Evolutionarily Stable Coalition Structure*

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Abstract

We analyze the formation and stability of coalitions for a situation where finitely many individuals form different coalitions and their payoffs depend on the consequence of a noncooperative game with different coalitions, and examine the moving path of individuals among various coalitions. Our main finding is to show that there exists at least one evolutionarily stable coalition equilibrium in $\Gamma^{n^{\pi}}$. When addressing the evolving path of coalitions, we conclude that in the case of symmetric strategies and symmetric population shares, if each coalition's population share is too small, the equilibrium requires a reduction of the number of coalitions, but if each coalition share is too large, the equilibrium requires an increase of the number of coalitions. Furthermore, when $u_i(.)s$ are symmetric but x is asymmetric, then (i) the highest payoffs are oscillatory across time still stability happens. (ii)In the evolutionarily stable structure, each group share the same population and the coalition numbers is hence $|D_t(x, \pi^t)|$.

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1 Introduction

The study of formation of coalitions has been one of the most intriguing and challenging problems open to game theorists. Many solution concepts were primarily designed as ways to solve the problem of joint determination of coalition structure and the allocation of the coalitional surplus among coalition members. However, the following social behavior seems to be neglected in the existing literature: In presidential election, different groups of civilians gather to support their favorite candidates to win the election game; Gamblers bet on different football teams and their rewards depend on the competing results among teams. These examples point out a situation where finitely many individuals form different coalitions and their payoffs depend on the consequence of a noncooperative game with different coalitions. The present paper intends to analyze the formation and stability of coalitions in this category, and analyze the moving path of individuals among various coalitions.

The first related literature is the discussion of coalition forming. Essentially, the process is viewed as two stage: players form coalitions in the first stage, and then decide on the allocation of the coalitional surplus, given a fixed coalition structure in the second stage. There are two settings as to why coalition structure form to begin with (Kurz, 1988). First, Aumann and Dreze (1974) regard the coalition structure to be such that each coalition forms, operates, and generates its coalition payoff. In this sense, the Aumann-Dreze coalition is a real entity to realize its potential value. Second, the view of Owen(1977), Myerson (1978), Shenoy (1979), Hart and Kurz (1983), and Aumann and Myerson (1988) is that the coalition structure is formed only as a bargaining tool aiming to increase the payoff of individual members. This entails a subtle bargaining among individual players within each coalition and among all the coalitions.

Our paper falls into the first category, in the sense that individual forms different coalitions in order to realize the payoff from the noncooperative game among coalitions. Consequently, we will not handle the various issues arising only from the second category, such as consistency (Hart and Kurz (1983)), which requires the same solution concept for the bargaining among the coalitions and within each coalition. However, we focus on the issue of stability which will be encountered in both settings.

Solution concepts in coalition structures are usually discussed in terms of valuations (or allocations). That is, let N be the set of players, v be the worth of coalitions, and a partition of player set $\pi = \{\pi_1, \pi_2, ...\pi_n\}$ be a coalition structure. A possible solution for the coalition game (N, v, π) is a payoff allocation among players, given the coalition structure π. The strategy aspect is ignored in the existing literature. The main difficulty comes from the fact that different coalitions of players will define different games. Hence, even if we assume symmetry among players, the stable cooperative solution concepts[1] require immunity to deviations in forming different subcoalitions, which in turns require modelling different games to take into account those deviating subcoalitions. The main concern of the existing cooperative solutions has been how to distribute the coalition values (represented by v) among and within each coalition, rather than how to obtain the coalition values, to be addressed in the present paper.

Our paper intends to explore the strategy issue in coalition structures, and ask if there exist strategies (among the coalitions) to support a stable coalition structure, in the sense that no single individual wishes to join in an already existing coalition or form a new coalition. However, to focus on our main concern and simplify the analysis, we make the following assumptions, which are different from that of the existing work. First, we assume that the player (population) set is large but finite (Weibull (1995)), where "largeness" implies that each individual is insignificant, and "finiteness" allows us to eschew the measuring problem. The insignificance of individual motivates us to assume further that each individual's payoff depends

[1] For example, the von Neumann-Morgenstern solutions, the core, the bargaining set, the nucleolus, the kernel and the Shapley value.(Aumann and Dreze (74)).

on the value of the coalition which she belongs to. Specifically, each individual's payoff is actually the coalition value[2] determined by the strategy profile (among the coalitions) and the relative size of coalition. A close to reality interpretation of this assumption is that each coalition's value is a public good, whose value can be appreciated by each member of the coalition.

The public good interpretation draws our attention to the literature (Tiebout (1956), Westhoff (1977), Greenberg and Weber (1986),) of "strong Tiebout equilibrium", which is a strong Nash equilibrium (Aumann (1959)) in the game of choosing communities. Aumann introduced the notion of strong Nash equilibrium, requiring that an allocation not be subject to improving deviation by any coalition of players. This requirement is too strong, as agreements must be resistant to deviations which are not themselves resistant to further deviations. Recognizing this problem, Bernheim, Peleg, and Whinston (1987) introduced coalition-proof Nash equilibrium which requires only an agreement be immune to improving deviations which are self-enforcing. A deviation is self-enforcing if there is no further self-enforcing and improving deviation available to a proper subcoalition of players.[3] Our paper resembles to this line of literature in characterizing equilibria in the context of noncooperative games, but we focus on games among coalitions, rather than among individuals. An equilibrium in a game among coalitions, if it also guarantees the stable coalition structure of individuals, is called an equilibrium to support an evolutionary stable coalition structure.

