國立政治大學應用數學系

碩士學位論文

The Tolerance Representations of Maximal Bipartite Outerplanar Graphs

最大，二分，外平面圖之容忍表示法

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Contents

Abstract

In this thesis, we prove a 2-connected graph G which is maximal outerplanar and bipartite is a tolerance graph if and only if there is no induced subgraphs $H_1,\ H_2,\ H_3$ and H_4 of G.

keywords: Tolerance Graphs; Maximal Outerplanar Graphs; Bipartite graphs.

中文摘要

在這篇論文中，我們針對2-連通的最大外平面圖而且是二分圖的圖形，討論其容忍表示法，並找到它的所有禁止子圖 H_1、H_2、H_3、H_4。

關鍵詞:最大外平面圖；二分圖；容忍表示法。

1 Introduction

1.1 History of Tolerance Graphs

In graph theory, *interval graphs* were introduced to study such problems of intersecting intervals on the real line. Each vertex v in an interval graph $G = (V, E)$ is associated with an interval I_v. If two vertices are connected by an edge in G, then the intersection of their associated intervals is nonempty. Those intervals could represent the durations of a set of events on a time line or sectors of consecutive elements of a linearly ordered set. When two intervals intersect, we could explain as their having something important in common.

Tolerance graphs which are a generalization of interval graph were introduced by Golumbic and Monma [1]. In a tolerance graph, each vertex can be represented by an interval and a tolerance. If the size of the intersection of two intervals is not at least as large as the tolerance associated with one of the vertices, then no edge is added between their vertices in the graph. Golumbic and Trenk [3] mention that their original motivation was the need to solve scheduling problems in which resources such as vehicles, support personnel, rooms, etc. may be needed on an exclusive basis, but where a measure of flexibility or tolerance would allow for sharing or relinquishing the resource if a solution is not otherwise possible. We can think of the tolerance graph as a model of conflicts for events occurring in a block of time, in which intervals represent a set of events on a time line and their associated tolerances are flexibility of the event.

Golumbic and Trenk [3] say that it is not even known how to get a tolerance representation when the graph is known to be a *tolerance graphs*. Similarly, the recognition problem for *bounded tolerance graph* is open when the graph is known to be a *bounded tolerance graph*. In spite of difficulty, we can also test whether it is not a *bounded tolerance graph* by some theorems. The complication of recognition

problem for the class of *tolerance graphs* is not solved. Hayward and Shamir [4] show that recognition of *tolerance graphs* is NP.

In our article, we discuss a 2-connected maximal outerplanar and bipartite graph G that is a *tolerance graph* and find all forbidden subgraphs of G.

At first, in Chapter 1, we introduce a interval graph and a tolerance graph which is generalization of interval graph. In Chapter 2, we introduce the definitions, theorems, and propositions of *tolerance graphs* and discuss some special tolerance graphs and their tolerance representations. In Chapter 3, we show a tolerance representation of a 2-connected maximal outerplanar and bipartite graph and we find all forbidden subgraphs to test whether if 2-connected maximal outerplanar and bipartite graphs are tolerance graphs. The graph in Figure 1, appeared in Golumbic and Trenk [3], shows the structure of *tolerance graphs*. It assist us to understand *tolerance graph* clearly. Finally, we bring up some open problems and further directions of research.

1.2 The Structure of Tolerance Graphs

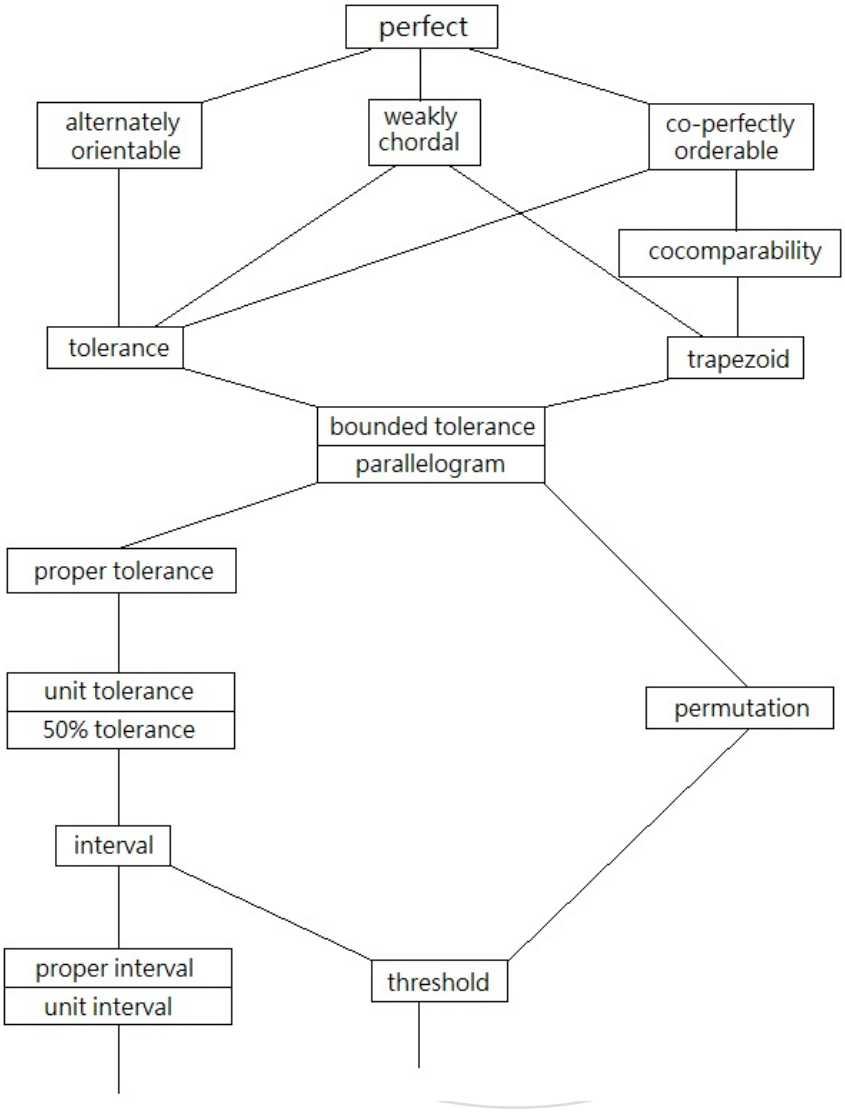

Figure 1: Structure of the tolerance graph, appeared in Golumbic and Trenk, 2004.

2 Tolerance Graphs

2.1 Definition and Theorem of Tolerance Graph

Definition 2.1. A graph $G = (V, E)$ is a *tolerance graph* if each vertex $x \in V$ can be assigned a closed interval I_x and a tolerance $t_x > 0$ and $t_x \in R$ so that $xy \in E$ if and only if $|I_x \cap I_y| \geq min\{t_x, t_y\}$. Such a collection of intervals and tolerances is called a *tolerance representation*.

Definition 2.2. A graph $G = (V, E)$ is a *bounded tolerance graph* if G has a tolerance representation with $t_v \leq |I_v|$, for all $v \in V$.

Example 2.3. Consider the graph in Figure 2 with tolerances $t_a = 1,\ t_b = 6,\ t_c = 1,\ t_d = 6$. Since

$$\begin{aligned}
ab \in E &\iff 5 = |I_a \cap I_b| > min\{t_a, t_b\} = 1 \\
bc \in E &\iff 1 = |I_b \cap I_c| \geq min\{t_b, t_c\} = 1 \\
cd \in E &\iff 4 = |I_c \cap I_d| > min\{t_c, t_d\} = 1 \\
ad \in E &\iff 1 = |I_a \cap I_d| \geq min\{t_a, t_d\} = 1 \\
ac \notin E &\iff 0 = |I_a \cap I_c| < min\{t_a, t_c\} = 1 \\
bd \notin E &\iff 3 = |I_b \cap I_d| < min\{t_b, t_d\} = 6,
\end{aligned}$$

G is a tolerance graph with tolerances $t_a = 1,\ t_b = 6,\ t_c = 1,\ t_d = 6$.

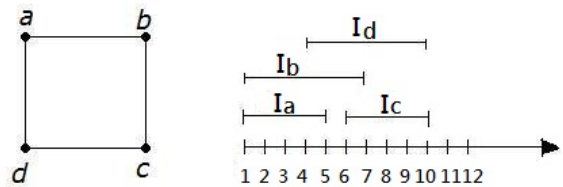

Figure 2: an example of tolerance graph and its tolerance representation

Definition 2.4. A *transitive orientation* F of graph $G = (V, E)$ is an assignment of a direction to each edge in E such that if $\overrightarrow{xy} \in F$ and $\overrightarrow{yz} \in F$ then $\overrightarrow{xz} \in F$.

