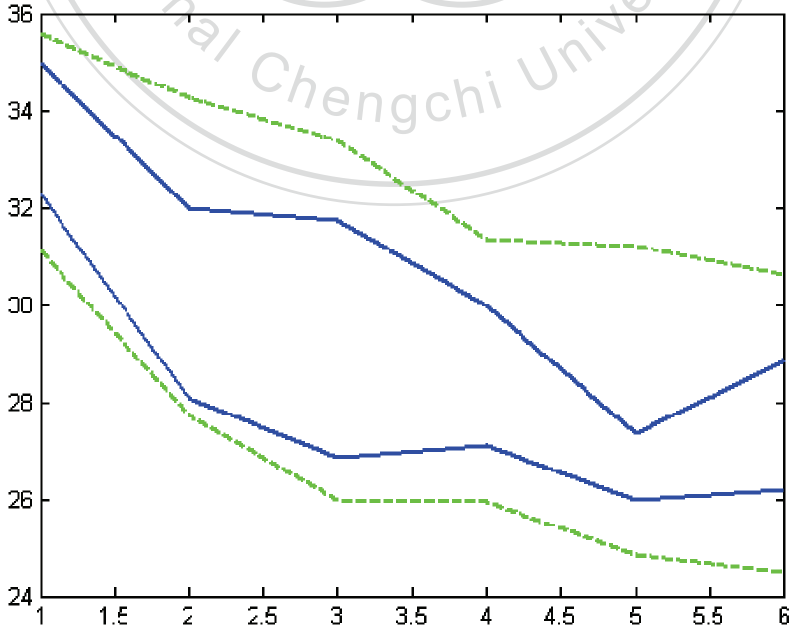

Figure 3.2 The forecast intervals are too wide.

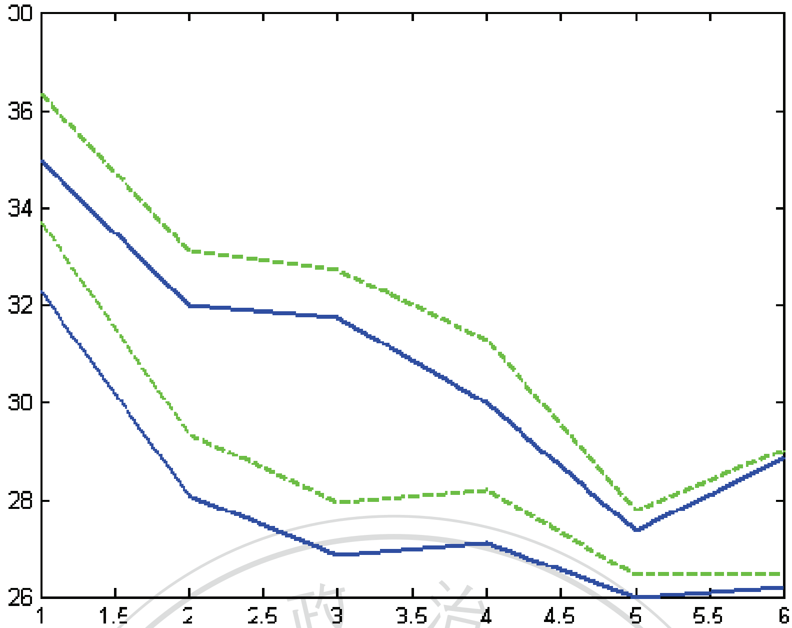

Figure 3.3　The forecast intervals incline to right.

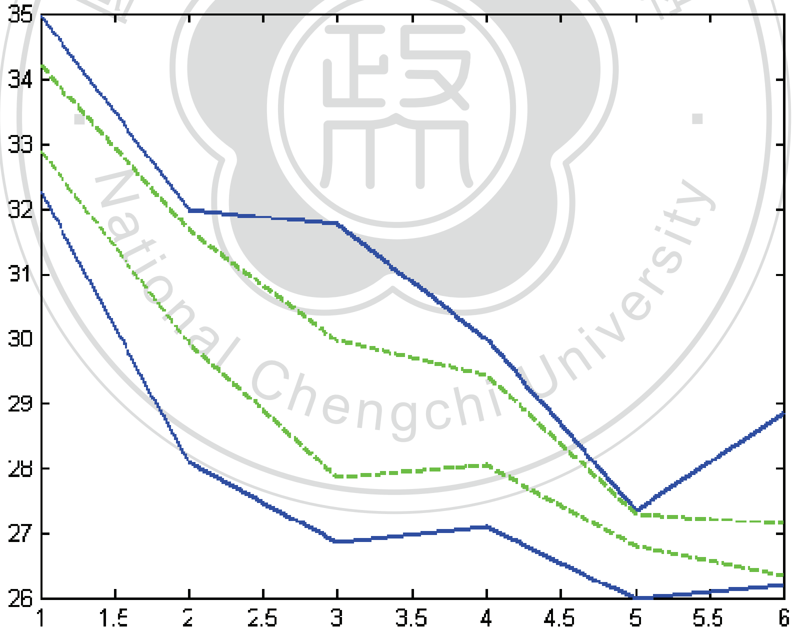

Figure 3.4　The forecast intervals are too narrow.

