行政院國家科學委員會專題研究計畫　成果報告

隨機成長系統之吉伯特定理與規模之謎

計畫類別：個別型計畫
計畫編號：NSC93-2415-H-004-015-
執行期間：93 年 08 月 01 日至 95 年 01 月 31 日
執行單位：國立政治大學經濟學系

計畫主持人：陳心蘋

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 95 年 3 月 29 日

行政院國家科學委員會補助專題研究計畫 ■ 成 果 報 告
□期中進度報告

隨機成長系統之吉伯特定理與規模之謎

計畫類別：■ 個別型計畫　　□ 整合型計畫
計畫編號：NSC　93　－　2415　－ H －　004　－ 015
執行期間：　2004　年　08　月　01 日至　2006　年　01　月　31　日

計畫主持人：陳心蘋
共同主持人：
計畫參與人員：

成果報告類型(依經費核定清單規定繳交)：■精簡報告　□完整報告

本成果報告包括以下應繳交之附件：
□赴國外出差或研習心得報告一份
□赴大陸地區出差或研習心得報告一份
□出席國際學術會議心得報告及發表之論文各一份
□國際合作研究計畫國外研究報告書一份

處理方式：除產學合作研究計畫、提升產業技術及人才培育研究計畫、列管計畫及下列情形者外，得立即公開查詢
□涉及專利或其他智慧財產權，□一年□二年後可公開查詢

執行單位：國立政治大學經濟學系

中　華　民　國　九十五　年　三　月　二十二　日

附件二

可供推廣之研發成果資料表

□ 可申請專利　□ 可技術移轉　日期：95 年 03 月 22 日

國科會補助計畫	計畫名稱：隨機成長系統之吉伯特定理與規模之謎 計畫主持人：陳心蘋 計畫編號：NSC　93－2415－H－004－015 學門領域：區域研究及地理
技術/創作名稱	
發明人/創作人	
技術說明	中文： （100~500 字）
	英文：
可利用之產業 及 可開發之產品	
技術特點	
推廣及運用的價值	

※　1. 每項研發成果請填寫一式二份，一份隨成果報告送繳本會，一份送 貴單位研發成果推廣單位（如技術移轉中心）。

※　**2. 本項研發成果若尚未申請專利，請勿揭露可申請專利之主要內容。**

※　3. 本表若不敷使用，請自行影印使用。

隨機成長系統之吉伯特定理與規模之謎

中文摘要

都市人口的成長與分佈俱有兩個顯著的特性，這兩個特性曾被不同時間與國家的資料印證過：(一)都市人口的分配符合普瑞夫定理（Zipf's law）：都市規模的分佈，實證顯著符合普瑞夫定理，為冪次法則的特殊情況（柏拉圖係數等於1）。實證研究顯示了該定理的普遍性與一致性（Losch 1954; Hoover 1954; Beckman 1958; Simon and Bonini 1958; Okuyama et al. 1999; Axtell 2001; Solomon et al. 2000, 2001），相關研究曾提出不同的模型與理論來解釋普瑞夫定理（Simon 1955; Fujita, Krugman, and Venables 1999; Gabaix et al. 1999）。其中 Gabaix（1999）提出以純統計觀點的吉伯特定理（Gibrat's law）解釋普瑞夫定理。(二)都市人口成長率的分配與都市人口數無關，人口成長率的成長為符合吉伯特定理的proportionate growth。許多實證研究發現，人口成長率分配的平均數和變異數，與都市人口數沒有顯著相關，成長率的分配與該變數無關的隨機成長過程稱之為proportionate growth。Gibrat(1931)曾推導證實這樣的隨機過程會導致 lognormal 型態的極限分配。然而，實證研究都市人口的資料並不是呈現 lognormal 分配，而是符合普瑞夫定理的柏拉圖分配(Pareto distribution)。Eeckhout（2004) 以完整的人口資料顯示實際的都市人口分佈是呈 lognormal 分配，而不是符合普瑞夫定理的柏拉圖分配。之前的研究會有柏拉圖分配之說是因為所用的資料只取部分，而非全部。這個研究解開了過去都市人口成長與分佈特性理論上不合之迷，Proportionate 成長過程的確可演化出 lognormal 的分配，理論實證皆證實。