Definition 2.5. A graph G is called a *comparability graph* if it has a transitive orientation.

Example 2.6. C_4 is a comparability graph but C_5 is not a comparability graph by the following graphs.
Let $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d\}$, $E_1 = \{ab, bc, cd, ad\}$ and $G_2 = (V_2, E_2)$ where $V_2 = \{a, b, c, d, e\}$, $E_2 = \{ab, bc, cd, ad, ea\}$. In any F_1 be an orientation to each edge of $G_1 = (V_1, E_1)$. We let $\overrightarrow{ab} \in F_1$, $\overrightarrow{cb} \in F_1$, $\overrightarrow{cd} \in F_1$ and $\overrightarrow{ad} \in F_1$. Hence, we can easily check F_1 is a transitive orientation of G_1.
Let F_2 be any transitive orientation of $G_2 = (V_2, E_2)$. Without loss of generality, we let $\overrightarrow{ea} \in F_2$. If $\overrightarrow{ab} \in F_2$, then $eb \in E_2$, but $eb \notin E_2$. Hence, if $\overrightarrow{ea} \in F_2$, then $\overrightarrow{ba} \in F_2$. Similarly, we know that $\overrightarrow{bc} \in F_2$, $\overrightarrow{dc} \in F_2$, $\overrightarrow{de} \in F_2$, and $\overrightarrow{ae} \in F_2$. We reach a contradiction to $\overrightarrow{ea} \in F_2$. Hence, G_2 has no transitive orientation.

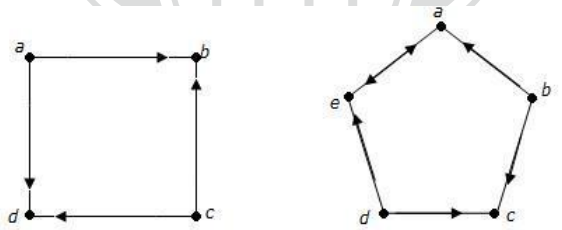

Figure 3: Graphs show that C_4 has a transitive orientation and C_5 has no transitive orientation.

Definition 2.7. A *cocomparability graph* is a graph whose complement is a comparability graph.

Definition 2.8. Three vertices $v_1, v_2, v_3 \in V(G)$ form an *asteroidal triple*(AT) of G if there is a path from v_i to v_j, for all $i, j \in \{1, 2, 3\}$ which avoids using any vertex in the closed neighborhood $N[v_k] = \{v_k\} \cup N(v_k)$.

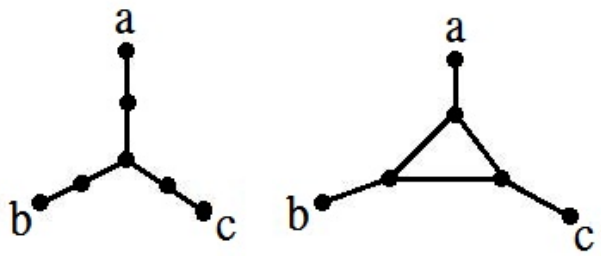

Figure 4: The vertices a, b, c form an asteroidal triple(AT) in both graphs.

Definition 2.9. A graph G is called *asteroidal triple free* ($AT-free$) if it contains no asteroidal triple(AT).

Lemma 2.10. *[2] All cocomparability graphs are AT-free.*

Theorem 2.11. *[3] If G is a bounded tolerance graph, then G is a cocomparability graph.*

Corollary 2.12. *If G is a bounded tolerance graph, G is AT-free.*

2.2 Bounded Tolerance Representations for Trees and Bipartite Graphs

Theorem 2.13. *[3] If T is a tree, the following are equivalent.*

(i) T is a bounded tolerance graph.

(ii) T has no subtree isomorphic to the graph T_2.

(iii) T is a caterpillar.

(iv) T is an interval graph.

(v) T is a permutation graph.

(vi) T is a cocomparability graph.

(vii) T has no asteroidal triple(AT).

Theorem 2.14. *[3] Let $G = (X, Y, E)$ be a bipartite graph. The following conditions are equivalent.*

(i) G is a bounded tolerance graph.

(ii) G is a trapezoid graph.

(iii) G is a cocomparability graph.

(iv) G is AT-free.

(v) G is a permutation graph.

2.3 A Tolerance Representation of C_4

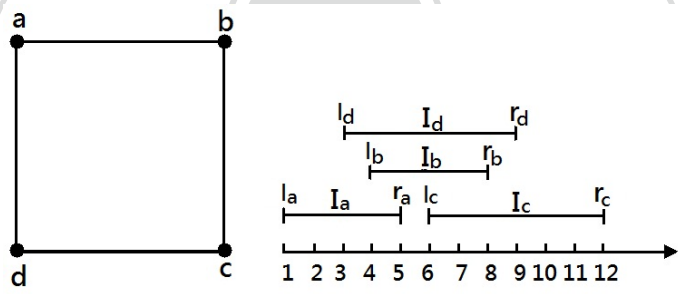

Figure 5: The graph G_1 and its tolerance representation.

Let $G_1 = (V_1, E_1)$, where $V_1 = \{a, b, c, d\}$, $E_1 = \{ab, bc, cd, ad\}$, $< I, t >$ be a tolerance representation of G_1 where $I = \{I_x | x \in V_1\}$, $t = \{t_x | x \in V_1\}$, $k_{xy} = |I_x \cap I_y|$ $x, y \in V_1$ and $I_x = (l_x, r_x)$.

Lemma 2.15. *If $xy \in E_1$ then $k_{xy} > k_{pq}$ for some $pq \notin E_1$.*

Proof. Because $ab \in E_1$, we have $k_{ab} = |I_a \cap I_b| \geq min\{t_a, t_b\}$. If $t_a \geq t_b$, then $k_{ab} = |I_a \cap I_b| \geq t_b > |I_b \cap I_d| = k_{bd}$. If $t_a < t_b$, then $k_{ab} = |I_a \cap I_b| \geq t_a > |I_a \cap I_c| = k_{ac}$. Therefore, $k_{ab} > k_{bd}$ or $k_{ab} > k_{ac}$. Similarly, $k_{xy} > k_{bd}$ or $k_{xy} > k_{ac}$, for all $x, y \in E_1$ and we can know that if $xy \in E_1$ then $k_{xy} > k_{pq}$, for some $pq \notin E_1$. □

By Lemma 2.15, we have the following lemma.

Lemma 2.16. *One of k_{ac}, k_{bd} is the smallest value in k_{xy}, for all $x, y \in V_1$.*

Without loss of generality, we assume that k_{ac} is the smallest value of k_{xy}, for all $x, y \in V_1$.

Proposition 2.17. *We have the following properties about $G_1 = (V_1, E_1)$ in Figure 5.*

(i) $t_a \leq k_{ab}$ *and* $t_c \leq k_{bc}$*;*

(ii) $t_a \leq k_{ad}$ *and* $t_c \leq k_{cd}$*;*

(iii) If $r_b > r_d$ *then* $t_b > k_{cd}$*;*

(iv) If $r_d > r_b$ *then* $t_d > k_{bc}$*.*

Proof. (i) At first, we discuss the position of I_a and I_c. If $I_a \subseteq I_c$, then $k_{ac} = |I_a \cap I_c| = |I_a|$. Because $ab \in E$, we have $k_{ab} > k_{ac} = |I_a|$. We reach a contradiction to $k_{ab} \leq |I_a|$. Hence, $I_a \nsubseteq I_c$. Similarly, $I_c \nsubseteq I_a$. Without loss of generality, we let $l_a < l_c$ and $r_a < r_c$. Next, we know that $k_{ab} > k_{ac}$ and $k_{bc} > k_{ac}$. Therefore, $l_b < l_c$ and $r_b > r_a$. The tolerance representation of G_1 is shown in Figure 6. Hence, we have the following four cases about t_a and t_c.

