行政院國家科學委員會專題研究計畫　成果報告

城市的大小分佈、最適規模與效率

計畫類別：個別型計畫
計畫編號：NSC91-2415-H-004-016-
執行期間：91 年 08 月 01 日至 93 年 01 月 31 日
執行單位：國立政治大學經濟學系

計畫主持人：陳心蘋

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 93 年 2 月 3 日

***JEL* Code:** R120

Distribution of Cities, Increasing Returns and Efficiency
城市大小的分佈.最適規模與效率

Abstract

Many things in the natural world consist of an ever larger number of ever smaller pieces. It is called a fractal, which can be an object in space or a process in time. This fractal system has been observed in various fields, such as in the physical, biomedical, and social sciences. In economics the size distribution of cities and the distribution of the number of AOL users empirically fit the fractal. The purpose of this paper is to investigate the possible underlying mechanisms of the distribution of cities, which can generate not only the general power law rather than the specific Zipf's law but also contain economic intuition. In the present paper we will introduce and simulate the proposed stochastic model to examine the feature that could generate power law which explains the regularity of the distribution of cities; furthermore, the extended features regarding the optimal scale and the efficiency prospect of the cities' distribution is also investigated. We find that the growth process with a diminishing returns' agglomeration economy or bounded an increasing returns' agglomeration economy converges to a stable limiting distribution with a constant expected proportion. On the contrary, the growth process with an unbounded increasing returns' agglomeration economy generates a fractal kind of limiting distribution with a time variant expected value. Given the assumption of agglomeration economies and robust evidence of Zipf's in city distribution, our result suggests the presence of unbounded agglomeration economies in residents' location benefit.

Keywords: Fractal, Diminishing returns, Increasing returns, Power law

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January 2003

國科會計畫編號：NSC 91-2415-H-004-016

1. Introduction

It is widely recognized that the size distribution of cities is surprisingly well described by Zipf's law across countries with various economic structures and histories. Zipf's law, which is a special case of the power law, essentially characterizes the size distribution of cities. The general power law not only appears in cities distribution, but also in other subjects. Shiode and Batty (2000) show that the most mature domains with the most pages follow the power law; moreover, Adamic (2001) shows the distribution of the number of AOL users' visits to various sites in 1997 fits the power law. Distribution, which follows power law, is a part of the family of fractal.

Different models have been applied to explain Zipf's law: economic models in Losch (1954), Hoover (1954), and Beckman (1958), and a spatial model in Fujita, Krugman, and Venables (1999). Although these efforts do provide different ways to analyze the possible theoretical foundation, the essential puzzle remains. Gabaix (1999) proposes Gibrat's law as an explanation of Zipf's law using a stochastic model. He finds that homogeneous growth processes in cities could lead the distribution to converge into the Zipf pattern. Although Gabaix's work proposes a general and neat interpretation for this regularity in a city distribution, the homogeneity assumption of growth processes in Gibrat's law shows a disregard of the agglomeration effect that is essential in economic interpretation.

The distribution of cities and the distribution of website users are different subjects; however, the dynamic generating processes in both cases may contain certain features that could result in a similar limiting distribution. The purpose of this paper is to investigate the possible underlying mechanisms that could generate not only the general power law rather than the specific Zipf's law, but also contain the economic intuition.

In the present paper we will introduce and simulate the proposed stochastic model to examine the features that could generate power law. In Section 2 we first introduce the fractal distribution and increasing returns. The proposed path dependent stochastic model is described and discussed in Section 3. In Section 4 simulation results are presented and concluding remarks are formulated in Section 5.

2. Fractal distribution and increasing returns

Fractal distribution

The assumption of normality implies that data can be meaningfully characterized by the constant mean and variance. However, much of the nature does not contain a unique mean and variance and is not "normal". Many distributions in the natural world consist of an ever larger number of ever smaller pieces. This is called a fractal, which can be an object in space or a process in time. This fractal system has been observed in various fields, such as in the physical, biomedical, and social sciences (Bunde and Havlin 1994; Liebovitch 1998; Bassingthwaighte et al. 1994; Lannaccone and Khokha 1995; Dewey 1997; Batty and Longley 1994; Peters 1994). For example, fractal systems have shown up in the timing of heart attacks, the blood vessels of the circulatory system, the surfaces of proteins, the durations of consecutive breaths, the distribution of cities, and the number of users visiting various websites.

A general distribution function of the fractal system has the power function form:

$$y = f(x) = Ax^{-\alpha}, \tag{1}$$

where $f(x)$ is the PDF of *x*; this can be transformed to

$$\log(y) = \log(A) - \alpha \log(x). \tag{2}$$

This equation (2) explains the essential feature of the fractal distribution that it is a straight line with a negative slope on a plot of the log[PDF(*x*)] versus log(*x*), which is called the power law. A fractal from a process in time could be characterized by a parameter, α, which measures the relative number of smaller values compared to the large values. An example of fractal distribution is shown in Figure 1.

In a fractal distribution, the population mean is not defined since the sample mean does not converge to a constant. Both mean and variance of a fractal distribution depend on the amount of data analyzed, and consequently, the average number and variance can no longer characterize data in fractal systems. Different from the normal distribution, fractal distribution is defined by the linearity of the power law form of the PDF, and the corresponding slope characterizes the fractal distribution. A constant slope α implies a constant size elasticity of PDF, ε_{fx}.

$$\alpha = \varepsilon_{fx} = \lim_{\Delta x \to 0} \frac{\Delta f(x) / f(x)}{\Delta x / x}. \tag{3}$$

In a fractal distribution, the percentage change in the size's PDF due to a percentage change in its size does not vary by size.

The Pareto distribution shows the probability that a value is greater than or equal to a certain value, which is given in terms of the cumulative distribution function (CDF). A power law distribution has the following Pareto distribution:

$$P[X > x] = (\frac{A}{\alpha - 1}) x^{-(\alpha-1)} = (\frac{A}{\alpha - 1}) x^{-\beta}, \tag{4}$$

where $\beta = \alpha - 1$. The cumulative distribution function could be interpreted as the rank of size *x*; thus, the Pareto distribution in (4) implies that the rank of the largest occurrence for size *x* is inversely proportional to size *x* with a constant exponent.