聚集經濟在都市的形成扮演重要的角色，產業與人口聚集所帶來的外部經濟對都市成長的影響在許多例子都可看到，聚集經濟隱含正回饋的報酬遞增特性。傳統的經濟理論以報酬遞減(decreasing returns)為基礎，推導出穩定的、可預測的單一均衡解。實際上很多經濟問題是衍化的(evolutionary)、不可預測的(unpredictable)以及可能有多個均衡解(multiple equilibrium)。這樣的經濟問題大部分具有報酬遞增（increasing returns）與正回饋（positive feedbacks）的特質。雖然並非所有的經濟問題都屬於報酬遞增的類型，傳統資源導向的經濟型態

(resources-based economy)仍是傾向於報酬遞減，但是近年來逐漸盛行的知識經濟（knowledge-based economy）產業則普遍具有報酬遞增與正回饋的特性（Authur 2002; Scheinkman 1994）。報酬遞增與正回饋的隨機成長模型會呈現路徑相依（path-dependent）、不可預測和鎖定效果（lock-in）的衍化過程，可能導致多個均衡解（multiple equilibrium）或不穩定的結果。具有報酬遞增特質的動態配置過程（dynamic allocation process），可以解釋多種經濟問題：例如消費者對產品的選擇行為，以及所導致的產品市場的最終佔有率；廠商與居民的區位選擇行為與所可能導致的都市規模分佈與都會結構。Authur（2002）應用一般的隨機波亞過程（Polya process）模型，分析具有報酬遞增特質的動態配置過程，解釋不同類型的經濟衍化問題。波亞過程模型假設每期新變數屬於各選項的機率是各選項當期佔有比率的函數。Chung（2003）以統計方法分析一般波亞過程的特性，該研究為純數學的推導，沒有經濟意義的解釋。

影響區位選擇重要因素之一的聚集經濟所具有的報酬遞增特性與吉伯特定理的都市成長為 proportionate growth 的特性似乎有所矛盾。理論只證明 Proportionate growth 是 lognormal 分配的充份而非必要條件。本研究的目的是探究聚集經濟發生在吉伯特定理（proportionate growth）的可能性，與探討報酬遞增的隨機成長過程是否可演化出 lognormal 的極限分配。本研究應用 Authur（2002）與 Chung（2003）的一般波亞過程模型為分析工具， 解釋居民的區位選擇，在不同的機率函數假設下，模擬動態隨機的波亞過程，比較極限分配，並分析波亞模型的特性。 研究結果顯示，在一般的波亞過程中，其動態過程與極限分配對機率函數中的係數 k 值的大小很敏感。當 k 大於 1 時，動態過程俱有報酬遞增的特性，都市大小的差異隨著原來大小的差異為權數擴大，某範圍的參數值下可導致極接近 lognormal 的極限分配，顯示在一般化的模型下，報酬遞增特性可產生符合實際資料的 lognormal 分配。

Gibrat’s Law and Scaling Puzzle in the Stochastic Growth Process

Abstract

Zipf’s law and proportionate growth process are two empirical regularities concerning the growth and resulting size distribution of cities. Zipf's law for cities and firms is one of the most robust empirical regularity in the social sciences generally. This striking pattern and its general form (power law) have been studied theoretically and empirically. It appears to hold in all countries and dates. Gabaix (1999) proposes Gibrat’s law as an explanation of Zipf's law. He shows that homogeneous growth processes of cities possibly lead the distribution of city size converging to Zipf's pattern.

The growth rate of city populations does not depend on the size of the city, which characterizes that the underlying stochastic process is the same for all cities. This is labeled the proportionate growth process. Empirical research has shown this regularity in different data. There is a puzzle regarding these two regularities about cities distribution: Gibrat (1931) has established a well known proposition that the proportionate growth process generates lognormal distribution, rather than the Pareto distribution. However, empirical works of city growth and distribution show the coexistence of the proportionate growth and Pareto. This puzzle is finally solved recently.

Eeckhout (2004) examines the untruncated Census 2000 data and finds that the size distribution of the entire sample is lognormal rather than Pareto, and the growth rate of cities is independent of city size. His work asserts a new look of the empirical regularities concerning that city growth is proportionate and the resulting distribution is lognormal.

Theoretically, proportionate growth process leads to lognormal distribution empirically which make it a sufficient condition of lognormal distribution; however, it has not been verified that proportionate growth process is also the necessary condition for the lognormal distribution.