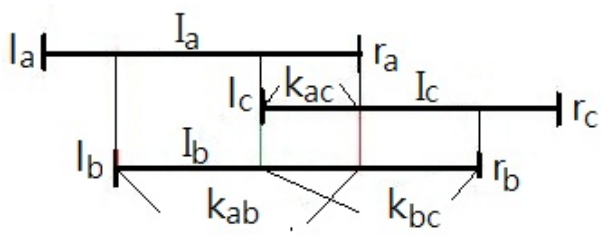

Figure 6: The tolerance representation of the vertices a, b and c.

case1: Suppose that the tolerance $t_a > k_{ab}$ and the tolerance $t_c > k_{bc}$.

Since $ab \in E_1$ and $bc \in E_1$, we have the result $t_b \leq k_{ab}$ and $t_b \leq k_{bc}$. Because $k_{ad} > k_{ac}$ and $k_{cd} > k_{ac}$, $l_d < l_c$ and $r_d > r_a$. Suppose that $l_d < l_b$. We obtain that $k_{ab} \leq k_{bd}$, but $k_{bd} = |I_b \cap I_d| < min\{t_b, t_d\} \leq t_b \leq k_{ab}$ which reaches a contradiction (see Figure 7). Hence, we know that $l_d \geq l_b$. The graph is shown in Figure 8. Because $ad \in E_1$, $k_{ad} = |I_a \cap I_d| \geq min\{t_a, t_d\}$. We know that $t_a > k_{ab} \geq k_{ad}$ (see Figure 8). Therefore, we obtain that $t_d \leq k_{ad}$. Hence, $|I_b \cap I_d| = k_{bd} \geq k_{ad} \geq t_d$, it implies that $bd \in E_1$, again, we reach a contradiction.

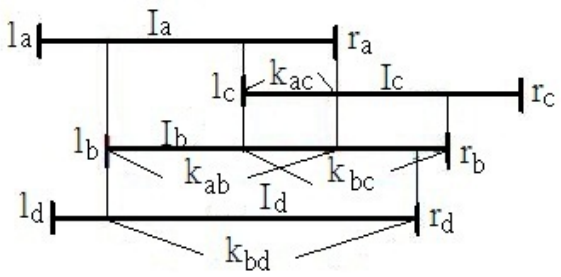

Figure 7: The tolerance representation of G_1 with $l_d < l_b$.

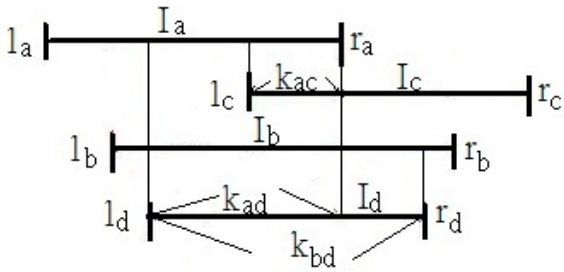

Figure 8: The tolerance representation of G_1 with $l_d \geq l_b$.

case2: Suppose that the tolerance $t_a > k_{ab}$ and the tolerance $t_c \leq k_{bc}$.

Since $ab \in E_1$, we get that $t_b \leq k_{ab}$. Because $k_{ad} > k_{ac}$ and $k_{cd} > k_{ac}$, $l_d < l_c$ and $r_d > r_a$. Suppose that $l_d < l_b$. We obtain that $k_{ab} \leq k_{bd}$, but $k_{bd} = |I_b \cap I_d| < min\{t_b, t_d\} \leq t_b \leq k_{ab}$ which reaches a contradiction (see Figure 7). Hence, we know that $l_d \geq l_b$. The graph is shown in Figure 8. Because $ad \in E_1$, $k_{ad} = |I_a \cap I_d| \geq min\{t_a, t_d\}$. We know that $t_a > k_{ab} \geq k_{ad}$ (see Figure 8). Therefore, we obtain that $t_d \leq k_{ad}$. Hence, $|I_b \cap I_d| = k_{bd} \geq k_{ad} \geq t_d$, it implies that $bd \in E_1$, again, we reach a contradiction.

case3: Suppose that the tolerance $t_a \leq k_{ab}$ and $t_c > k_{bc}$.

Since $bc \in E_1$, we obtain that $t_b \leq k_{bc}$. Similarly, we also reach a contra-

diction.

Therefore, we have $t_a \leq k_{ab}$ and $t_c \leq k_{bc}$.

(ii) We can exchange two points b and d and use the result of (i) to get that $t_a \leq k_{ad}$ and $t_c \leq k_{cd}$.

(iii) If $r_b > r_d$, we can easily obtain that $t_b > k_{bd} = |I_b \cap I_d| \geq |I_c \cap I_d| = k_{cd}$ from Figure 9.

(iv) If $r_b < r_d$, we can easily obtain that $t_d > k_{bd} = |I_b \cap I_d| \geq |I_b \cap I_c| = k_{bc}$ from Figure 10.

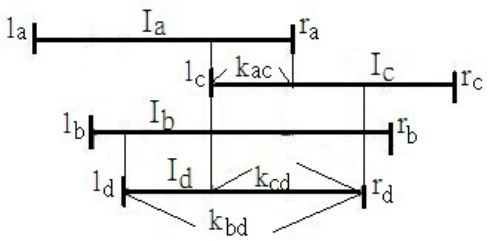

Figure 9: The tolerance representation of G_1 with $r_b > r_d$.

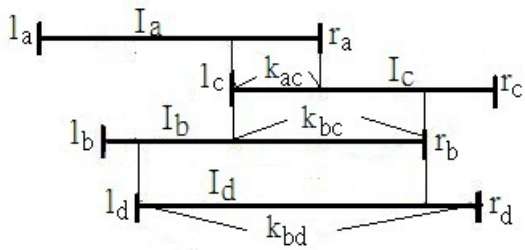

Figure 10: The tolerance representation of G_1 with $r_b < r_d$.

□

2.4 A Tolerance Representation of Concatenation of Two 4-cycles

Let G_2 be the concatenation of two 4-cycles (see Figure 11).

Because G_2 is AT-free and by Theorem 2.14, it is a bounded tolerance graph. Hence it has a tolerance representation. Let $G_2 = (V_2, E_2)$, $V_2 = \{a, b, c, d, e, f\}$, $E_2 = \{ab, bc, cd, ad, be, ef, cf\}$ and $< I, t >$ be a tolerance representation of G_2 where $I = \{I_x | x \in V_2\}$, $t = \{t_x | x \in V_2\}$, $k_{xy} = |I_x \cap I_y|$, for all $x, y \in V_2$ and $I_x = (l_x, r_x)$.

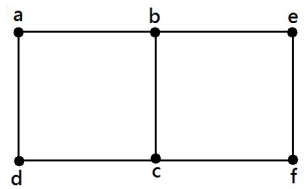

Figure 11: The graph G_2.

Proposition 2.18. *If k_{ac} is the smallest value in all of k_{xy}, for all $x, y \in \{a, b, c, d\}$, then k_{ce} is the smallest value in all of k_{ij}, for all $i, j \in \{b, c, e, f\}$.*

Proof. Let k_{ac} is the smallest value in all of $k_{xy}s$, for all $x, y \in \{a, b, c, d\}$. Because the subgraph $G_{\{b,e,f,c\}}$ of G_2 is also a C_4, one of k_{ce} and k_{bf} is the smallest value in all of $k_{ij},\ i, j \in \{b, c, e, f\}$ by *Lemma* 2.16. If k_{bf} is the smallest value in all of $k_{ij},\ i, j \in \{b, c, e, f\}$. Because the subgraph $G_{\{a,b,c,d\}}$ of G_2 is a C_4, we can draw a tolerance representation of a subgraph G_1 of G_2, where $G_1 = (V_1, E_1),\ V_1 = \{a, b, c, d\},\ E_1 = \{ab, bc, cd, ad\}$. The tolerance representation is shown in Figure 12. Next, we discuss the right endpoints of the interval I_b and I_d. If $r_b \leq r_d$, we get that $k_{bd} \geq k_{bc} \geq t_b$ (see Figure 13). Therefore, we obtain that $bd \in E_2$. It contradicts to the fact that $bd \notin E_2$. If $r_b > r_d$, by Proposition 2.17 (ii) and (iii), we have $t_b > k_{cd}$ and $t_c \leq k_{cd}$. Therefore, $t_b > k_{cd} \geq t_c$. Now, we discuss the left endpoints of I_c and I_e. If $l_c \geq l_e$, we get that $k_{ce} \geq k_{cd} \geq t_c$(see Figure 14). Therefore, we obtain $ce \in E_2$, but $ce \notin E_2$. We get a contradiction. If $l_c < l_e$, we can easily see that $k_{ce} \geq k_{be} \geq t_b > t_c$(see Figure 15). Hence, we also get a contradiction. Thus, k_{ce} is the smallest value in all $k_{ij},\ i, j \in \{b, c, e, f\}$.