This is called the rank size rule:

$$Rank = B * Size^{-\beta} \quad (5)$$

$$\log(Rank) = \log(B) - \beta \log(Size). \quad (6)$$

The rank size rule becomes Zipf's law when the exponent $\beta = 1$.

Data overall from a fractal system is defined by the form called power law and is characterized by its slope. In addition, the data also fulfills the Pareto law and the rank size rule. If the exponent in the rank size rule (5) equals one, then the data also fits Zipf's law. In short, fractal distribution implies both the power law and rank size rule.

Increasing returns

Both the equilibrium and optimal solution in conventional economic theory are derived from the assumption of diminishing returns. Diminishing returns imply stabilization and a single equilibrium point for an economy. In many parts of an economy, unstabilizing forces do appear. Arthur (1984) conducts work on the problem of increasing returns in an economy and mentions that western economies have undergone a transformation from processing of resources to processing of information. The resource-based part of an economy appears to have diminishing returns, while the knowledge-based economy is largely subject to increasing returns. The underlying mechanisms of economic behavior have shifted from diminishing returns to increasing returns, and increasing returns, driven by self-reinforcement and positive feedback, generate not only equilibrium, but instability. The evolution process of increasing returns is non-predictable, lock-in, and historically dependent. It is modeled as dynamic and non-linear rather than static and deterministic.

3. A non-linear path-dependent Polya process

Due to fractal's property, the limiting distribution of the increasing returns' dynamic process is examined. A path-dependent dynamic process is applied in this section to investigate the possible relation between the dynamic increasing returns process and the static fractal distribution. A locational choice model of residents is set up similar to the location model in Arthur (2000).

Assume residents decide on locating in one of N possible cities in the region. Let $s_t^i (i = 1,...,N)$ describe the city size for each city at time t; and $x_t^i (i = 1,...,N)$ describes the proportion of population of city i in the region at time t.

Assume the benefits, $r_j^{\ i}(i=1,...,N)$, of resident *j* for locating in city *i*, consist of two components: geographical benefit and the agglomeration benefit:

$$r_j^{\ i} = q_j^{\ i} + g(x^i), \tag{7}$$

where $q_j^{\ i}$ is the geographical benefit to resident *j* for locating in site *i*; and $g(x^i)$ represents the agglomeration benefit of resident in site *i*.

The location attractiveness due to geographical considerations is independent of the current location's shares. The agglomeration benefit is the external benefit resulting from the gathering residents represented by the location's shares (relative size), $x_t^{\ i}$. Assume that the geographical benefit is not resident specific (the homogeneity in tastes of the geographical benefit). The probability that the next resident prefers site *i* over all other sites is:

$$p^i = \Pr ob\{[q^i + g(x^i)] > [q^j + g(x^j)] \quad all \quad j \neq i\}. \tag{8}$$

Consequently, given the time invariant geographical benefit, q^i, the probabilities of the locational choice for city *i* at time *t*, $p_t^{\ i}(x_t^{\ i})$, depends on the current location's shares, $x_t^{\ i}$.

The change of size at city *i* follows the dynamic process:

$$s_{t+1}^{\ i} = s_t^{\ i} + z_t^{\ i}(x_t^{\ i}), \qquad i = 1,...,N., \tag{9}$$

where

$$z_t^{\ i} = \begin{cases} 1 & with \quad probabilit\,y \quad p_t^{\ i}(x_t^{\ i}) \\ 0 & with \quad probabilit\,y \quad 1 - p_t^{\ i}(x_t^{\ i}) \end{cases},$$

$E[z_t^{\ i}] = p_t^{\ i}(x_t^{\ i})$, $Var[z_t^{\ i}] = E[(z_t^{\ i})^2] = p_t^{\ i}(x_t^{\ i})$. Consequently,

$$s_t^{\ i} = s_0^{\ i} + z_1^{\ i} + z_2^{\ i} + \cdots + z_{t-1}^{\ i}. \tag{10}$$

Each random variable, $z_t^{\ i}$, has an expected value, $p_t^{\ i}(x_t^{\ i})$, which is a function of the current proportion rather than a time invariant constant.

$$E[s_t^{\ i}] = s_0^{\ i} + p_1^{\ i}(x_1^{\ i}) + p_2^{\ i}(x_2^{\ i}) + \cdots + p_{t-1}^{\ i}(x_{t-1}^{\ i}). \tag{11}$$

$$Var[s_t^{\ i}] = \sum_{k=1}^{t-1} p_k^{\ i}(x_k^{\ i}) + 2\sum_{k\neq l}^{t-1} Cov(z_k^{\ i}, z_l^{\ i})\,. \tag{12}$$

Both the strong law of large numbers and the central limit theorem cannot be applied in this general Polya process, as the limiting size proportion does not exist. Both the mean and variance of city size actually varies by time, and the expected value of city size is not defined. In addition, according to equation (9), the evolution of the relative city size at city *i* is:

$$\begin{aligned} x_{t+1}^{\ i} &= x_t^{\ i} + \frac{1}{(w+t)}[z_t^{\ i}(x_t^{\ i}) - x_t^{\ i}] \\ &= x_t^{\ i} + \frac{1}{(w+t)}[p_t^{\ i}(x_t^{\ i}) - x_t^{\ i}] + \frac{1}{(w+t)}u_t^{\ i}(x_t^{\ i}), \qquad i = 1,\ldots, N\,. \end{aligned} \tag{13}$$

where $w = \sum_i s_1^{\ i}$, which is the total population initially; and the disturbance term, $u_t^{\ i}(x_t^{\ i}) = z_t^{\ i}(x_t^{\ i}) - p_t^{\ i}(x_t^{\ i})$, is with zero conditional expectation.

This path-dependent process consists of a determinate part, $x_t^{\ i} + \frac{1}{(w+t)}[p_t^{\ i}(x_t^{\ i}) - x_t^{\ i}]$; and a perturbation part, $\frac{1}{(w+t)}u_t^{\ i}(x_t^{\ i})$. The determinate part includes the preceding proportion and the difference between the probability and the preceding proportion. In addition, the expected motion of the locational share depends on the determinate part, which contains the choice probability function.

$$E[x_{t+1}^{\ i} | x_t^{\ i}] = x_t^{\ i} + \frac{1}{(w+t)}[p_t^{\ i}(x_t^{\ i}) - x_t^{\ i}]\,. \tag{14}$$

The features of the location choice probability function, $p_t^{\ i}(x_t^{\ i})$, essentially characterize the limiting proportion. In addition, the expected motion tends to be directed by the term $[p_t^{\ i}(x_t^{\ i}) - x_t^{\ i}]$ in the determinate part. A positive term would drive the expected motion to grow.