The term "evolution" is meant to capture the movement of individuals among coalitions. Since an equilibrium can be viewed as a steady state in the population and strategy adjusting process, the study of the process will provide interesting implications to the equilibrium. This topic is related to multi-population models in

[2]Another setting to avoid the discussion of distributing within coalitions is to assume a fix division rule (Bolch (1996)).

[3]We do not consider coalition-proof equilibrium with correlated strategies. (See Moreno and Wooders (96)).

the evolutionary game theory literature (see for example Weibull (1995)), however, we use a simpler setting as will be described in the text. Except for this, our model distinguishes form the standard setting in that individuals are free to move across populations (that is, coalitions).

Finally, we investigate the cooperation possibility among coalitions and address a small implication to folk theorem, which has been excessively discussed in the game theory literature (see for example, Osborne and Rubinstein (1994)). Folk theorem says that the efficient payoffs can be supported as average payoffs of subgame perfect equilibria. The rationale for this result is that players can punish the deviators by switching to the minimax equilibrium (the reservation utility). However, when individuals are free to move, the punishment decision will force individuals move toward the deviating coalition which ensures the highest payoff. Hence the partition will change and the game is not the same each period. When a coalition really deviates, other coalitions can only find the best reply to this deviation, and hence would not be the minimax strategy. As a result, the only possible payoff is which can be supported by the equilibrium with evolutionary stable coalition structure.

The rest of the paper is organized as follows. Section 2 presents the model of coalition forming. Section 3 is the discussion of evolutionary path. Section 4 is the concluding remarks and an implication to folk theorem.

2 Evolutionary stable coalition equilibrium

Let $\mathcal{L}$ be a finite but large[4] population of individuals. Individuals make their coalition decisions independently and simultaneously,[5] and the induced partition of population defines a coalition structure. That is, $\forall i \in \mathcal{L}$, i's coalition strategy k^i is to choose which coalition to belong to, or equivalently of which role to play in resulted coalition game.[6] $k^i \in K^i := \{K \subset \mathcal{L} | i \in K\}$. Let $K = \underset{i \in \mathcal{L}}{\times} K^i$ be the set of coalition strategy profiles, and k as one element, $k = \{k^i\}_{i \in \mathcal{L}} \in K$. Let $\pi(k)$ be the induced coalition structure, i.e., $\pi(k) := \{\pi^i(k) \subset N | i, j \in \pi^i(k)$ if and only if $k^j = k^i$, for $k^j, k^i \in k\}$.[7] Let $n^{\pi(k)}$ denote the number of subsets in $\pi(k)$, that is, $n^{\pi(k)} := |\pi(k)|$. Fix a k, $\pi(k) = \{\pi_1, \pi_2, ...\pi_{n^{\pi(k)}}\}$. Let Π be the set of all possible coalitions, $\Pi := \{\pi(k) | k \in K\}$. To index each coalition, let $N(\pi)$ be the set of indexes for each coalition in π, i.e., $N(\pi) := \{1, 2, ..., n^{\pi}\}$. For future inference, denote Π^{n^π} as a subset of Π such that the numbers of every coalition in Π^{n^π} is n^π, i.e., $\Pi^{n^\pi} = \{\pi \in \Pi || \pi| = n^\pi\}$.

Given π, we define an n^π-player coalition game in a triplet containing the player set, strategy profiles and payoffs, that is, $\Gamma^{n^\pi} := (N(\pi), (x_j)_{j \in N(\pi)}, (u_j(x, \pi))_{j \in N(\pi)})$. In this game each coalition π_j, $j \in N(\pi)$, plays the role j in Γ^{n^π}. Let C_j be coalition j's strategy set and $x_j \in \Delta(C_j)$ be the associated mixed strategy set.

[4]The same setting can be found in Weibull (95). The assumption of "finiteness" enables us to easily calculate the proportion of each coalition's members; The "largeness" implies that each individual is insignificant so that mutual deviation in terms of correlated strategies is not appropriate in this case.

[5]The alternative assumption of sequential formation of coalitions can be found in Selten (81), Chatterjee et al. (93), Moldovanu (92), Winter (93) and Bloch (96) in the case of small number of individuals.

[6]Each individual is assumed to choose only one coalition and the detail of individuals' coalition decision rule will be discussed in Section 4.

[7]We use the definition of δ coalition. The alternative setting is γ coalition. The difference concerns with different settings to the behavior of the complementary coalition. (Hurt and Katz (1983), p. 1060).