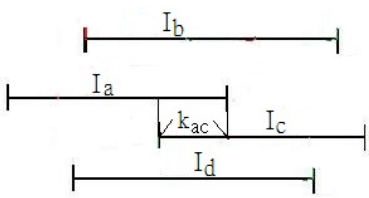

Figure 12: A tolerance representation of a subgraph G_1 of G_2.

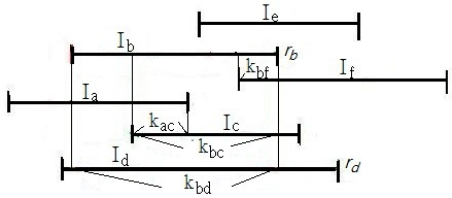

Figure 13: A tolerance representation of G_2 with $r_b \leq r_d$.

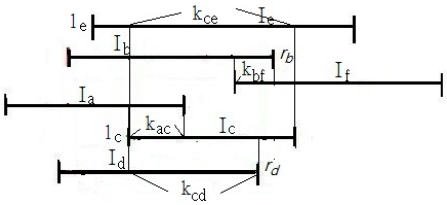

Figure 14: A tolerance representation of G_2 with $r_b > r_d$ and $l_c \geq l_e$.

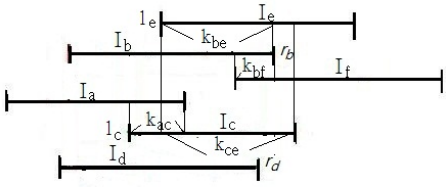

Figure 15: A tolerance representation of G_2 with $r_b > r_d$ and $l_c < l_e$.

□

2.5 A Tolerance Representation of Concatenation of Three 4-cycles

There are two types of the concatenation of three 4-cycles. The first type is shown in Figure 16 which will be mentioned later. Now, we discuss the second type, say G_3 (see Figure 17).

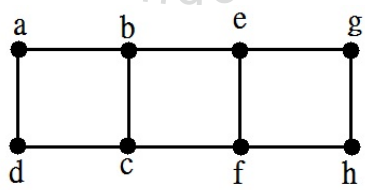

Figure 16: The first type of graph G_3.

By Figure 17, we know that G_3 is a tolerance graph. Hence, it has a tolerance representation. Let $G_3 = (V_3, E_3)$, $V_3 = \{a, b, c, d, e, f, g, h\}$, $E_3 = \{ab, bc, cd, ad, be, ef,$

$cf, ch, fg, gh\}$, $< I, t >$ be a tolerance representation of G_3, where $I = \{I_x | x \in V_3\}$, $t = \{t_x | x \in V_3\}$, $k_{xy} = |I_x \cap I_y| x, y \in V_3$ and $I_x = (l_x, r_x)$. By Proposition 2.18, we know that if k_{ac} is the smallest value in all k_{xy}, for all $x, y \in \{a, b, c, d\}$, then k_{ce} is the smallest value in all k_{ij}, for all $i, j \in \{b, c, e, f\}$. We can use Proposition 2.18 again to get that k_{cg} is the smallest value in all k_{pq}, for all $p, q \in \{c, f, g, h\}$.
Now, we want to prove that it is impossible that k_{ac} is the smallest value of k_{xy}, for all $x, y \in \{a, b, c, d\}$ of G_3.

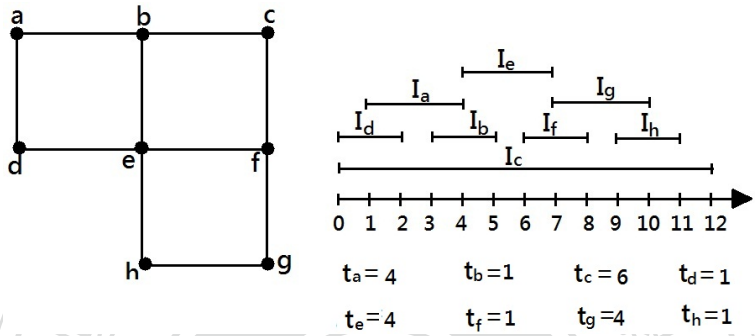

Figure 17: The second type of graph G_3 and its tolerance representation.

Proposition 2.19. *If $< I, t >$ is a tolerance representation of G_3 and k_{xy} is defined by above, then k_{ac} is not the smallest value of k_{xy}, for all $x, y \in \{a, b, c, d\}$ in G_3.*

Proof. For a contradiction, suppose that k_{ac} is the smallest value of k_{xy}, for all $x, y \in \{a, b, c, d\}$. By Proposition 2.18, we obtain that k_{ce} is the smallest value of k_{ij}, for all $i, j \in \{b, c, e, f\}$ and k_{cg} is the smallest value of k_{pq}, for all $p, q \in \{c, f, g, h\}$. Therefore, we can draw a tolerance representation of subgraph G_2 of G_3, where $G_2 = (V_2, E_2)$, $V_2 = \{a, b, c, d, e, f\}$, $E_2 = \{ab, bc, cd, ad, be, ef, cf\}$. The tolerance representation of subgraph G_2 of G_3 is shown in Figure 18.

Now, consider the interval I_g, it has two conditions in the tolerance representation of G_3. One is that $I_g \subseteq I_c$ and the other is that $I_g \nsubseteq I_c$. At first, let $I_g \subseteq I_c$, it implies that $k_{cg} = |I_c \cap I_g| = |I_g| \geq |I_f \cap I_g| = k_{fg}$. We get a contraction to the fact that k_{cg} is the smallest value of k_{pq}, for all $p, q \in \{c, f, g, h\}$. So we obtain

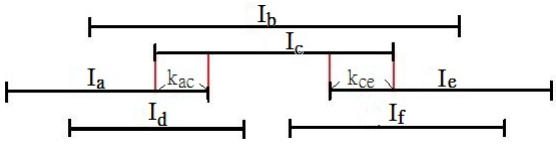

Figure 18: The tolerance representation of subgraph G_2 of G_3.

that $I_g \nsubseteq I_c$. Because $fg \in E_3$, that is, $I_f \cap I_g$ is not empty. Next, we continue to consider the size of k_{cg} and k_{ce}.

If $k_{cg} \geq k_{ce}$, then $l_g \leq l_e$. We can add an interval I_g in Figure 18 and is shown in the following figures. From Figure 19, we get that if $r_g \leq r_b$ then $|I_g \cap I_b| = |I_g| \geq |I_g \cap I_f| \geq t_g$. It implies that $gb \in E_3$, but $gb \notin E_3$. Therefore, $r_g > r_b$. Similarly, $r_g < r_e$ and $r_b \leq r_e$. Hence, we can get that $r_b \leq r_g \leq r_e$. Next, consider the interval I_f, then we have some cases in the right endpoints of I_b, I_e, I_f and I_g. We use figures to explain the contradictions of the following cases.

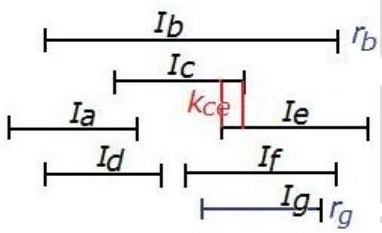

Figure 19: The tolerance representation of G_3 with $r_g \leq r_b$.

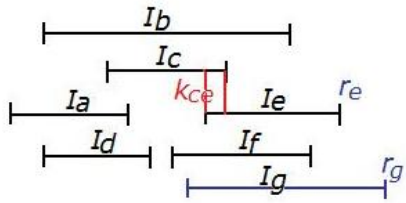

Figure 20: The tolerance representation of G_3 with $r_g \geq r_e$.

Case1: Suppose that $r_f \leq r_b \leq r_g \leq r_e$ or $r_b \leq r_f \leq r_g \leq r_e$.

From the following figures, we can easily get that $|I_g \cap I_e| \geq |I_e \cap I_f| \geq t_e$. Hence, we obtain that $ge \in E_3$. But, it contradicts to the fact that $ge \notin E_3$.

Case2: Suppose that $r_b \leq r_g \leq r_f \leq r_e$ or $r_b \leq r_g \leq r_e \leq r_f$.