Case 1. If there are no economies or diseconomies of agglomeration in the location choice ($g(x^i) \equiv 0$ in (7)), which means that the location benefit is independent of the location's share, then the location choice probability function depends only on the predetermined geographical attributes, $p_t^{\ i}(q^i)$. The vector of the limiting proportion of *N* cities in the region is just the vector of the fixed location specific probability, $p_t^{\ i}(q^i)(i = 1,\ldots, N)$, determined by the given geographical attributes. The location share hence tends to converge to a single equilibrium point

Case 2. There are economies of agglomeration in the location choice ($g(x^i) \neq 0$ in (7)), and the choice probability equals the current proportion

($p_t^i(x_t^i) = x_t^i$). This is called the standard Polya process. The determinate part in (13) disappears, and the perturbation part dominates the motion. It is therefore proved that the vector of limiting expected proportions tends to be a fixed vector with a probability of one. (Polya 1931)

Case 3. There are economies or diseconomies of agglomeration in the location choice. The probability function, $p_t^i(x_t^i)$, is assumed to be non-linear. The stochastic process (13) with a non-linear probability function is called a non-linear Polya process (Arthur 2000). In the case of the non-linear Polya process, a negative first derivative of the probability function characterizes a diminishing returns process (such as Figure2(e)), $p_t^i{}'(x_t^i) < 0$; the limiting expected proportion converges to a single equilibrium point, $\overline{x^i}$. Thus, the limiting corresponding probability is time invariant, $p^i(\overline{x^i})$. As a result, the limiting proportion of city *i* is $p^i(\overline{x^i})$ according to the strong law of large numbers.

Assuming the probability function is not city specific, the limiting proportion is a constant, $p^1(\overline{x^1}) = \ldots = p^N(\overline{x^N}) = p(\overline{x})$. The mean of the size proportion is defined, and a positive first derivative of the probability function refers to an increasing returns process, $p_t^i{}'(x_t^i) > 0$; the tendency of the city size proportion is to be attracted toward one or several fixed points depending on the functional form of the probability function. The positive feedback feature of the increasing returns process consists of two major categories: bounded agglomeration economies and unbounded agglomeration economies.

Increasing returns with bounded agglomeration economies (Figure2(b)(c)), which describes the economies of agglomeration with a ceiling, can be presented by a diminishing increasing returns of the probability function ($p_t^{i\prime}(x_t^i) > 0$, and $p_t^{i\prime\prime}(x_t^i) < 0$). The limiting expected proportion converges to a single equilibrium point. Consequently, similar to the case of the diminishing returns process discussed above, the limiting proportion is a constant both in time and site under the condition of the homogeneous probability function, and hence the mean of size proportion is defined.

Increasing returns with unbounded agglomeration economies (Figure2(a)(d)) are presented by an rising increasing returns of the probability function ($p_t^{i\prime}(x_t^i) > 0$, and $p_t^{i\prime\prime}(x_t^i) > 0$). The limiting expected proportion does not converge to a single equilibrium point, and therefore a fixed expected proportion does not exist.

The question that we are interested in this paper is to ask whether these various possible equilibrium states according to the different features of the path-dependent process could characterize their limiting distribution? Moreover, is the limiting fractal distribution associated with certain features of the dynamic stochastic process?

4. Simulation

We simulate the proposed non-linear path-dependent Polya process in Section 4 in order to analyze the asymptotic distribution properties of particular classes of stochastic equations, especially increasing returns. In this model one resident is added into the region at each time; the probabilities of an addition to a city depend on their current proportion. The functional form of the probability function is essential in characterizing both the growth process and its limiting distribution. Both cases of increasing returns and decreasing returns are simulated in this section.

4.1 Increasing returns with unbounded agglomeration economies: function (a)

A Polya process given the probability function (a) in Figure 2 characterizes increasing returns with unbounded agglomeration economies. The larger the size proportion in the region is, the higher the probability will be that the city will grow. Furthermore, as the city size proportion passes one half of the region, the probability that this city will grow is greater than 0.5 at a diminishing rate, showing a tendency toward 0 or 1.

Assume a region of 50 cities starts with a uniformly-distributed city size; the simulating dynamic processes for all cities after 3000 iterations are shown in Figure 3. The relative size distributions at different time periods are shown in Figure 4. This states that as the time increases, the location shares distribute closer to the fractal distribution. This distribution tendency can be observed in Figure 5. The plot of log(Rank) versus log(Proportion) tends to be linear.

Table 1 lists the estimated slopes and R-square of the plots as in Figure 5.2. The R-square value is increasing, while the absolute value of the estimated slope is diminishing. A smaller absolute value of the estimated slope represents a more diversely-distributed city size in the region. The diminishing tendency of the absolute value of the slope and the increasing linearity of the log(Rank) versus log(Proportion) plot is consistent with the experimental city distribution.

4.2 Increasing returns with bounded agglomeration economies: function (b)

The probability function shown in Figure 2(b) characterizes increasing returns for a diminishing increasing rate. It shows a tendency toward a fixed point $\bar{x}$. The simulating dynamic process of the region with 15 cities after 3000 iterations is produced in Figure 6.1. The expected size proportion of all cities in the region tends to converge to a stable ratio. Moreover, the expected value of the limiting location exists, which is very different from the fractal distribution.

4.3 Increasing returns with bounded agglomeration economies: function (c)

The probability function shown in Figure 2(c) characterizes diminishing increasing returns, and it shows a tendency toward 1. The simulating dynamic process of the region with 15 cities after 3000 iterations is displayed Figure 6.2. The expected size proportion converges to a stable point. Similar to the case of function (b), the expected value of the limiting location shares does exist.

4.4 Increasing returns with unbounded agglomeration economies: function (d)

The probability function shown in Figure 2(d) characterizes rising increasing returns , and shows a tendency toward 0. The simulating dynamic process of the region with 50 cities after 3000 iterations is shown in Figure 6.3. It is similar to the dynamic process of function (a), which offers a tendency toward a fractal distribution.