$x = (x_j)_{j \in N(\pi)}$ is hence one strategy profile. For $j \in N(\pi)$, $u_j(x, \pi)$ denotes coalition j's payoff in Γ^{n^π} and is continuous in x. Here we encounter some problems for the discontinuity of payoff function. Intuitively, $u_j(x, \pi)$ is not continuous in π. Hence, despite that we consider mixed strategies for each coalition, there does not necessarily exist an equilibrium. To simplify and ensure the existence of a coalition structure,[8] we make the following explicit assumptions. First, we assume that each member in a coalition shares the same payoff as the coalition, i.e., $u^i(x, \pi) = u^{i'}(x, \pi) = u_j(x, \pi)$, $\forall i, i' \in \pi_j$. As mentioned in the Introduction, this could be justified by the public good argument. Second, we assume coalition values to depend on the relative attraction in the structure. That is, $u_j(x, \pi) = u_j(x, p_j)$, where $p_j = \frac{|\pi_j|}{|L|}$ be the population share of coalition j. Third, similar to Tiebout (1956) in assuming the per capita cost of a local public good to be a U-shaped function of community size, we assume that $u_j(x, p_j)$ is concave in p_j. The concavity assumption reflects the congestion cost of crowd. Without it, the equilibrium structure is trivially the grand coalition.

To keep the generality, the following definition is presented in the general form $u_j(x, \pi)$.

Definition 1 *Given a* $\pi \in \Pi$, x *is an evolutionarily stable coalition equilibrium of* Γ^{n^π} *if* x *is an equilibrium of* Γ^{n^π} *and* $\forall i \in \pi_j$, $\pi_j \in \arg\max_{k^i \in K^i} u^i(x, \pi(k^i, k_{-i}))$.

A coalition structure is said to be evolutionarily stable if it is supported by a strong equilibrium of Γ^k. The notion requires an agreement not be subject to an improving deviation by any coalition of individuals. That is, there exists a strategy profile x such that $u^i(x, \pi(k^i, k_{-i})) > u^i(\widehat{x}, \widehat{\pi}(\widehat{k^i}, k_{-i}))$. Note that

[8] Non-superadditivity is the most compelling explanation for the formation of a coalition structure. Aumann and Dreze (1974) described a situation in which certain exogenous conditions result in a partition of individuals into a finite numbers of jurisdictions. Other groups of causes include the existence of legal limitation (Guesnerie and Oddou (1981)), geographic and spatial limitations (Westhoff (1977)), and communication problem.

since different strategy combinations lead to different coalition structures, we have $\widehat{x} \in \underset{j \in N(\widehat{\pi})}{\times} \Delta(C_j)$ and $x \in \underset{j \in N(\pi)}{\times} \Delta(C_j)$. But fortunately since individuals are assumed insignificant, given the coalition decisions of the rest of the population, it suffices to check if $u^i(x, \pi(k^i, k_{-i})) > u^i(x, \pi(\widehat{k}^i, k_{-i}))$ (as requested in Definition 1). In what follows, we turn to the explicit assumption of $u_j(x, p_j)$.

We first examine the extreme case when each coalition's payoff is not related to the size of each coalition, that is, $u_j(x, p_j) = u_j(x)$, $\forall j \in N(\pi)$, $\forall \pi \in \Pi$. This is equivalent to treating the coalition value as a pure public good. In such case, the set of evolutionarily stable coalition equilibria is the set of equilibria for Γ^{n^π} that result in identical payoffs for each coalition. That is, x is an evolutionarily stable coalition equilibrium if $x_j \in \arg\max_{\widehat{x}_j \in \Delta(C_j)} u_j(\widehat{x}_j, x_{-j})$ and $u_j(x) = \overline{u}(x), \forall j \in N(\pi)$, $\forall \pi \in \Pi$. Examples with evolutionarily stable coalition equilibria can be found in the prisoner dilemma games and the battle of sex games. The main problem with this class of games is that the existence is obviously not generally guaranteed, for instance the hawk and chicken games. Notice also that in this case the set of evolutionarily stable coalition equilibria is strictly included in the set of equilibria for Γ^{n^π}.

When each coalition's payoff is related to the size of each coalition, Proposition 1 shows the existence of an evolutionarily stable coalition equilibrium. Recall that $u_j(x, p_j)$ is continuous in x and p_j, and concave in $p_j, \forall j \in N(\pi)$, $\forall \pi \in \Pi$. To proceed our proof for the existence, Lemma 1 and Lemma 2 are required.