From the following figures, we can easily get that $|I_g \cap I_e| \geq |I_e \cap I_b| \geq t_e$. Hence, we obtain that $ge \in E_3$. But, it contradicts to the fact that $ge \notin E_3$.

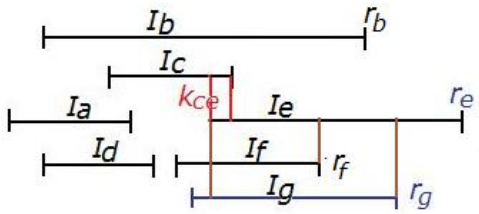

Figure 21: The tolerance representation of G_3 with $r_f \leq r_b \leq r_g \leq r_e$.

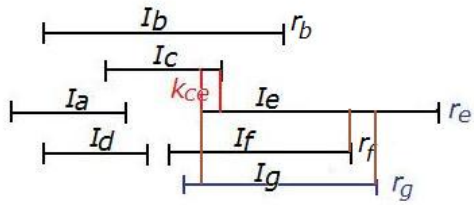

Figure 22: The tolerance representation of G_3 with $r_b \leq r_f \leq r_g \leq r_e$.

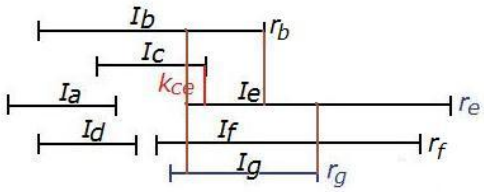

Figure 23: The tolerance representation of G_3 with $r_b \leq r_g \leq r_f \leq r_e$.

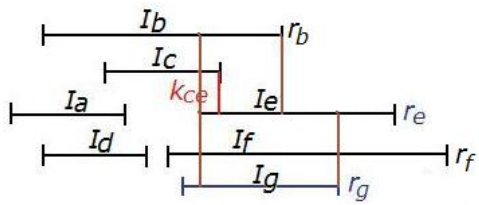

Figure 24: The tolerance representation of G_3 with $r_b \leq r_g \leq r_e \leq r_f$.

If $k_{cg} < k_{ce}$, then $l_g > l_e$. We add an interval I_g in Figure 18 and is shown in the following figures. In the Figure 25, we can prove that $r_g > r_b$ and $r_g > r_e$ by similar way in the case $k_{cg} \geq k_{ce}$. Next, we also discuss the condition of the right endpoints of I_b, I_e, I_f and I_g.

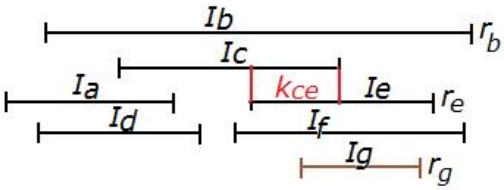

Figure 25: The tolerance representation of G_3 with $r_g \leq r_b$ and $r_g \leq r_e$.

Case1: Suppose that $r_f \leq r_b \leq r_e < r_g$, $r_f \leq r_e < r_b < r_g$ or $r_b \leq r_f \leq r_e < r_g$.

From Figure 26, Figure 27 and Figure 28, we see that $|I_g \cap I_e| > |I_f \cap I_g| \geq t_g$. Therefore, we get that $ge \in E_3$, but in fact $ge \notin E_3$. Hence, we obtain a contradiction.

Case2: Suppose that $r_b \leq r_e \leq r_f < r_g$.

We can draw a graph shown in Figure 29 and in the same figure the interval I_f will not cause any contradiction. Next, we continue to discuss left endpoint and right endpoint of the interval I_h. If $l_e < l_h$, then $|I_c \cap I_h| < |I_c \cap I_e| < t_c$ from Figure 29. This contradicts to the fact that $t_c \leq |I_c \cap I_h|$.Therefore, we get that $l_e \geq l_h$. There are three cases about the position of the right endpoint r_h of the interval I_h.

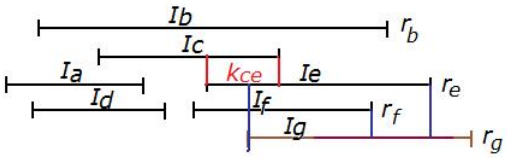

Figure 26: The tolerance representation of G_3 with $r_f \leq r_b \leq r_e < r_g$.

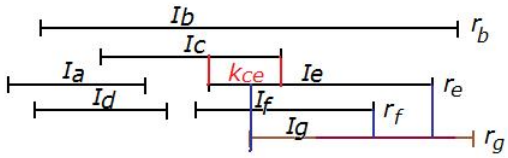

Figure 27: The tolerance representation of G_3 with $r_f \leq r_e < r_b < r_g$.

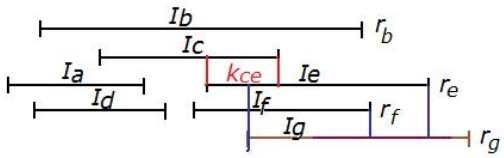

Figure 28: The tolerance representation of G_3 with $r_b \leq r_f \leq r_e < r_g$.

(i) If $r_h \geq r_e$, we can easily get that $|I_h \cap I_e| = |I_e| \geq |I_e \cap I_f| \geq t_e$ from the following figure. It implies that $he \in E_3$, but $he \notin E_3$. We obtain a contradiction.

(ii) If $r_b < r_h < r_e$, we can easily see that $|I_h \cap I_e| \geq |I_b \cap I_e| \geq t_e$ from

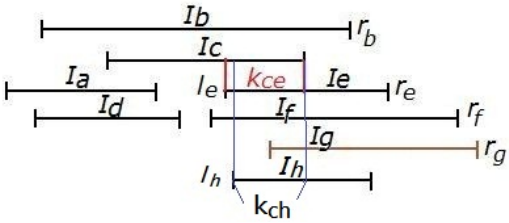

Figure 29: The tolerance representation of G_3 with $r_b \leq r_e \leq r_f < r_g$ and $l_e < l_h$.

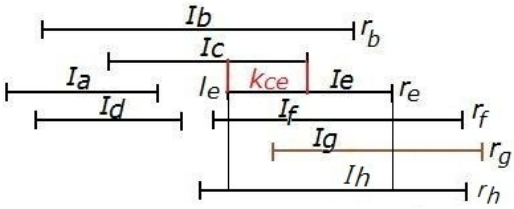

Figure 30: The tolerance representation of G_3 with $r_h \geq r_e$.

the following figure. It implies that $he \in E_3$, but $he \notin E_3$. We reach a contradiction.

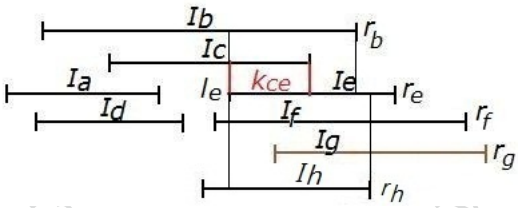

Figure 31: The tolerance representation of G_3 with $r_b < r_h < r_e$.

(iii) If $r_h \leq r_b$, we can easily see that $|I_g \cap I_e| \geq |I_h \cap I_g| \geq t_g$ from the following figure. It implies that $ge \in E_3$, but $ge \notin E_3$. We obtain a contradiction.

Case3: Suppose that $r_b \leq r_e < r_g \leq r_f$.

A contradiction of this case is the same as Case2.

Case4: Suppose that $r_e \leq r_f < r_b < r_g$.

From Figure 33, we get that $|I_g \cap I_b| \geq |I_g \cap I_f| \geq t_g$. Therefore, we get that

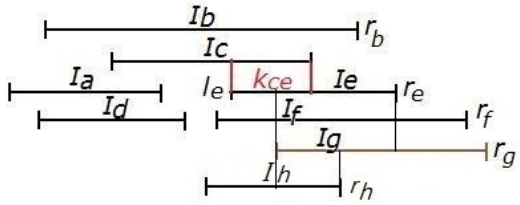

Figure 32: The tolerance representation of G_3 with $r_h \leq r_b$.

$gb \in E_3$, but in fact $gb \notin E_3$. We reach a contradiction.

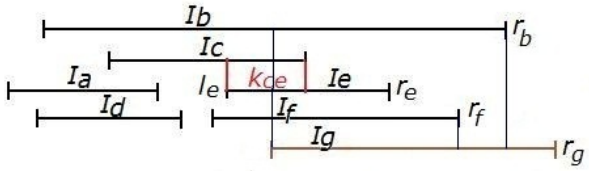

Figure 33: The tolerance representation of G_3 with $r_e \leq r_f < r_b < r_g$.