4.5 Diminishing returns: function (e)

The probability function shown in Figure 2(e) characterizes diminishing returns. It appears to have a tendency toward a fixed point $\bar{x}$. The simulating dynamic process of the region with 15 cities after 10 iterations is displayed in Figure 6.4. The expected size proportion converges much faster than the process of function (b) to a stable point. Consequently, the expected value of the limiting location shares does exist, and the limiting location shares do not distribute as a fractal.

The simulation results show that the dynamic process of diminishing returns and increasing returns with bounded agglomeration economies tend to converge to a stable point; also there exists a fixed expected locational pattern of proportions. By contrast, in the case of increasing returns with unbounded agglomeration economies, a constant expected proportion does not exist, and the limiting distribution of the location shares tends to be fractal and displays the power law. The growth process of increasing returns does not necessarily generate a distribution with the power law.

5. Concluding remarks

If the benefits from agglomeration economies in a residents' location benefit are absent, then the size distribution depends only on the geographical benefit that does not contain positive feedback and path-dependent properties. Given the geographical endowment in the region, suppose the residents' location preferences are homogeneous, and residents cluster according to the given geographical benefits; on the other hand, if residents' location tastes are heterogeneous, then the distribution of the city size is more dispersed than in the homogeneous case. Furthermore, both the size evolution and limiting distribution tend to be stable and predictable.

In the general case where agglomeration economies are present, the probability of adding residents depends upon any past addition, so that the standard strong law is not usable and the size evolution is historically dependent and non-predictable. The limiting distribution is also closely related to the feature of the dynamic evolution process, especially the mechanism of the agglomeration economies. If the addition of residents confers a net benefit on a location, under upper limit-bounded agglomeration economies, then the dynamic process for each city's share tends to converge to a stable point and there exists a fixed expected value in the limiting distribution. If the addition of residents always confers a net benefit on a location, under no upper limit-unbounded agglomeration economies, then the dynamic process of each city's share tends to diverge to a fractal distribution that follows the power law. The expected locational share is time variant and undefined. As a result, given the assumption of agglomeration economies and robust evidence of a city’s distribution being fractal, our result suggests the presence of the unbounded agglomeration economies for the benefit Of the residents' location.

Table 1. The regression result of the plot of log(Rank) versus log(Proportion) of function (a)

Estimated Slope	R-square
-0.5358	0.5072
-0.5129	0.5625
-0.4950	0.6098
-0.4805	0.6508
-0.4683	0.6870
-0.4579	0.7190
-0.4488	0.7474
-0.4408	0.7727
-0.4336	0.7950
-0.4271	0.8145
-0.4211	0.8314
-0.4157	0.8460
-0.4108	0.8584
-0.4063	0.8688
-0.4023	0.8774

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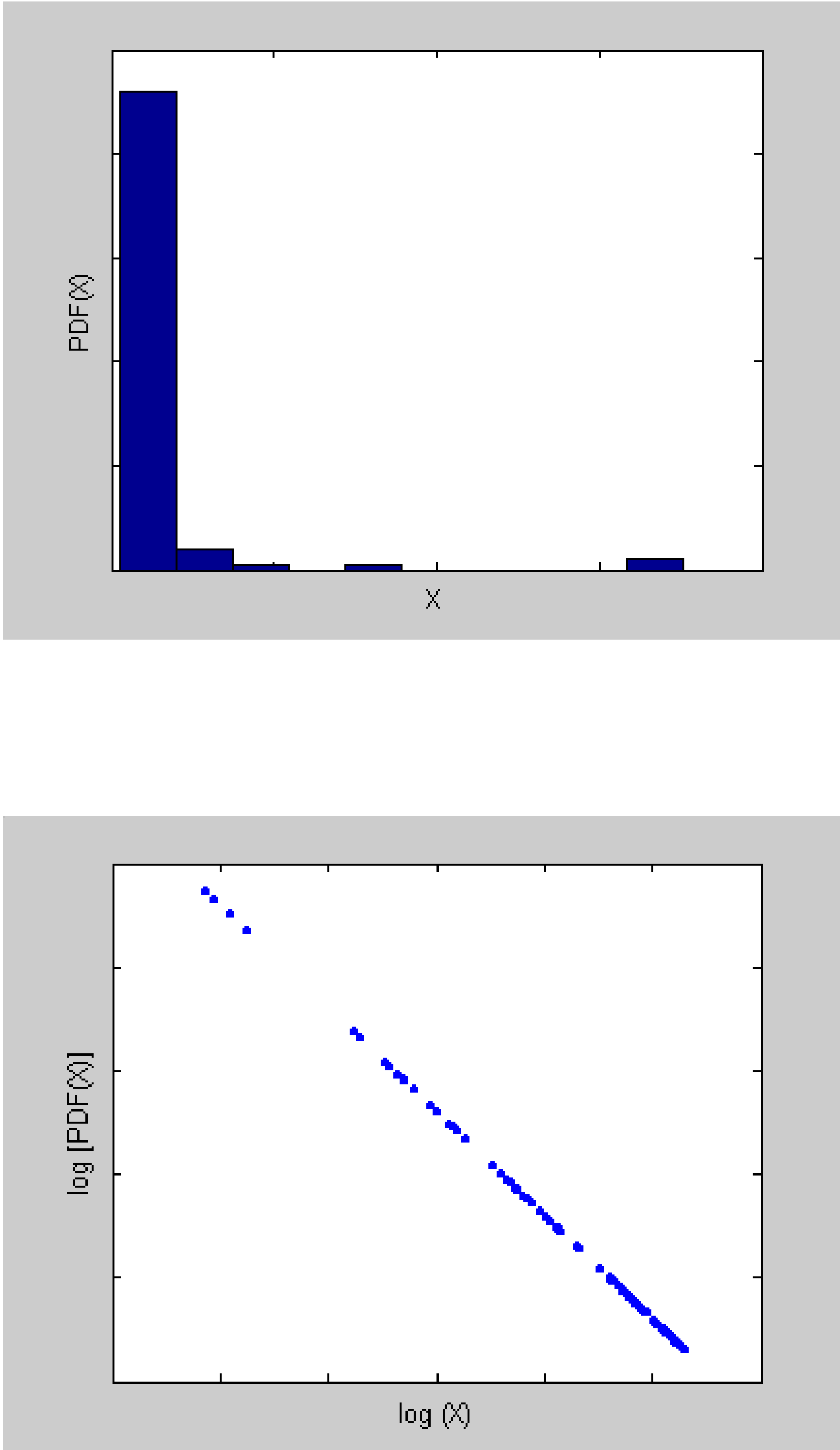
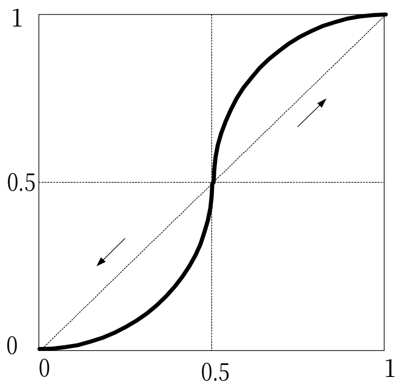
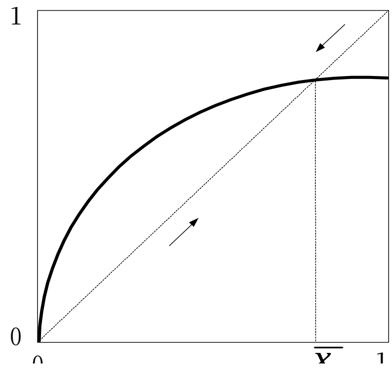