As described in Section 3.4.1, the forecast results have too wide interval lengths and cover actual data completely in Figure 3.2. Case 1 presents the minimum *MRIE* than other cases. But the forecast intervals contain too much noisier message in Case 1, it has the worst *MSEI* and

MRXOR. Case 2 performs better *MSEI* than Case 1. The reason is the lengths of the forecast intervals almost equal to the actual interval lengths. Owing to the forecast results are *FIR*s in Figure 3.3, the centers of forecast intervals deviate to the centers of actual intervals badly. Then *MRIE* in Case 2 is made larger than the others.

When we evaluate their *MRXOR*s, data in Case 3 has the smallest amount of *MRXOR*. Not because their centers are near to the actual centers, but also the relative length of non-overlap between forecast intervals and the actual intervals are less than the others. The radii of intervals in Case 3 are small so that *MSEI* of Case 3 is larger than *MSEI* of Case 2. But it is still better than that of Case 1. The Case 3 is a good forecasting technique by means of surveying among those *MESI*s, *MRIE*s and *MRXOR*s. As shown in Table 3.2, if the value of *MRXOR* is small, then *MSEI* and *MRIE* are not too large.

3.5.2 **The Forecasting Performance of Temperature**

According to the daily temperatures in Taipei mentioned in Section 2.5.2, this example use the forecast results of the interval moving average, the weighted interval moving average, the *ARIMA* interval forecasting of $(c, r)_{LR}$, and the *ARIMA* interval forecasting of $(m, l, u)_{LR}$ to illustrates their *MSEI*, *MRIE* and *MRXOR*. Figure 3.5 demonstrates the actual intervals and the forecast intervals obtained by the interval moving average. The actual intervals and the forecast intervals of the weighted interval moving average are illustrated in Figure 3.6. The actual intervals and the forecast intervals of the *ARIMA* interval forecasting of $(c, r)_{LR}$, and the *ARIMA* interval forecasting of $(m, l, u)_{LR}$ are demonstrated in Figure 3.7 and Figure 3.8 respectively. The comparison of the forecasting performance for the interval moving average, the weighted interval moving average, the *ARIMA* interval forecasting of $(c, r)_{LR}$, and the *ARIMA* interval forecasting of $(m, l, u)_{LR}$ are shown in Table 3.3.

Table 3.3 The comparison of evaluating forecasting performance for temperature interval data.

	***IMA* (*c*, *r*)**	***WIMA* (*c*, *r*)**	***ARIMA* (*c*, *r*)**	***ARIMA* (*m*, *l*, *u*)**
MSEI	13.39*	16.34	17.89	14.03
MRIE	0.26	0.22	0.22	0.20*
MXOR	0.61	0.56*	0.77	0.78

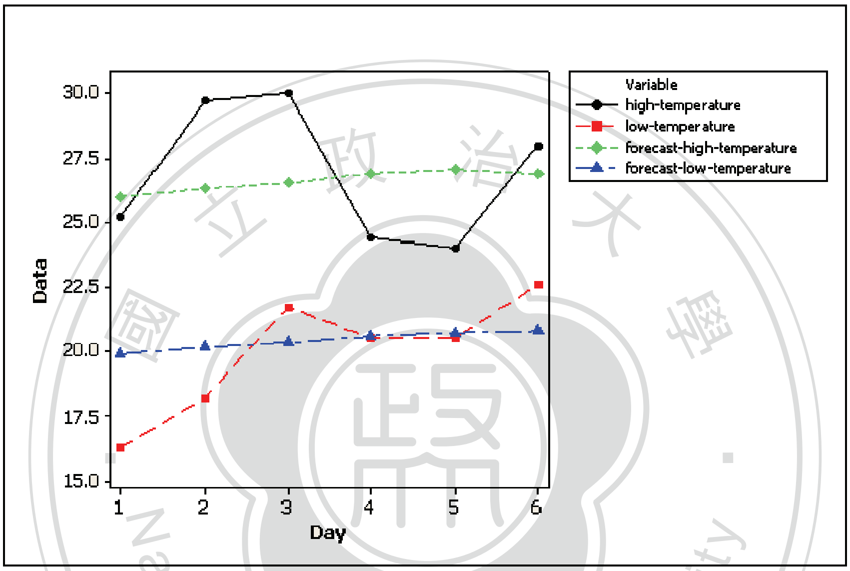

Figure 3.5 The forecast intervals of IMA.