Case5: Suppose that $r_e < r_b \leq r_f < r_g$.

This case and Case2 are the same. We also discuss the left endpoint and the right endpoint of the interval I_h. If $l_e < l_h$, then we get that $|I_c \cap I_h| < k_{ce} = |I_c \cap I_e| < t_c$ from the Figure 34. This contradicts to the fact that $t_c \leq |I_c \cap I_h|$. Therefore, we know that $l_h \leq l_e$. Next, consider the right endpoint of I_h. There are two cases in the right endpoint r_h of the interval I_h.

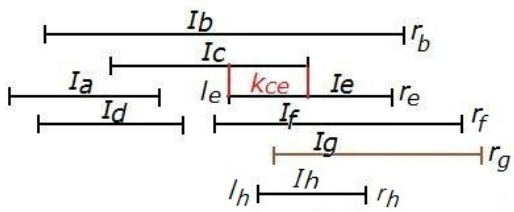

Figure 34: The tolerance representation of G_3 with $l_e < l_h$.

(i) If $r_h \geq r_e$, we can easily get that $|I_h \cap I_e| = |I_e| \geq |I_e \cap I_f| \geq t_e$ from the following figure. It implies that $he \in E_3$, but $he \notin E_3$. We obtain a contradiction.

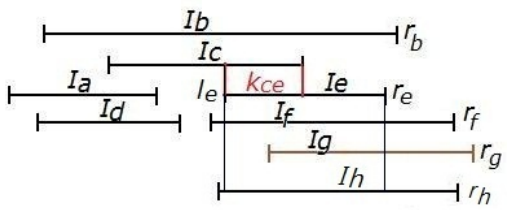

Figure 35: The tolerance representation of G_3 with $r_h \geq r_e$.

(ii) If $r_b < r_h < r_e$, we can easily get that $|I_h \cap I_e| \geq |I_b \cap I_e| \geq t_e$ from the following figure. It implies that $he \in E_3$, but $he \notin E_3$. We reach a contradiction.

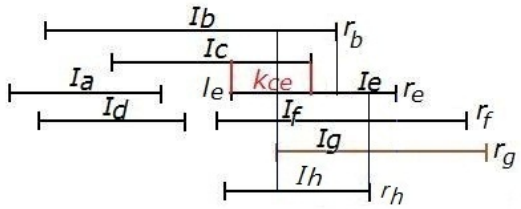

Figure 36: The tolerance representation of G_3 with $r_h < r_e$.

Case6: Suppose that $r_e < r_b < r_g \leq r_f$.

A contradiction of this case is the same as Case5.

□

3 Some Results on Maximal Outerplanar Graphs

3.1 A 2-connected Graph Which Is Maximal Outerplanar Graph and Bipartite Is Not Necessarily a Tolerance Graph.

Definition 3.1. A graph G is an *outerplanar graph* if it has an embedding in the plane with every vertex on the boundary of the unbounded face. A *maximal outerplanar graph* is a simple outplanar graph that is not a spanning subgraph of a large simple outplanar graph.

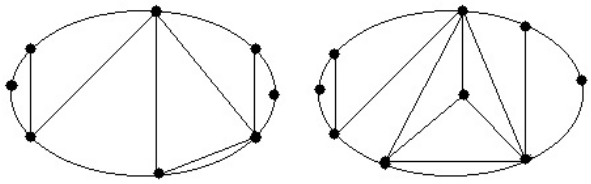

Figure 37: The left is an example of outerplanar graph and the right is an example of nonouterplanar graph.

In this article, we discuss a graph that is maximal outerplanar and bipartite. Furthermore, we want to know that whether this graph is a tolerance graph or not. We adopt that every maximal outerplanar graph in this article is 2-connected.

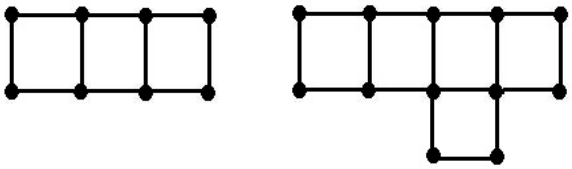

Figure 38: Some graphs are maximal outerplanar and bipartite.

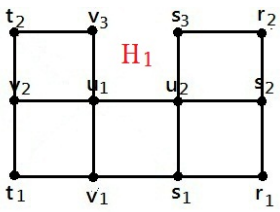

Figure 39: The graph H_1.

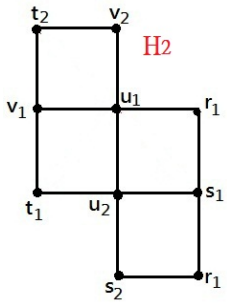

Figure 40: The graph H_2.

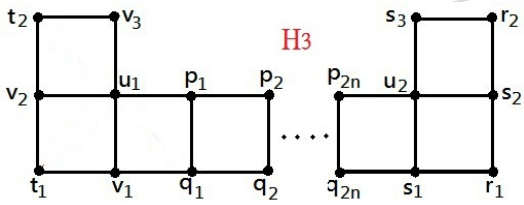

Figure 41: The graph H_3.

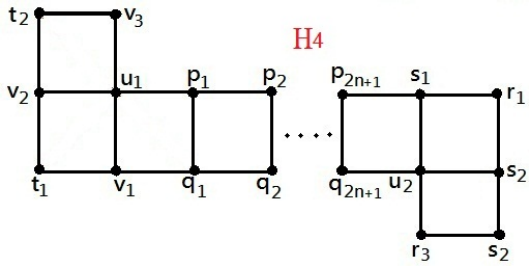

Figure 42: The graph H_4.

Theorem 3.2. *Let G be a maximal outerplanar and bipartite graph with vertices number $n(G) \geq 4$. G is a tolerance graph if and only if G has no induced subgraphs H_1, H_2, H_3 and H_4.*

Proof. Let G=(V, E) be a graph that is maximal outerplanar and bipartite. Then we have the following cases of the degree of v where $v \in V$.

Case1: Suppose that the degree of v is not more than 3 for all $v \in V$.

In this case, it is easy to know that G is AT-free. Therefore, by Theorem 2.14, we obtain that G is a bounded tolerance graph. The graph G is shown in Figure 43.

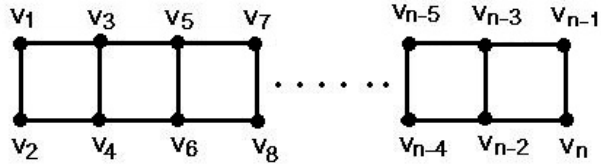

Figure 43: The graph is maximal outerplanar and bipartite with deg(v) ≤ 3 for all $v \in V$.

Case2: Suppose that there is the only one vertex u$\in V$ whose degree is greater than 3.

Let u be the common neighbor of v_1, v_2, ..., v_k. Let t_1, t_2, ..., t_{k-1} be vertices between v_i and v_{i+1}, $i = 1, 2, ..., k-1$, respectively. The graph G is shown in Figure 44. We have a tolerance representation of G that is shown in Figure 45. Hence, G is a tolerance graph with tolerances $t_u = \infty$, $t_{v_i} = 1$, for all $i = 1, 2, ..., k$ and $t_{t_j} = \infty$, for all $j = 1, 2, ..., k-1$.

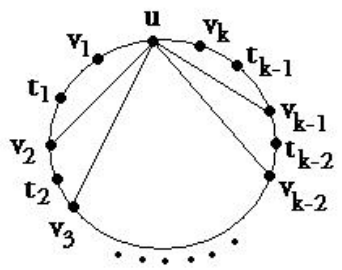

Figure 44: The graph is maximal outerplanar and bipartite with deg(u)$\geq$ 4. for the only vertex $u \in V$.

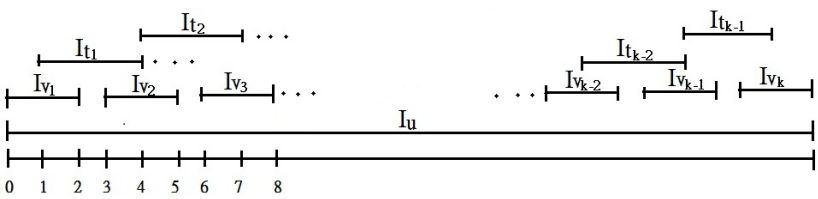

Figure 45: The tolerance representation of G

Case3: Suppose that there are more than two vertices whose degree are greater than 3.