Lemma 1 *If the payoff function is symmetric with respect to the maximal argument p^*, then in an evolutionarily stable coalition equilibrium (if exists), (1) $u_i(x, p_i) = u_j(x, p_j)$ for $\forall i, j \in N(\pi)$; (2) it cannot be $p_i < p^*$ for $\forall i \in N(\pi)$.*

Proof: Firstly, let $\widehat{x}$ be an equilibrium[9] of Γ^{n^π}. Suppose $u_i(\widehat{x}, p_i) \neq u_{i'}(\widehat{x}, p_{i'})$, and without loss of generality, assume $u_i(\widehat{x}, p_i) > u_{i'}(\widehat{x}, p_{i'})$. Then given the coalition choices of the rest of the population k_{-j}, individual j is better off deviating

[9]The finiteness of $N(\pi)$ is supported by the assumption of finiteness of $\mathcal{L}$.

to coalition i. But then $\widehat{x}$ will not support an evolutionarily stable coalition structure. Secondly, let $\widehat{x}$ be an equilibrium and suppose $p_i < p^*$, for $\forall i \in N(\pi)$. In other words, equilibrium happens in the increasing part of the utility function. We are left to show that an individual's deviation in her coalition choice is always beneficial. Consider a group deviation from coalition i' to i such that $u_i(\widehat{x}, \widehat{p}_i) = u_{i'}(\widehat{x}, \widehat{p}_{i'}) + \epsilon$. Since $u_l(\widehat{x}, p_l) = u_m(\widehat{x}, p_m)$ for arbitrary l, m, and we know from the presumption that $u_i(\widehat{x}, \widehat{p}_i) > u_m(\widehat{x}, p_m)$ for all $m \in N(\pi)\backslash i$. Since this deviation is beneficial, $\widehat{x}$ cannot be an evolutionarily stable coalition equilibrium. □

For each coalition game Γ^{n^π}, denote $\overline{\Gamma}^{n^\pi}$ as the corresponding normal form. $\overline{\Gamma}^{n^\pi} := (N(\pi), (x_j)_{j \in N(\pi)}, (\overline{u}_j(x)_{j \in N(\pi)}))$, where $\overline{u}_j(.)$ is not related to the coalition size. Denote $R_i(x_{-i}, \pi)$ as the set of coalition i's best response to the strategy profile of the rest coalitions x_{-i} in Γ^{n^π}, and $\overline{R}_i(x_{-i})$ as player i's best response set to x_{-i} in $\overline{\Gamma}^{n^\pi}$.

Lemma 2 $\overline{R}_i(x_{-i}) \subset R_i(x_{-i}, \pi)$ *for some* $\pi \in \Pi$.

Proof: (1) Let $e_i \in \overline{R}_i(x_{-i})$, then there exists a π such that $e_i \in R_i(x_{-i}, \pi)$. Suppose $e_i \notin R_i(x_{-i}, \pi)$, that is, $u_i((e_i, x_{-i}), p_i) < u_i((\widehat{e}_i, x_{-i}), p_i)$ for some $\widehat{e}_i \notin \overline{R}_i(x_{-i})$. Since $u_i(.)$ is continuous, there exist p_i and p_i' such that $u_i((e_i, x_{-i}), p_i) = u_i((\widehat{e}_i, x_{-i}), p_i')$. Lemma 1 says that it must be $p_i < p_i' \leqslant p^*$ or $p^* < p_i' < p_i$ in equilibrium. The former is not possible since it violates the individual's coalition decision. The latter indicates that $u_i((e_i, x_{-i}), p_i') > u_i((\widehat{e}_i, x_{-i}), p_i')$, which implies that coalition i is better off deviating to e_i.

(2) Consider $e_k \notin \overline{R}_i(x_{-i})$ such that $\overline{u}_i(e_k, x_{-i}) \leqslant \overline{u}_i(e_i, x_{-i})$. By continuity of the utility function, there exists a population state such that $u_i((e_k, x_{-i}), p_i) > u_i((e_i, x_{-i}), \widehat{p}_i)$ for $p_i < p^* < \widehat{p}_i$. □

Proposition 1 *There exists at least an evolutionarily stable coalition equilibrium in* Γ^{n^π}.

The existence is guaranteed by the Kakutani fixed point theorem, and by Lemma 2, the set of evolutionary stable equilibrium includes the set of equilibrium of $\overline{\Gamma}^{n^{\pi}}$.

3 Evolutionary stable coalition structure-the stability condition

In this section, we assume that there is only a proportion δ of each coalition that will reconsider their coalition choices at each point of time. δ can be interpreted as the birth/death rate. We hope to examine the movement of each coalition's population share across time, from any initial partition π^0. The movement is taken place in a discrete time version. Throughout this section, we examine the paths for a given strategy profile. One can interpret this as the following. Given a new provision of each jurisdiction's public good, how will people move from one jurisdiction to another? The case when each coalition's strategy varies with its population is left for further discussion.

Consider a partition of the population at time t, $\pi^t := \{\pi_1^t, \pi_2^t, ...\pi_{n^{\pi(k)}}^t\}$. Let x be a strategy profile in $\Gamma^{N(\pi^t)}$. Denote $u_i(x, p_i^t)$ as coalition i's payoff given the strategy profile x and the population share p_i^t at time t. Denote k^j as individual j's coalition choice if it is j's turn to reconsider her coalition choice. Upon renewing, individual j observes each coalition's payoff from the previous period, and chooses myopically the coalition(s) with the highest payoff. We assume that if there are more than one coalitions with the highest payoff, the renewing individual randomly chooses one with equal probability. That is, $k^j = i$, if $i \in D(x, \pi^t) := \arg\max_{h \in N(\pi)} u_h(x, p_h^t)$. Denote $D_t(x) := |D(x, \pi^t)|$ as the number of coalitions which possess the highest payoff at time t. The population shares dynamics are

$$p_i^{t+1} = p_i^t + \frac{\delta}{D_t(x)}(\sum_{k \in -i} p_k^t + p_i^t) - \delta p_i^t = (1-\delta)p_i^t + \frac{\delta}{D_t(x)}, \quad if\ i \in D(x, \pi^t), \ (1)$$

$$p_i^{t+1} = (1-\delta)p_i^t, \qquad \qquad if\ i \notin D(x, \pi^t). \quad (2)$$