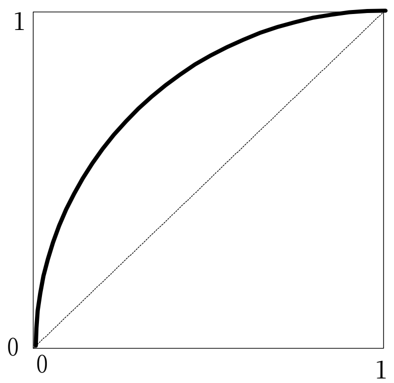
Figure 1. Fractal distribution



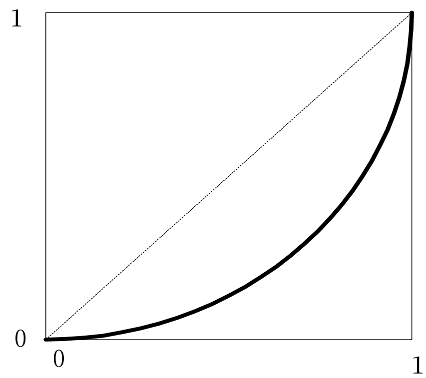
(a) Increasing returns
(Unbounded agglomeration economies)



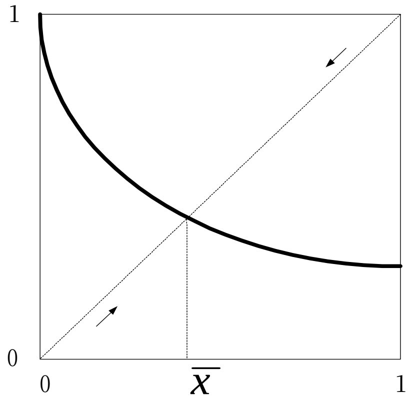
(b) Increasing returns
(Bounded agglomeration economies)



(c) Increasing returns
(Bounded agglomeration economies)



(d) Increasing returns
(Unbounded agglomeration economies)



(e) Diminishing returns

Figure 2. Probability functions assumed in the Polya process

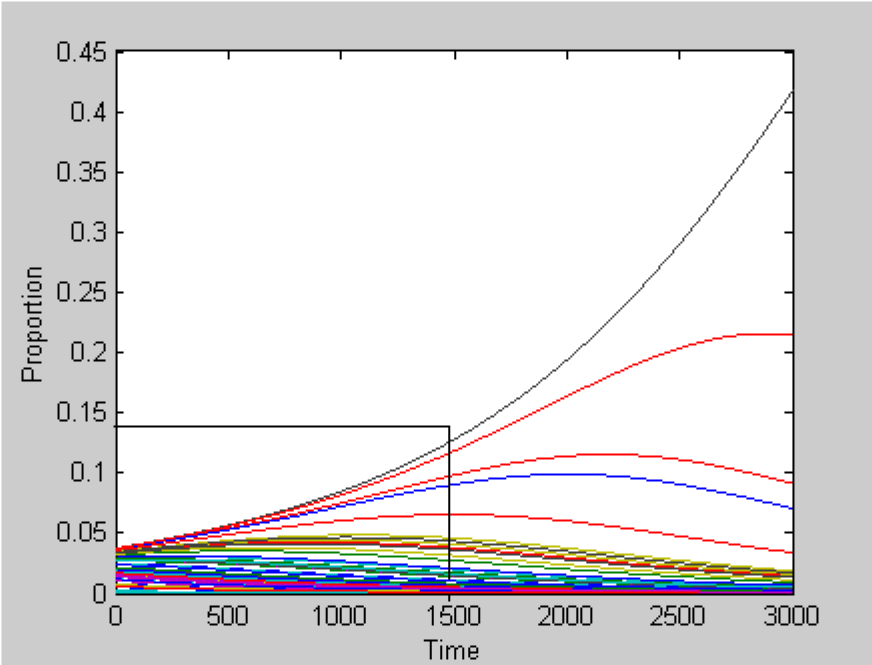


Figure 3.1 The dynamic process of function (a): 50 cities after 3000 iterations starting form uniform distribution.