Let u_1 and u_2 be two vertices of G with deg(u_1)$\geq$ 4 and deg(u_2)$\geq$ 4. We have the following relation in u_1 and u_2.

(i) If the edge $u_1u_2 \in E$ and u_1u_2 on the boundary of the unbounded face. Let u_1 be the common neighbor of $v_1,\ v_2,\ ...,\ v_k$, and u_2 be the common neighbor of $s_1,\ s_2,\ ...,\ s_i$. Let $t_1,\ t_2,\ ...,\ t_{k-1}$ be vertices between v_i and $v_{i+1},\ i = 1, 2, ..., k-1$, respectively and $r_1,\ r_2,\ ...,\ r_{i-1}$ be vertices between s_j and $s_{j+1},\ j = 1, 2, ..., i-1$, respectively. The graph G is shown in Figure 46.

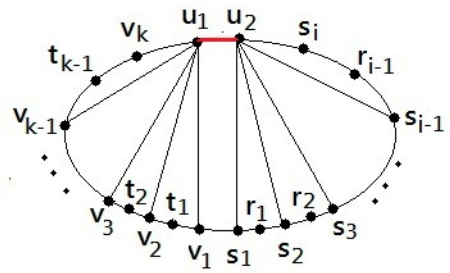

Figure 46: The graph G.

For a contradiction, suppose that G is a tolerance graph. Therefore, G has a tolerance representation $< I, t >$, where $I = \{I_x | x \in V\}$, $t = \{t_x | x \in V\}$, $k_{xy} = |I_x \cap I_y|, x, y \in V$. We consider the subgraph H_1 of G that is shown in Figure 47 and Figure 48. Because G is a tolerance graph, the subgraph H_1 of G also has a tolerance representation. By Lemma 2.16, we know that one of $k_{t_2u_1}$ and $k_{v_2v_3}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G. By Proposition 2.19, we know that $k_{t_2u_1}$ is not the smallest value of k_{xy}, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G. Hence, we get that $k_{v_2v_3}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G. By Proposition 2.18, we can obtain that $k_{v_1u_2}$

is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_1, s_1, u_2\}$ in G. Since $k_{v_1u_2}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_1, s_1, u_2\}$ in G, but $k_{v_1u_2}$ is not the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_1, s_1, u_2\}$ in G by Proposition 2.19. We reach a contradiction. Therefore, G is not a tolerance graph.

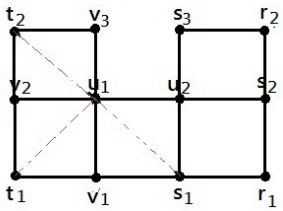

Figure 47: The subgraph H_1 of G that $k_{t_2u_1}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

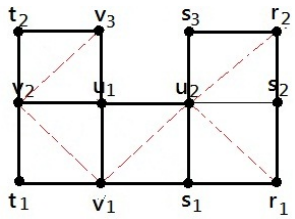

Figure 48: The subgraph H_1 of G that $k_{v_2v_3}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

(ii) If the edge $u_1u_2 \in E$ and u_1u_2 is not on the boundary of the unbounded face.

Let u_1 be the common neighbor of $v_1,\ v_2,\ ...,\ v_k$, and u_2 be the common neighbor of $s_1,\ s_2,\ ...,\ s_i$. Let $t_1,\ t_2,\ ...,\ t_k$ be vertices between v_{i-1} and $v_i,\ i = 1, 2, ..., k$, respectively and $r_1,\ r_2,\ ...,\ r_i$ be vertices between s_{j-1} and $s_j,\ j = 1, 2, ..., i$, respectively. The graph G is shown in Figure 49.

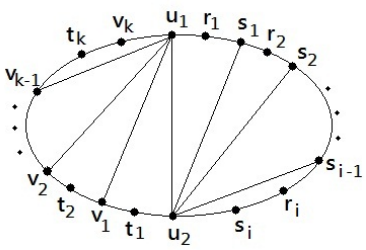

Figure 49: The graph G.

Similarly, we consider the subgraph H_2 of G that is shown in the following figures. By Proposition 2.18 and Proposition 2.19, we also prove that G is not a tolerance graph.

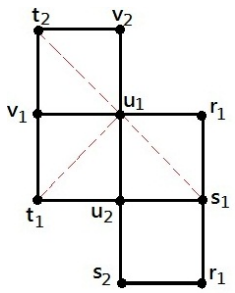

Figure 50: The subgraph H_2 of G that $k_{t_2u_1}$ is the smallest value of k_{xy}, for all $x, y \in \{u_1, v_1, t_2, v_2\}$ in G.

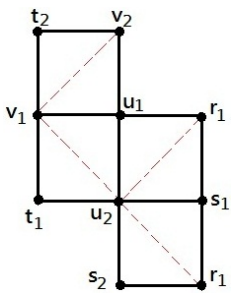

Figure 51: The subgraph H_2 of G that $k_{v_1v_2}$ is the smallest value of k_{xy}, for all $x,\ y \in \{u_1,\ v_1,\ t_2,\ v_2\}$ in G.

(iii) If the edge $u_1u_2 \notin E$ and the shortest path from u_1 to u_2 with odd length is on the boundary of the unbounded face.

Let u_1 be the common neighbor of $v_1,\ v_2,\ ...,\ v_k$, and u_2 be the common neighbor of $s_1,\ s_2,\ ...,\ s_i$. Let $t_1,\ t_2,\ ...,\ t_{k-1}$ be vertices between v_{i-1} and $v_i,\ i = 1, 2, ..., k$, respectively and $r_1,\ r_2,\ ...,\ r_{i-1}$ be vertices between s_{j-1} and $s_j,\ j = 1, 2, ..., i$, respectively. Let $p_1,\ p_2, ...,\ p_{2n}$ be vertices of the shortest path from u_1 to u_2 and $q_1,\ q_2,\ ...,\ q_{2n}$ be vertices of the shortest path from v_1 to s_1 where $p_jq_j \in E$, for all $j = 1, ..., 2n$. The graph G is shown in Figure 52.

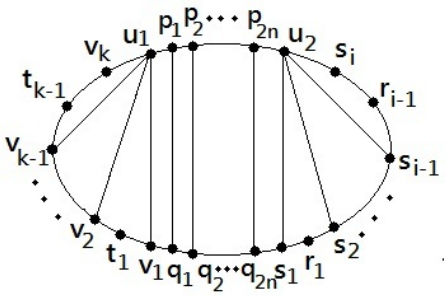

Figure 52: The graph G.

Similarly, we consider the subgraph H_3 of G that is shown in the following figures. By Proposition 2.18 and Proposition 2.19, we also prove that G is not a tolerance graph.

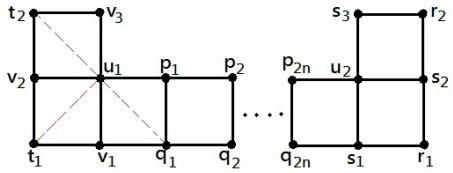

Figure 53: The subgraph H_3 of G that $k_{t_2u_1}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

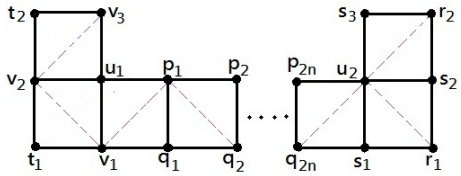

Figure 54: The subgraph H_3 of G that $k_{v_2v_3}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

(iv) If the edge $u_1u_2 \notin E$ and the shortest path from u_1 to u_2 with even length is on the boundary of the unbounded face.

Let u_1 be the common neighbor of v_1, v_2, ..., v_k, and u_2 be the common neighbor of s_1, s_2, ..., s_i. Let t_1, t_2, ..., t_{k-1} be vertices between v_{i-1} and v_i, $i = 1, 2, ..., k$, respectively and r_1, r_2, ..., r_{i-1} be vertices between s_{j-1} and s_j, $j = 1, 2, ..., i$, respectively. Let p_1, p_2, ..., p_{2n+1} be vertices of the shortest path from u_1 to u_2 and q_1, q_2, ..., q_{2n+1} be vertices of the shortest path from v_1 to s_1 where $p_jq_j \in E$, for all $j = 1, ..., 2n+1$. The graph G is shown in Figure 55.