For those coalitions whose previous period payoffs are not among the highest ($i \notin D(x, \pi^t)$), there will be only outflows of population. For those whose previous period payoffs are among the highest, there will still be a proportion δ of population reconsidering their coalition choices, but, there will be a proportion $\frac{1}{D_t(x)}$ of them (after reconsidering) choosing to stay in their original coalitions. Notice from equation (1) that even among the highest, if $p_i^t > \frac{1}{D_t(x)}$, there will still be outflows from coalition i.

We first examine the case where $u_i(.)$ is identical for each coalition i, and x is symmetric. There are mainly two groups: when the population partition is symmetric and when it is not. If the partition is symmetric, i.e., $p_1^t = ... = p_n^t$, then $D_t(x, \pi^t) = N(\pi^t)$. Hence for each $i \in N(\pi^t)$, the population share dynamics is as equation (1). Stability[10] requires $(1-\delta)p_i^t + \frac{\delta}{D_t(x)} = p_i^t$ for $i \in N(\pi^t)$, or equivalently, $p_i^t = \frac{1}{N_t}$ (where $N_t := |N(\pi^t)|$). In other words, when $p_i^t > \frac{1}{N_t}$, there is only an outflow of population from each coalition, and when $p_i^t < \frac{1}{N_t}$, there is an inflow of population to each coalition.

We discuss three possible scenarios for equilibrium. Firstly, if $p_1^t = ... = p_n^t < \frac{1}{N_t}$, then by equation (1) $p_i^{t+1} > p_i^t$ for each $i \in N(\pi^t)$. This would not be possible for a fix population. By concavity and continuity of the utility function, there will exist an $M(\widehat{\pi}^t) \subset N(\pi^t)$ (i.e., $\frac{1}{M_t} > \frac{1}{N_t}$) such that $p_1^t = ... = p_{M_t}^t = \frac{1}{M_t}$ and $D_t(x, \widehat{\pi}^t) = M(\widehat{\pi}^t)$, and hence $p_i^{t+1} = p_i^t$ for $i \in M(\widehat{\pi}^t)$. That is, in the case of symmetric strategies and symmetric population shares, if each coalition's population share is too small, the equilibrium requires a reduction of the number of coalitions. Secondly, if $p_1^t = ... = p_n^t > \frac{1}{N_t}$, then by equation (1) $p_i^{t+1} < p_i^t$ for each $i \in N(\pi^t)$. Like the previous case, this would not be possible for a fix population. Hence slightly differently, there will exist an $\widehat{M}(\widetilde{\pi}^t) \supset N_t(\pi^t)$ such

[10]By stability, we mean that each coalition's population share does not vary with time. Since each individual is insignificant, we assume that whether each coalition consists of the same individuals will not affect the payoff.

that $p_1^t = ... = p_{\widehat{M}_t}^t = \frac{1}{\widehat{M}_t}$ and $D_t(x, \pi^t) = \widehat{M}_t(\widetilde{\pi}^t)$, and $p_i^{t+1} = p_i^t$ for $i \in \widehat{M}_t(\widetilde{\pi}^t)$. That is, if each coalition share is too large, the equilibrium requires an increase of the number of coalitions. Finally, if $p_1^t = ... = p_n^t = \frac{1}{N_t}$, then $p_i^{t+1} = p_i^t$, $\forall i \in N(\pi^t)$.

We now consider the general case when the population is asymmetric. The development of each group depends not only on its initial location in distribution but also its strategical advantage among coalitions. Accordingly, we can identify two cases by the size of $D_t(x, \pi^t)$. Firstly, if $D_t(x, \pi^t) = N(\pi^t)$ and without loss of generality,[11] let $p_i^t = \underline{p} < p^*$ for $i \leqslant z$, and $p_i^t = \overline{p} > p_i^*$ for $i > z$. Recall that p^* is the maximal argument of $u(.)$. For this case, by equation (1), we have $p_i^{t+1} > p_i^t$ for $i \leqslant z$ and $p_i^{t+1} < p_i^t$ for $i > z$. If the utility function is symmetric to p^* (so that $D_{t+1}(x, \pi^{t+1}) = N(\pi^t)$), then both groups of coalitions are converging to the size of $\frac{1}{N}$.