Positive spillovers (agglomeration economies) in production are important reasons that firms and workers locate in cities. It is suggested that the concentration of particular industries is the result of some set of cumulative processes involving some form of increasing returns generated from self-reinforcing feature. Increasing returns imply positive feedbacks, multiple equilibrium, nonpredictability, and lock-in properties. The external benefit (agglomeration) generated from concentration of activities is essential features of the formation of city. Nonetheless, the empirical suggestion of city growth as proportionate growth process seems not congruous with the feature of agglomeration economy in the city.

The objective of this paper is to study the possibility of comprising the feature of increasing returns into the city growth process and still resulting in a lognormal distribution. We empirically investigate the property of the limiting distribution of the general *Polya process* based on the *process* in Chung et al (2003) and Arthur (2000). The feature of the growth process and size distribution of cities in Taiwan is examined. Moreover, we investigate the emergence condition of Gibrat's law in the proposed stochastic nonlinear *Polya process*.

The simulations of the general *Polya process* find that the features of the dynamic process as well as the limiting distribution are highly sensitive to the power coefficient *k* of size in probability function. When *k* is greater than one, the dynamic process is increasing returns. The degree of size differences is reinforcing by corresponding size proportion. The limiting distribution is more diverged than the initial distribution. It tends to lead to a positively skewed distribution depends on the value of *k*. The limiting distribution appears lognormal given some parameters values. The growth rate of size is positively related to corresponding location size. The larger the size of location, the faster it grows. This finding states that a stochastic process with increasing returns may lead to a lognormal limiting distribution. On the contrary, when *k* is less than one, the dynamic process is decreasing returns. The degree of size differences is diminishing. The limiting distribution of population proportion tends to converge to be much more uniformly distributed than the initial distribution. The population distribution in Taiwan approximates lognormal distribution; this is consistent with Eeckhout (2004).

1. Introduction

Zipf's law and proportionate growth process are two empirical regularities concerning the growth and resulting size distribution of cities. Zipf's law for cities and firms is one of the most robust empirical regularity in the social sciences generally. This striking pattern and its general form (power law) have been studied theoretically and empirically (Losch 1954; Hoover 1954; Beckman 1958; Simon and Bonini 1958; Okuyama et al. 1999; Axtell 2001; Solomon et al. 2000, 2001). Different models have been applied to explain power law and its special case, Zipf's law (Simon 1955; Fujita, Krugman, and Venables 1999; Gabaix et al. 1999). It appears to hold in all countries and dates. Gibrat's law was first applied in explaining the size distribution of firms (Marcus 1969; McCloughan 1995; Sutton 1997; Lotti et al. 2003). Gabaix (1999) proposes Gibrat's law as an explanation of Zipf's law. He shows that homogeneous growth processes of cities possibly lead the distribution of city size converging to Zipf's pattern.

The growth rate of city populations does not depend on the size of the city, which characterizes that the underlying stochastic process is the same for all cities. This is labeled the proportionate growth process. Empirical research has shown this regularity in different data (Edward Glaeser et al. 1996; Jonathan Eaton and Zvi Eckstein 1997;and Yannis M. Ioannides and Henry G. Overman 2003). There is a puzzle regarding these two regularities about cities distribution: Gibrat (1931) has established a well known proposition that the proportionate growth process generates lognormal distribution, rather than the Pareto distribution. However, empirical works show the coexistence of the proportionate growth and Pareto. This puzzle is finally solved recently.

Eeckhout (2004) examines the untruncated Census 2000 data and finds that the size distribution of the entire sample is lognormal rather than Pareto. The stochastic kernel density of the normalized growth rate is observed for various deciles. It shows that both mean and the variance of the growth rate appear fairly constant over different deciles. The growth rate of cities is confirmed to be independent of city size empirically, which is consistent of other works. Their work provides a new look of the empirical regularities concerning that city growth is proportionate and the resulting distribution is lognormal. Eeckhout proposes a local externality model to explain the empirical city growth process.

Eeckhout's work empirically investigates the property of the city growth process. They conclude that the distribution of growth rate is independent of city size labeled as proportionate growth. However, in the surface plot of the kernel density estimation of normalized growth rates, variances of the distributions do not appears constant in all sizes.