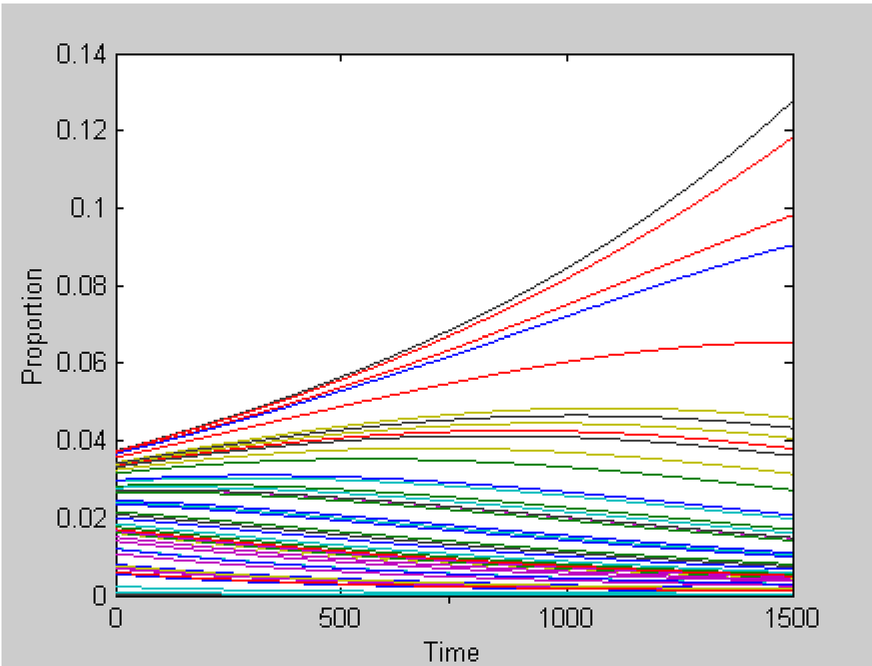
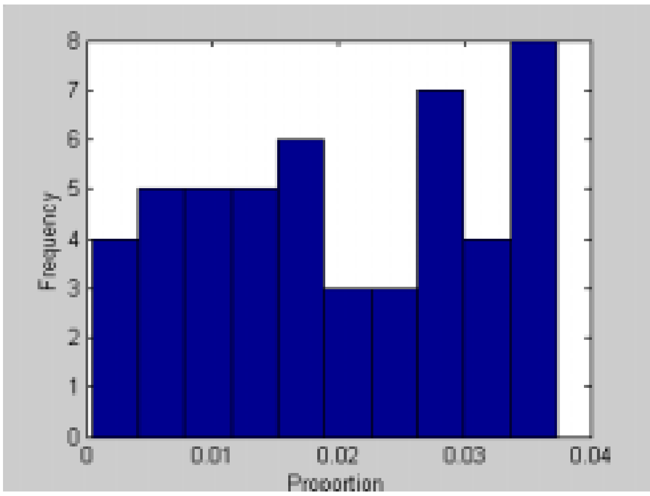
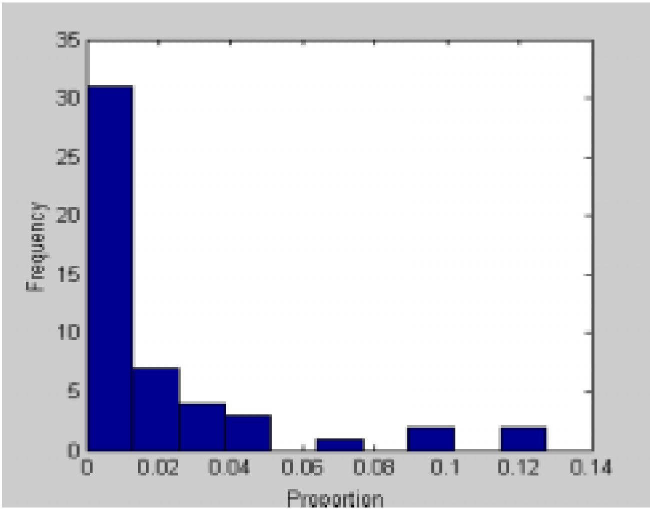


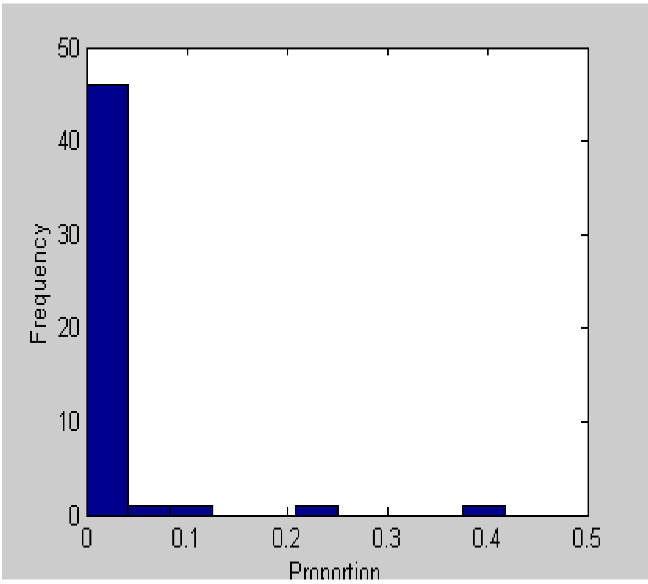
Figure 3.2 The dynamic process of function (a): 50 cities after 1500 iterations starting form uniform distribution.



(a) t = 1



(b) t=1500



(c) t=3000

Figure 4. The frequency distribution of proportion

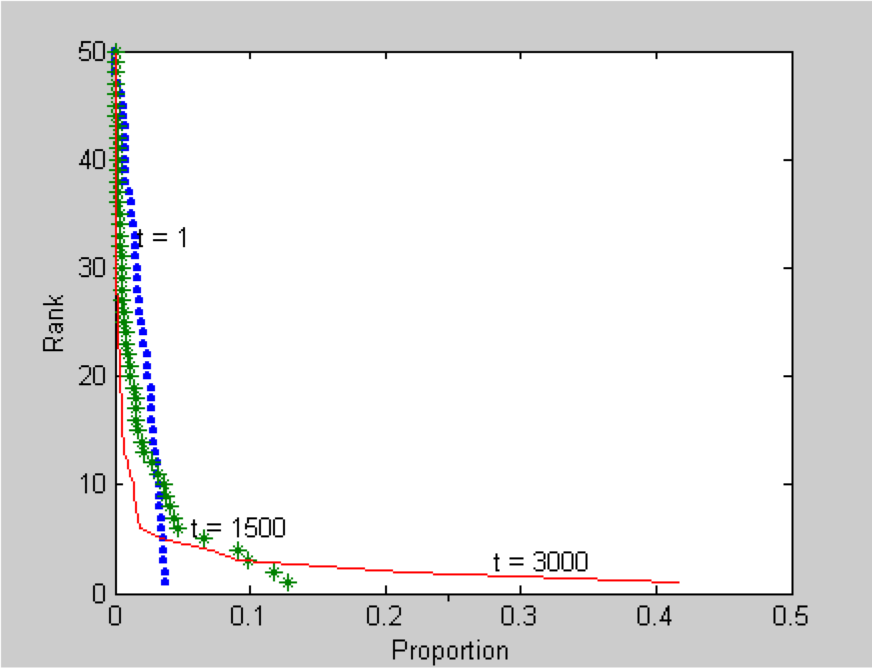


Figure 5.1. The plot of proportion and rank of function (a)