Secondly, if $D_t(x, \pi^t) = M_t(x) \subset N(\pi^t)$ and without loss of generality, we consider four subgroups: A_1, A_2, A_3, A_4. The connections are: $A_1, A_2 \subset M_t(x)$ and $A_3, A_4 \subset N(\pi^t) \backslash M_t(x)$, and for each $i \in A_1, A_3$, we have $p_i^t < p_i^*$ and for $i \in A_2, A_4$, we have $p_i^t > p_i^*$. The movement of coalitions in each subgroup is as follows: (i) In group A_3 since $p_i^t < p_i^*$, by equation (2) the size of coalition $i \in A_3$ is decreasing through time ($p_i^{t+1} < p_i^t$). Coalitions $i \in A_3$ will ultimately diminish as t increases.

(ii) In group A_4, by equation (2) the size of coalition $i \in A_4$ will be decreasing. However, since $p_i^t > p_i^*$ for this case, the process of decreasing ceases at time $\widetilde{t}$ (to be defined), from which each coalition in A_4 then joins in the group of $D_{\widetilde{t}}(x, \pi^{\widetilde{t}})$.

(iii) Notice that since $A_1, A_2 \subset M_t(x)$, for some $\underline{p}$ and $\overline{p}$ with $\underline{p} < \overline{p}$, it must be true that $p_i^t = \underline{p}$ for all $i \in A_1$ and $p_i^t = \overline{p}$ for all $i \in A_2$. (Otherwise, the payoff level will not be equivalent). Therefore in group A_2, we can further identify two cases: if $p_i^t < \frac{1}{M_t}$ for $\forall i \in A_2$, then by equation (1) $p_i^{t+1} > p_i^t$ till some point of time $\widehat{t}$, from which $p_i^{\widehat{t}} > \frac{1}{M_{\widehat{t}}}$ then $p_i^{\widehat{t}+1} < p_i^{\widehat{t}}$. If $p_i^t > \frac{1}{M_t}$, we have $p_i^{t+1} > p_i^t$.

(iv) In group A_1, we have $p_i^{t+1} > p_i^t$ but the extent of increase varies with the

[11] If the equality of the population share for both groups $i \leqslant z$ and $i > z$ is not satisfied, then $D_t(x, \pi^t) \neq N(\pi^t)$.

population shares of group A_2 in the following sense. If $p_j^t < \frac{1}{M_t}$ for $j \in A_2$, then by equation (1) $p_j^{t+1} > p_j^t$ for $j \in A_2$. Combining the movement in group A_1, we have $D_{t+1}(x, \pi^{t+1}) = A_1$. Accordingly, we have $p_i^{t+2} > p_i^{t+1}$ for $i \in A_1$, but $p_j^{t+2} < p_j^{t+1}$, for $j \in A_2$. Since the size of group A_1 keeps on growing, we know $D_{t+2}(x, \pi^{t+2}) = A_1$. The process will continue till some time $\widehat{t}$ where $D_{\widehat{t}}(x, \pi^{\widehat{t}}) = M_t$.[12] From $\widehat{t}$ and so forth, $p_i^t > p_i^*$ for all $i \in A_1, A_2$.

If $p_j^t > \frac{1}{M_t}$ for $j \in A_2$, then $p_j^{t+1} < p_j^t$. By symmetry of the utility function, the number of coalitions with the highest payoffs at $t+1$ is the same as time t, that is, $D_{t+1}(x, \pi^{t+1}) = M_t(x)$.

If $p_j^t = \frac{1}{M_t}$ for all $j \in A_1, A_2$, the population share remains unchanged but this temporary stability stays till time $\widetilde{t}$, at which $j \in A_4$ joins in the group of $D_{\widetilde{t}}(x, \pi^{\widetilde{t}})$. The existence of time $\widetilde{t}$ is guaranteed by the continuity of the utility function. Proposition 2 summarizes our finding.

Proposition 2 *When both $u_i(.)$ and x are symmetric, (i) if the partition is symmetric, the only evolutionarily stable structure is $\pi^t = \{\pi_1^t, \pi_2^t, ...\pi_{n^{\pi(k)}}^t\}$, where each $|\pi_i^t| = \frac{1}{n^{\pi(k)}}$. (ii) if the partition is asymmetric, those coalitions with $|\pi_i^t| < p^*$ and which are not among the highest profit will ultimately diminish; the rest will form another stable structure at some time $\widetilde{t}$, where all coalitions have the share $|\pi_i^{\widetilde{t}}| = \frac{1}{D_{\widetilde{t}}(x, \pi^{\widetilde{t}})}$.*

As the end of our discussion, we now assume that $u_i(.,.)$ is identical for each coalition i, but x is asymmetric. Without loss of generality, we rank the coalitions according to the levels of payoff in $\overline{\Gamma}^{n^\pi}$, and let $u_j(x, 0)$ be the payoffs$\overline{u}_j(x)$ for $j \in N(\pi)$ in $\overline{\Gamma}^{n^\pi}$. That is, $u_1(x, 0)$ denotes the highest coalition payoff $\underset{\forall i \in N(\pi)}{argmax} \overline{u}_j(x)$, and $u_2(x, 0)$ denotes the second highest coalition payoff, and so on. Fig. 1 gives an illustration for this arrangement. By definition, $u_1(x, 0) > u_2(x, 0) > ... > u_{\widehat{n}}(x, 0)$,

[12] $\widehat{t}$ is the same as defined in the discussion of group A_2, and $M_t = |D_t(x, \pi^t)|$ at time t. Notice further that Group A_4 will not join in $D_t(x, \pi^t)$ earlier than group A_2, whose population share must have been lower to be in $D_t(x, \pi^t)$ in the first place.