Theoretically, proportionate growth process leads to lognormal distribution which is empirically observed in city size distribution. Proportionate growth is the sufficient condition of the lognormal distribution; it has not been verified that it is also the necessary condition for the lognormal distribution.

Positive spillovers in production are important reasons that firms and workers locate in cities (Guy Dumais et al. 1997). This external benefit from the clustering of economic activity named agglomeration economies includes localization and urbanization economies. It is suggested that the concentration of particular industries is the result of some set of cumulative processes involving some form of increasing returns generated from self-reinforcing feature (Fujita, Krugman, and Venables 1999). Eeckhout propose a model with local externalities to explain the empirical size distribution of cities. Unfortunately, the feature of local economy in the work is not crucial in generating the proportionate growth process.

Conventional economic theory is built on the assumption of diminishing returns. Diminishing returns imply stable and predictable evolution process and result in a single equilibrium point. However, diminishing returns do not apply everywhere. Parts of economies especially knowledge-based economies are subject to increasing returns (Scheinkman 1994). Increasing returns imply positive feedbacks, multiple equilibrium, nonpredictability, and lock-in properties.

Arthur (2000) applies a simple nonlinear stochastic process, called *Polya process*, to model the dynamic allocation process under positive feedback and increasing returns. *Polya process* models the probability of addition to the categories is a function of current proportions, which is an essential feature contained in various allocation process in reality. This nonlinear dynamic allocation process is applied to explain various problems of evolutions and corresponding limiting structures: including consumer behavior and corresponding industries market share; industries location decision and corresponding urban structure. As the improvement of symmetric information and the unavoidable trend of global economy, the knowledge-based economy becomes essential in the economy. Consequently, the feature of knowledge-based economy is worthwhile to analyzed. Investigating the classic *nonlinear Polya process* is a promising step to examine the feature of increasing returns. Chen (2004) simulates *Polya process* given various kinds of probability functions and concludes that a growth process with a diminishing returns agglomeration economy or a bounded increasing returns agglomeration economy would converge to a stable limiting distribution. On the contrary, the growth process with an unbounded increasing returns agglomeration economy would generate a fractal kind (power law) limiting distribution. Chung et al.(2003) theoretically analyze generalizations of the classical Polya urn problem. They derive power law distribution from the classical Polya urn problem given certain parameter conditions.

Empirical evidence shows that the growth process is proportionate; this is reconfirm by the limiting lognormal distribution. However, proportionate growth implies that the distribution of growth rate of city is independent of city size. This is very much an opposite idea of the external economics from clustering activities. Furthermore, there has not been confirmed that the proportionate growth is the necessary condition for lognormal distribution.

The objective of this paper is to study the possibility of comprising the feature of increasing returns into the city growth process and still resulting in a lognormal distribution. We empirically investigate the property of the limiting distribution of the general *Polya process* based on the general *Polya process* in Chung et al (2003) and Arthur (2000). The feature of the growth process and size distribution of cities in Taiwan is examined. Moreover, we investigate the emergence condition of Gibrat's law in the proposed stochastic nonlinear *Polya process*.

2. The Polya processes

The model applied in this paper is developed and introduced in Arthur (1984, 2000). The long-run limiting behavior of this nonlinear Polya type path-dependent process is examined to investigate the possible features of the dynamic increasing returns process.

Given finite set of locations (cities) $i \in N$. Each city at time *t* has population of size $s_{i,t}(i=1,...,N)$; and $x_{i,t}(i=1,...,N)$ describes the proportion of population of city *i* in the region at time *t*. Let $r_i(i=1,...,N)$ be the benefits of resident for locating in city *i*, consisting of geographical benefit q_i and the agglomeration benefit $g(x_i)$. The utility of resident for locating in city *i* is a function of the resident's benefit:

$$U_i = V(r_i) + e_i = V(q_i + g(x_i)) + e_i = v(x_i) + e_i, \qquad (1)$$

where e_i is the unknown part of the utility The location attractiveness due to geographical considerations is independent of the current location's shares. Consequently, given the time invariant geographical benefit, q_i, the probabilities of the locational choice of resident for city *i*, depends on the current location's shares, $x_{i,t}$.

$$p_i(x_i) = \Pr ob\{U_i > U_j \quad all \quad j \neq i\} = \frac{e^{v_i(x_i)}}{\sum e^{v_j(x_j)}}. \qquad (2)$$