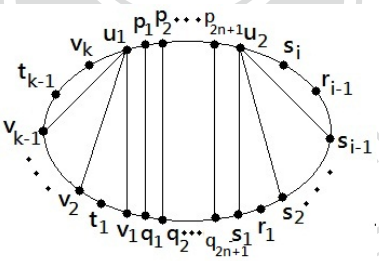

Figure 55: The graph G.

We have a tolerance representation of G that is shown in Figure 56. Hence, G is a tolerance graph with tolerances $t_{u_1} = \infty$, $t_{u_2} = \infty$, $t_{v_j} = 1$, for all $j = 1, 2, ..., k$, $t_{s_j} = 1$, for all $j = 1, 2, ..., i$, $t_{t_j} = \infty$, for all $j = 1, 2, ..., k-1$, $t_{r_j} = \infty$, for all $j = 1, 2, ..., i-1$, $t_{p_j} = 1$, for all $j = 1, 3, ..., 2n+1$, $t_{p_j} = \infty$, for all $j = 2, 4, ..., 2n$, $t_{q_j} = 1$, for all $j = 2, 4, ..., 2n$ and $t_{p_j} = \infty$, for all $j = 1, 3, ..., 2n+1$.

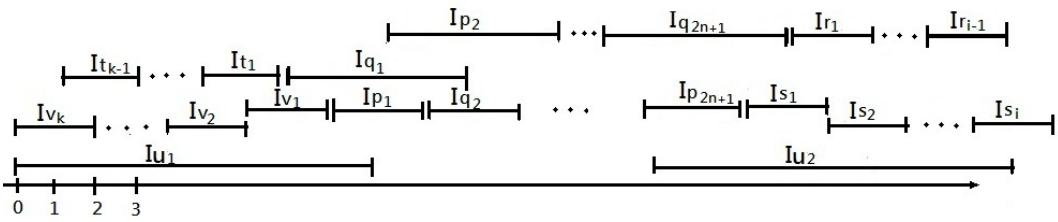

Figure 56: The tolerance representation of graph G.

(v) If the edge $u_1u_2 \notin E$ and the shortest path from u_1 to u_2 with odd length is not on the boundary of the unbounded face.

Let u_1 be the common neighbor of $v_1,\ v_2,\ ...,\ v_k$, and u_2 be the common neighbor of $s_1,\ s_2,\ ...,\ s_i$. Let $t_1,\ t_2,\ ...,\ t_{k-1}$ be vertices between v_{i-1} and $v_i,\ i = 1, 2, ..., k$, respectively and $r_1,\ r_2,\ ...,\ r_{i-1}$ be vertices between s_{j-1} and $s_j,\ j = 1, 2, ..., i$, respectively. Let $p_1,\ p_2,\ ...,\ p_{2n+1}$ be vertices of the shortest path from u_1 to s_1 and $q_1,\ q_2,\ ...,\ q_{2n+1}$ be vertices of the shortest path from v_1 to u_2 where $p_jq_j \in E$, for all $j = 1, ..., 2n+1$. The graph G is shown in Figure 57.

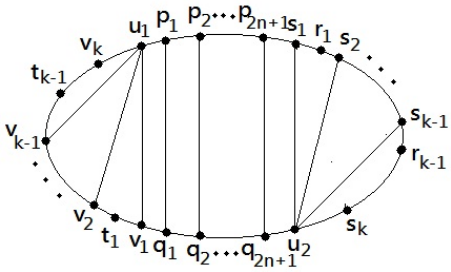

Figure 57: The graph G.

Similarly, we consider the subgraph H_4 of G that is shown in the following figures. By Proposition 2.18 and Proposition 2.19, we also prove that G is not a tolerance graph.

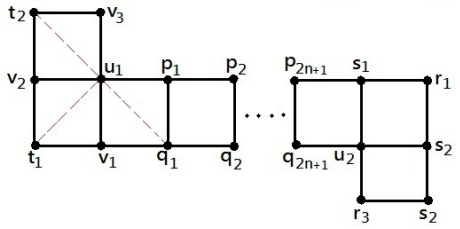

Figure 58: The subgraph H_4 of G that $k_{t_2u_1}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

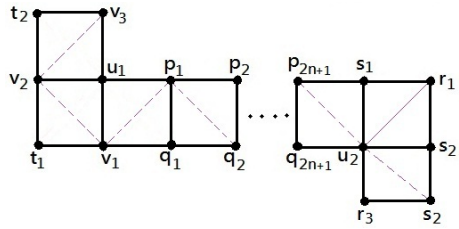

Figure 59: The subgraph H_4 of G that $k_{v_2v_3}$ is the smallest value of $k_{xy}s$, for all $x, y \in \{u_1, v_2, t_2, v_3\}$ in G.

(vi) If the edge $u_1u_2 \notin E$ and the shortest path from u_1 to u_2 with even length is not on the boundary of the unbounded face.

Let u_1 be the common neighbor of v_1, v_2, ..., v_k and u_2 be the common neighbor of s_1, s_2, ..., s_i. Let t_1, t_2, ..., t_{k-1} be vertices between v_{i-1} and v_i, $i = 1, 2, ..., k$, respectively and r_1, r_2, ..., r_{i-1} be vertices between s_{j-1} and s_j, $j = 1, 2, ..., i$, respectively. Let p_1, p_2, ..., p_{2n} be vertices of the shortest path from u_1 to s_1 and q_1, q_2, ..., q_{2n} be vertices of the shortest path from v_1 to u_2 where $p_jq_j \in E$, for all $j = 1, ..., 2n$. The graph G is shown in Figure 60.

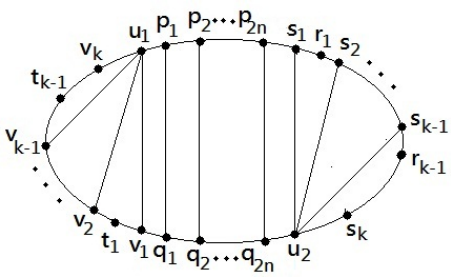

Figure 60: The graph G.

We have a tolerance representation of G as Figure 61 shown. Hence, G is a tolerance graph with tolerances $t_{u_1} = \infty$, $t_{u_2} = \infty$,
$t_{v_j} = 1$, for all $i = 1, 2, ..., k$, $t_{s_j} = 1$, for all $j = 1, 2, ..., i-1$,
$t_{t_j} = \infty$, for all $j = 1, 2, ..., k-1$, $t_{r_j} = \infty$, for all $j = 1, 2, ..., i-1$,

$t_{p_j} = 1$, for all $j = 1, 3, ..., 2n-1$, $t_{p_j} = \infty$, for all $j = 2, 4, ..., 2n$,

$t_{q_j} = 1$, for all $j = 2, 4, ..., 2n$ and $t_{p_j} = \infty$, for all $j = 1, 3, ..., 2n-1$.

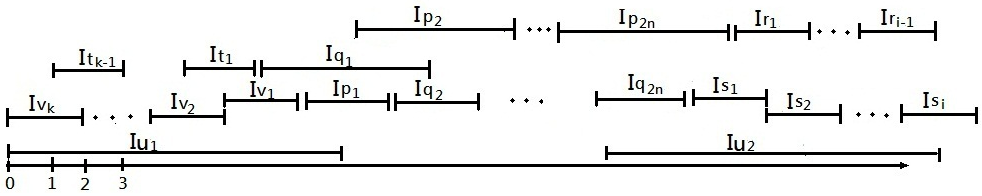

Figure 61: The tolerance representation of graph G.

□

4 Open Problems and Further Directions of Studies

In this article, we have presented the tolerance representations and all forbidden subgraphs of 2-connected graphs which are maximal outerplanar graphs and bipartite. There are still some open problems for future studies.

1. In Figure 1, we have known a hierarchy of classes of tolerance graph. Furthermore,

a. We would like to characterize the graphs which are both tolerance and cocomparability.

b. We would like to characterize the graphs which are both tolerance and trapezoid.

2. The general question of characterizing bipartite bounded tolerance is proved by the following theorem. [3]
Let $G = (X, Y, E)$be a bipartite graph. The following conditions are equivalent.

(i) G is a bounded tolerance graph.

(ii) G is a trapezoid graph.

(iii) G is a cocomparability graph.

(iv) G is AT-free.

(v) G is a permutation graph.

Furthermore, we would like to prove the theorem for tolerance graph instead of bounded tolerance graph.

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[3] M. Golumbic and A. Trenk, *Tolerance graphs*, Cambridge Univ Pr, 2004.

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