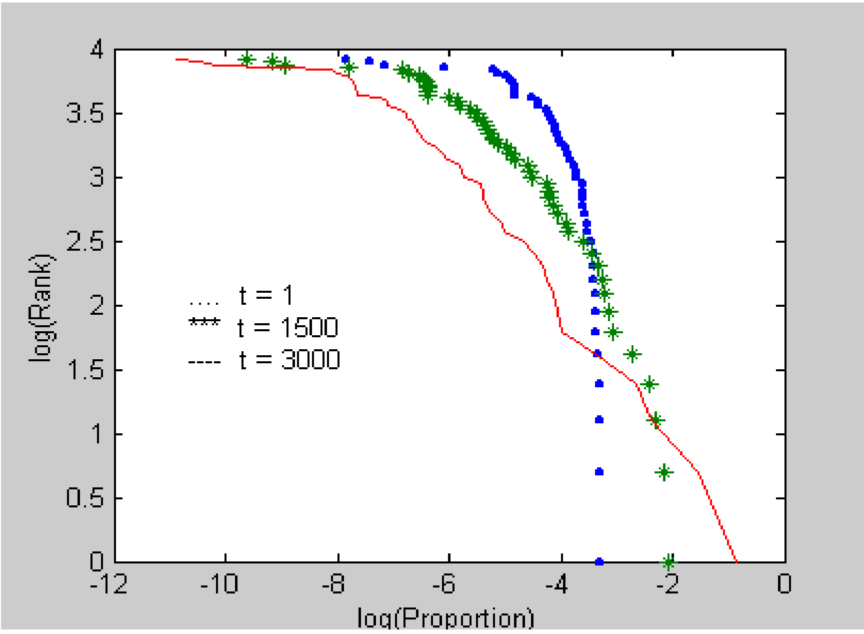


Figure 5.2. The plot of log(Proportion) and log(Rank) of function (a)

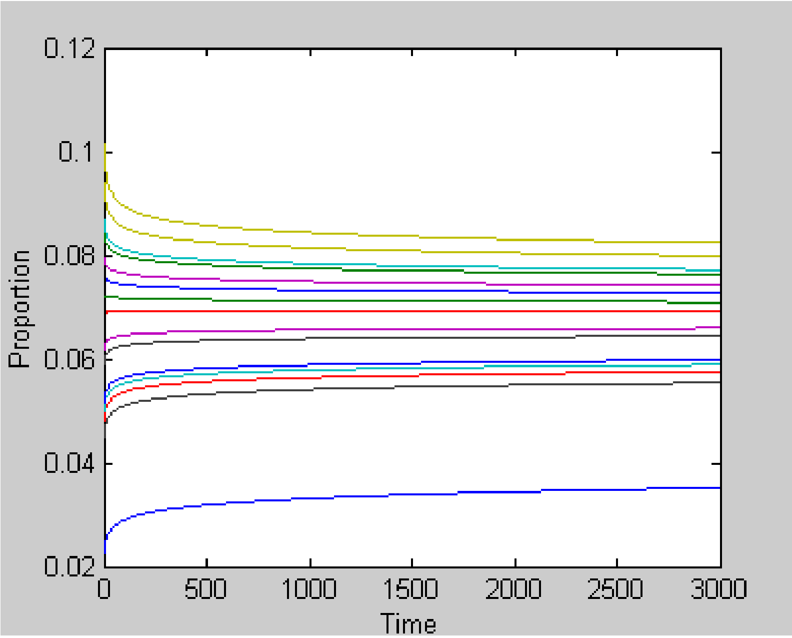


Figure 6.1. The dynamic processes of probability function (b):
15 cities after 3000 iterations starting form uniform distribution

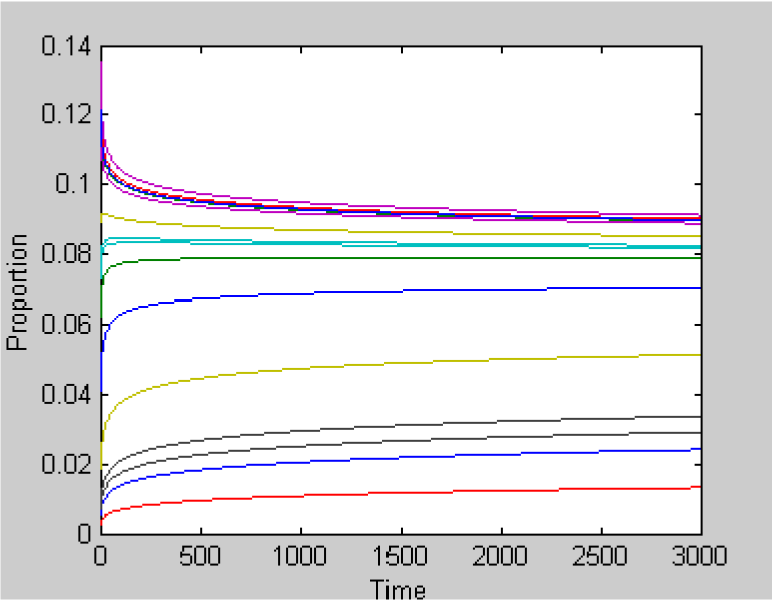


Figure 6.2 The dynamic processes of probability function (c):
15 cities after 3000 iterations starting form uniform distribution