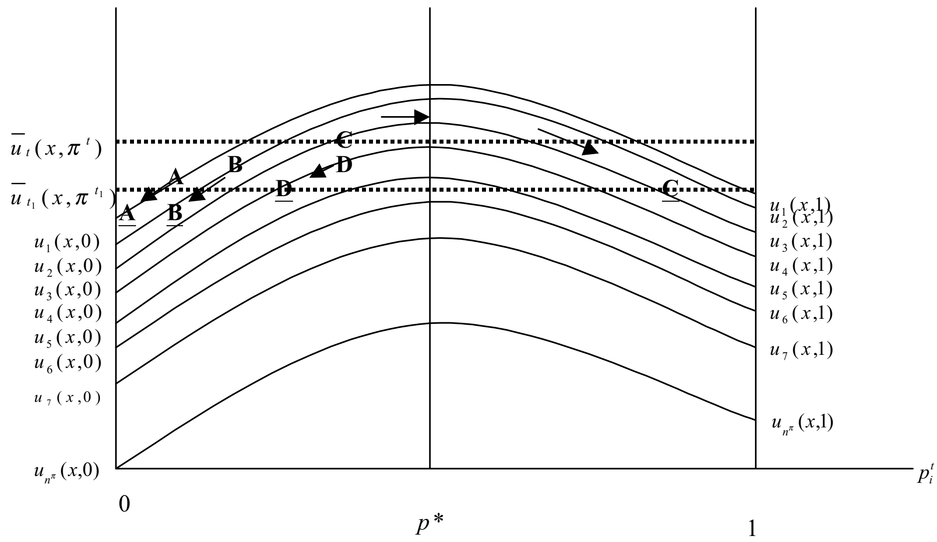

Figure 1: Ranking the coalitions according to the levels of payoff

where $\widehat{n} \leqslant n^\pi$ (with "strictly less" if more than one coalitions have the same level of payoff). Every $u_\phi(x, p_\phi^t)$ for $\phi = 1, 2, ..\widehat{n}$ is concave in the size of coalition p_ϕ^t, and recall p^* as the value of p when $u_i(., .)$ is maximized. To ensure the existence of a coalition structure, we assume that $u_1(x, 1)$ is not too high, which is similar to assume non-superadditivity to avoid grand coalition to be the only consequence. To examine the movement of each coalition, we classify the following cases.

Firstly, when $p_\phi^t < p^*$ for $\phi = 1, 2, ..\widehat{n}$, all coalitions' payoffs are increasing in the size of the population. If $D_t(x, \pi^t) = M_t(x) \subset \{1, 2, ..\widehat{n}\}$,[13] then for every $\phi \in M_t(x)$ the population movement follows equation (1) and hence $p_\phi^{t+1} > p_\phi^t$; for $\phi \notin M_t(x)$, the movement follows equation (2) and hence $p_\phi^{t+1} < p_\phi^t$. Therefore at time $t + 1$, we have $D_{t+1}(x, \pi^{t+1}) = M_t(x)$ and the highest payoff is firstly increasing then decreasing till time $\overline{t}$ at which either of the following situations happens:

[13]We eschew the discussion of $D_t(x, \pi^t) = N(\pi^t)$ only for simplification. The alternative assumption is to assume that some coalition's payoffs are much less than the others.

(a) If $D_{\overline{t}}(x, \pi^{\overline{t}}) = V_{\overline{t}}(x) \subset \{1, 2, ..\widehat{n}\} \backslash M_t(x)$, then for $\phi \in V_{\overline{t}}(x)$ we have $p_{\phi}^{\overline{t}+1} > p_{\phi}^{\overline{t}}$; for $\phi \notin V_{\overline{t}}(x)$ (including $M_t(x)$), the movement follows equation (2) and we have $p_{\phi}^{\overline{t}+1} < p_{\phi}^{\overline{t}}$. Therefore at time $\overline{t}+1$, if $D_{\overline{t}+1}(x, \pi^{\overline{t}+1}) = V_{\overline{t}}(x)$, then the movement is as just described; if $D_{\overline{t}+1}(x, \pi^{\overline{t}+1}) = M_t(x)$, meaning that the coalitions in $M_t(x)$ retain their leadership in payoffs), the movement is as earlier.