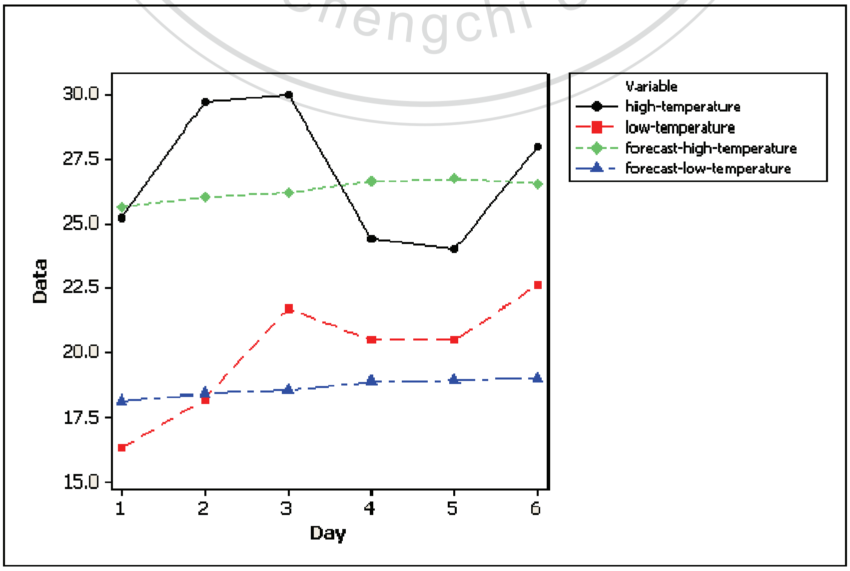

Figure 3.6 The forecast intervals of WIMA.

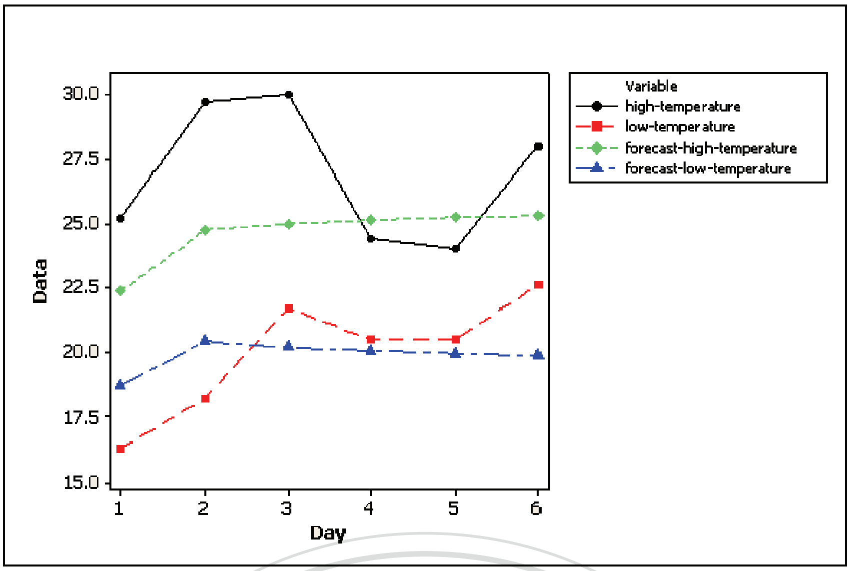

Figure 3.7 The forecast intervals of ARIMA of $(C, R)_{LR}$.