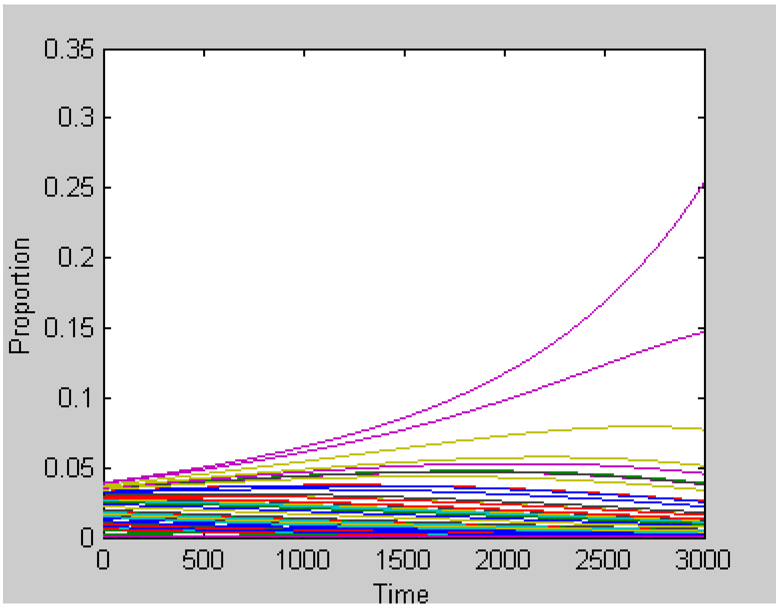


Figure 6.3. The dynamic processes of probability function (d): 50 cities after 3000 iterations starting form uniform distribution

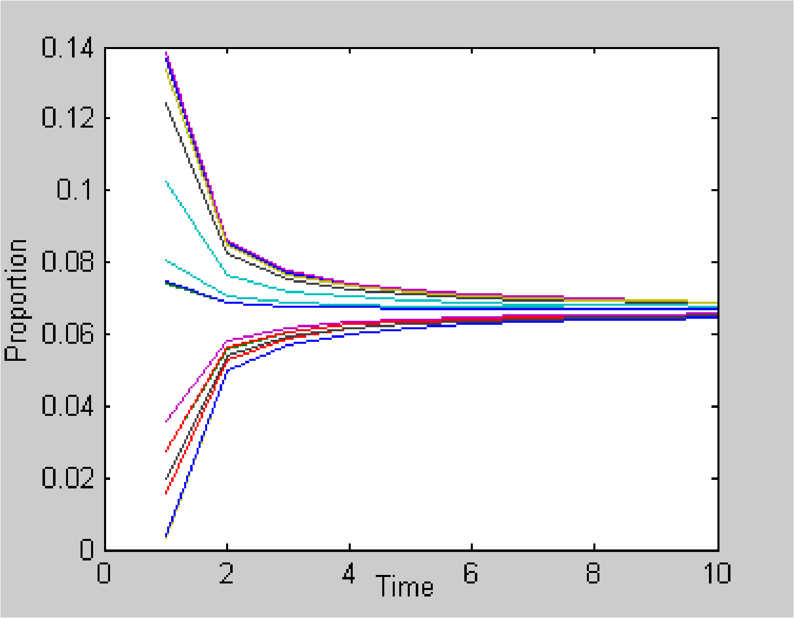


Figure 6.4. The dynamic processes of probability function (e): 15 cities after 10 iterations starting form uniform distribution

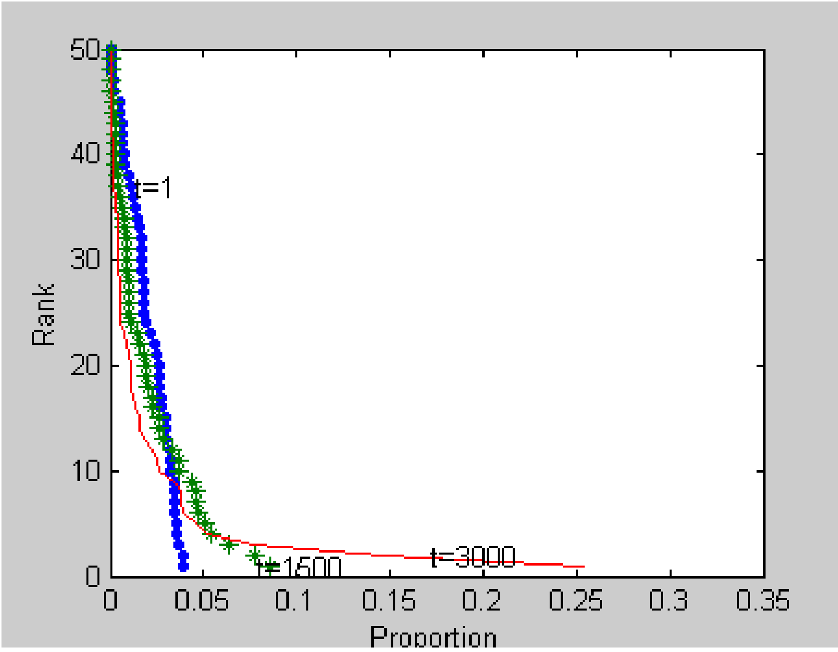


Figure 7.1 The plot of proportion and rank of function (d)

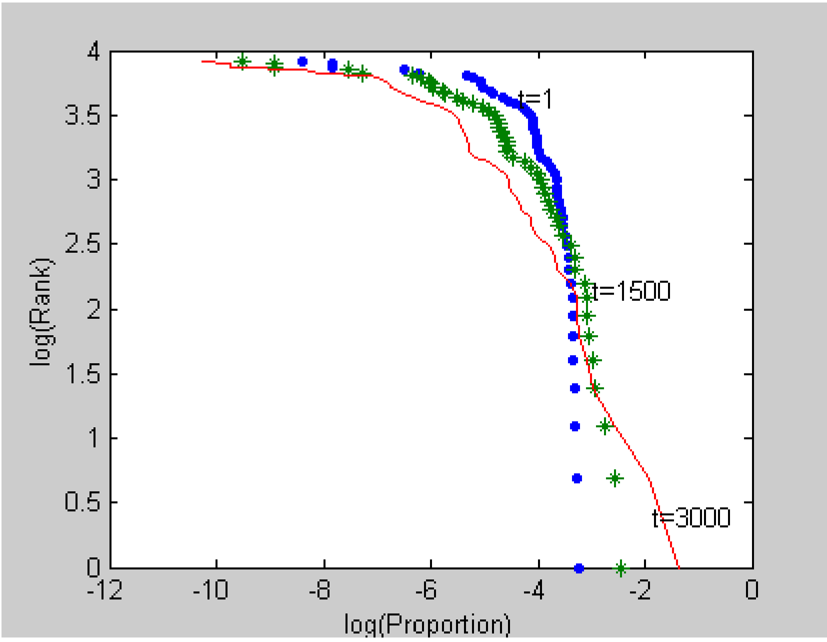


Figure 7.2 The plot of log(Proportion) and log(Rank) of function(d)