(b) If $D_{\overline{t}}(x, \pi^{\overline{t}}) = V_t(x) \supset M_t(x)$, then since $p_{\phi}^{\overline{t}} > \frac{1}{V_{\overline{t}}}$ for $\phi \in M_t(x)$ and $p_{\phi}^{\overline{t}} < \frac{1}{V_{\overline{t}}}$ for $\phi \in V_{\overline{t}}(x) \backslash M_t(x)$, we have $p_{\phi}^{\overline{t}+1} < p_{\phi}^{\overline{t}}$ for $\phi \in M_t(x)$ and $p_{\phi}^{\overline{t}+1} > p_{\phi}^{\overline{t}}$ for $\phi \in V_{\overline{t}}(x) \backslash M_t(x)$. By symmetry of the utility function, $D_{\overline{t}+1}(x, \pi^{\overline{t}+1}) = V_{\overline{t}}(x)$, and the movement at $\overline{t}+2$ repeats the process at $\overline{t}+1$ till time $\widetilde{t}$ at which all members of $V_{\overline{t}}(x)$ are in the decreasing part and some new members will join in the group with the highest payoff and the process repeats (a) or (b).

For an illustration, suppose in Fig. 1 that $D_t(x, \pi^t) = \{3\}$, (point C), with the highest payoff $\overline{u}_t(x, \pi^t)$. So $p_3^{t+1} > p_3^t$, till time $\overline{t}$, when $D_{\overline{t}}(x, \pi^{\overline{t}}) = \{3, 4\}$ (point C, D), with the highest payoff of level $\overline{u}_{\overline{t}}(x, \pi^{\overline{t}})$. From equation (1) we have $p_3^{\overline{t}+1} < p_3^{\overline{t}}$ and $p_4^{\overline{t}+1} > p_4^{\overline{t}}$. Note that from time t to $\overline{t}$, the highest payoffs are decreasing from $\overline{u}_t(x, \pi^t)$, with maximum at point p^* and decreasing till $\overline{u}_{\overline{t}}(x, \pi^{\overline{t}})$. Step (a) and (b) stop at a time t when $p_i^t = \frac{1}{|D_t(x, \pi^t)|}$, $\forall i \in D_t(x, \pi^t)$, and $p_i^t = 0$, $\forall i \notin D_t(x, \pi^t)$. The numbers of coalitions in an evolutionarily stable structure is hence $|D_t(x, \pi^t)|$.

Proposition 3 *When $u_i(.)$s are symmetric but x is asymmetric, (i) the highest payoffs are oscillatory across time still stability happens. (ii)In the evolutionarily stable structure, each group share the same population and the coalition numbers is hence $|D_t(x, \pi^t)|$.*

(ii) When[14] π^t is asymmetric. The analysis is similar to the case with symmetric x.

[14]We eschew the discussion of $p_{\phi}^t > p^*$, $\phi = 1, 2, ..\widehat{n}$, which is unlikely.

4 The concluding remarks and implication

The present paper analyzes the formation and stability of coalitions for a situation where finitely many individuals form different coalitions and their payoffs depend on the consequence of a noncooperative game among different coalitions, and examine the moving path of individuals among various coalitions. The model actually consists of two stages. In the first stage, each individual chooses which coalition to belong to, and this leads to the formation of a coalition structure π. All coalitions in π then play an n^{π}-player noncooperative game in the second stage. The payoff for each individual depends on the strategy combinations of all participating coalitions and the proportion of population for the coalition which he belongs to.

We defined an evolutionarily stable coalition equilibrium of $\Gamma^{n^{\pi}}$, which will support an evolutionarily stable coalition structure not subject to an improving deviation by an coalition of individuals. Our specific findings are as follows. Firstly, we show that there exists at least one evolutionarily stable coalition equilibrium in $\Gamma^{n^{\pi}}$. This result is based on an explicit assumption on utility function, that is, we assume that coalition values depend on the population share of each coalition and is concave in p_j. Secondly, when addressing the evolving path of coalitions, we conclude that in the case of symmetric strategies and symmetric population shares, if each coalition's population share is too small, the equilibrium requires a reduction of the number of coalitions, but if each coalition share is too large, the equilibrium requires an increase of the number of coalitions. Thirdly, in Proposition 2 we show that when both $u_i(.)$ and x are symmetric, (i) if the partition is symmetric, the only evolutionarily stable structure is $\pi^t = \{\pi_1^t, \pi_2^t, ...\pi_{n^{\pi(k)}}^t\}$, where each $|\pi_i^t| = \frac{1}{n^{\pi(k)}}$. (ii) if the partition is asymmetric, those coalitions with $|\pi_i^t| < p^*$ and which are not among the highest profit will ultimately diminish. Finally, when $u_i(.)s$ are symmetric but x is asymmetric, then (i) the highest payoffs are oscillatory across time still stability happens. (ii)In the evolutionarily stable structure, each group share the same population and the coalition numbers is hence $|D_t(x, \pi^t)|$.

Our model can draw an interesting implication regarding Folk theorem in repeated games. Folk theorem says that the efficient payoffs can be supported as average payoffs of subgame perfect equilibria. The rationale for this result is that players can punish the deviators by switching to the minimax equilibrium (the reservation utility). However, when individuals are free to move, the punishment decision will force individuals move toward the deviating coalition which ensures the highest payoff. Hence the partition will change and the game is not the same each period. When a coalition really deviates, other coalitions can only find the best reply to the deviation, and hence would not be the minimax strategy. As a result, the only possible payoff is which can be supported by the equilibrium with evolutionary stable coalition structure.

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