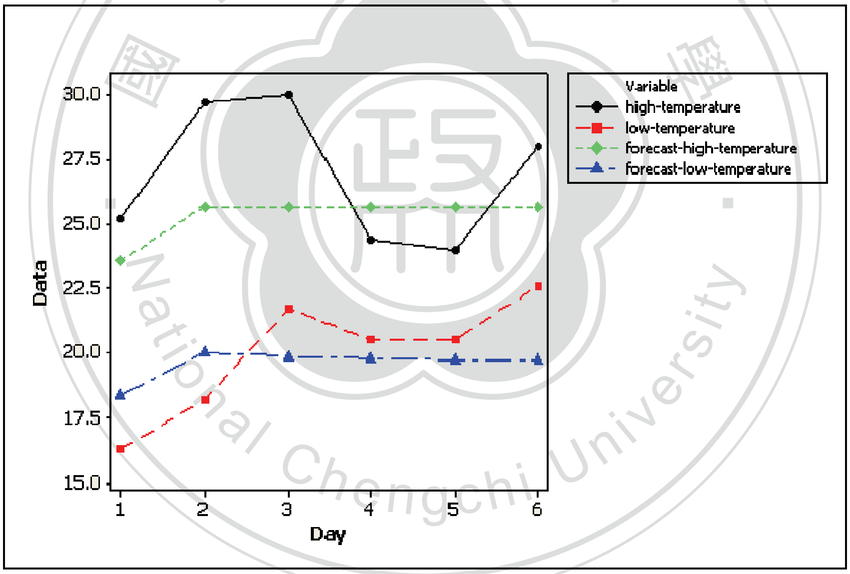

Figure 3.8 The forecast intervals of ARIMA of (m, l, u).

In Figure 3.7, the centers of forecast intervals diverge from the centers of actual intervals badly and the forecast intervals are too narrow to cover the actual intervals completely. Then *MSEI* of the *ARIMA* interval forecasting of $(c, r)_{LR}$ is larger than *MSEI*s of the others. Since *IMA* has the satisfied forecast radii in Figure 3.5, it obtains the superior *MSEI* to the others in Table 3.3. When evaluating *MRIE*s, the *ARIMA* interval forecasting of $(m, l, u)_{LR}$ generates smaller *MRIE*s. The reason is the ranges of the forecast intervals cover the most of the actual intervals. In Table 3.3, we found that *WIMA* obtains a better result of *MRXOR* than the others because the relative length of non-overlap parts between forecast intervals and the actual intervals are small. In fact, the *MRXOR*s which are evaluated by those forecast results are very close. So are the *MRIE*s. Not only the *MRXOR* of the *WIMA* is small, but also the *MSEI* and the *MRIE* of the *WIMA* are not too large. Consequently, the *WIMA* is a better forecast method among those forecast methods in this temperature forecasting case. Although the *MRXOR*s of the *ARIMA* interval forecasting of $(c, r)_{LR}$ and the *ARIMA* interval forecasting of $(m, l, u)_{LR}$ in efficiency analysis are somewhat larger than the others, their fourth and fifth forecast intervals are very close to the actual intervals.

3.6 CONCLUSIONS

This chapter discusses the quality of the forecast result through evaluating forecasting performance, such as *MSEI*, *MRIE* and *MRXOR*. They had advantages and disadvantages as illustrated in Section 3.3 and Section 3.4. From the example in Section 3.5, we find *MRXOR* provides an important efficiency analysis for interval forecasting. Based on the value of *MRXOR* in different forecasting situations, such as *FIW, FIN, FIR* and *FIL*, it may modify the forecasting method of the center and radius respectively. It is noteworthy that if we can establish a good efficiency process, we can make a superior interval forecasting for the interval time series.

Although the approaches in this chapter proposed the efficiency evaluations of interval forecasting, there are some problems still remaining to be solved and some improvement can be done for further research, which is described respectively as follows.

(1) There are so many factors associated with interval data. Consequently, we only consider the boundaries of the intervals and their centers and the radiuses caused by all factors of

efficiency analysis in this chapter. If it needs to make the result more accurate, it can consider finding out the key factors of influencing the interval data.

(2) Besides *FIW, FIN, FIR* and *FIL*, a forecasting situation was not discussed in this chapter. This situation is that the forecast interval and the actual interval do not overlap at all. There are two cases: the forecast interval is certainly greater than the actual interval. And the forecast interval is certainly smaller than the actual interval (Interval FAQ from Dominque Faudot[39]). They are not good forecast outcomes at all. We don't like such forecast result happened certainly. Once it occurs. Computing their *MRXOR*s may reveal what drawbacks of the forecast system does? And how is the forecast scheme made improvements?

(3) What is a good forecast? When the forecast results have the same forecasting situations, they are easily judged which one is better forecast among those forecasting methods. While the forecast consequences are not in the same situation such as *FIW* and *FIN*, it is hard to choose between them. Especially their *MRXOR*s are equal; they always make us in confusion. Is the interval containing entire actual data and extra noisy message superior? Or is the interval which is not disturbed by the boisterous message but losing some data fit? It should be defined a criterion which is more sufficient to show the efficiency of interval forecasting.