Assuming the change of size at city *i* follow the dynamic process:

$$s_{i,t+1} = s_{i,t} + z_{i,t}(x_{i,t}), \qquad i = 1,...,N., \qquad (3)$$

where

$$z_{i,t} = \begin{cases} 1 & with \quad probabilit\ y \quad p_{i,t}(x_{i,t}) \\ 0 & with \quad probabilit\ y \quad 1 - p_{i,t}(x_{i,t}) \end{cases},$$

$$v(x_i) = \log(x_i^{\kappa}) + e_i,$$

The probability of the locational choice is a function of power of size share x_i^k. The expected motion of the locational share depends on the determinate part, which contains the choice probability function.

$$E[x_{i,t+1} | x_{i,t}] = x_{i,t} + \frac{1}{(w+t)}[p_{i,t}(x_{i,t}) - x_{i,t}]. \tag{4}$$

$$r_{i,t+1} = \frac{x_{i,t+1} - x_{i,t}}{x_{i,t}} = \left(\frac{1}{w+n}\right)\left[\frac{P_{i,t}(x_{i,t})}{x_{i,t}} - 1\right] = f\left(P_{i,t}(x_{i,t}), x_{i,t}\right) = G(x_{i,t}) \tag{5}$$

$$E(r_{i,t+1}) = \left(\frac{1}{w+n}\right) E\left[\frac{P_{i,t}(x_{i,t})}{x_{i,t}} - 1\right] = f\left(E\left[\frac{P_{i,t}(x_{i,t})}{x_{i,t}}\right]\right) = G(x_{i,t}) \tag{6}$$

$$Var(r_{i,t+1}) = E\left[\left(\frac{1}{w+n}\right)\left(\frac{P_{i,t}(x_{i,t})}{x_{i,t}} - 1\right)\right]^2 = \left(\frac{1}{w+n}\right)^2 E\left(\frac{P_{i,t}(x_{i,t})}{x_{i,t}} - 1\right)^2 \tag{7}$$

3. Simulation

The data used in this study is the population of 369 city, Township and District in Taiwan in 2004 and 2005. We simulate the finite general Polya process in section 2 given various parameter values to analyze the asymptotic distribution properties. Given finite number of locations *N*, the probability of location choice depends on the current size share. The value of parameter in probability function is essential in characterizing both the growth process and its limiting distribution.

3.1 *k*=1

The Polya process given the parameter *k*=1 implies that the location choice probability at time *t* exactly equal to its current size proportions in the region. The process remains in the initial distribution. The growth rate of size is independent of location size.

$$p_{i,t}(x_i) = x_{i,t}.$$

3.2 *k*<1

The Polya process given the parameter *k*<1 generate a process with decreasing returns. The degree of size differences is diminishing gradually depends on the value of k. The limiting distribution tends to converge to uniform distribution with small k. The growth rate of size is inversely related to location size. The larger the size of location, the slower its size grow.

3.3 $k>1$

The Polya process given the parameter $k>1$ generate a process of increasing returns. The degree of size differences is reinforcing by corresponding size proportion characterized by the value of k. The limiting distribution is more diverged than the initial distribution. It tends to lead to a positively skewed distribution based on the value of k. The limiting distribution appears lognormal given some parameters values. The growth rate of size is positively related to corresponding location size. The larger the size of location, the faster it grows.

4. Concluding remarks

The simulations of the general *Polya process* find that the features of the dynamic process as well as the limiting distribution are highly sensitive to the power coefficient k of size in probability function.

When k is greater than one, the dynamic process is increasing returns. The degree of size differences is reinforcing by corresponding size proportion. The limiting distribution is more diverged than the initial distribution. It tends to lead to a positively skewed distribution depends on the value of k. The limiting distribution appears lognormal given some parameters values. The growth rate of size is positively related to corresponding location size. The larger the size of location, the faster it grows. This finding states that a stochastic process with increasing returns may lead to a lognormal limiting distribution.

On the contrary, when k is less than one, the dynamic process is decreasing returns. The degree of size differences is diminishing. The limiting distribution of population proportion tends to converge to be much more uniformly distributed than the initial distribution. Moreover, the growth rate of size is inversely related to corresponding location size. The larger the size of location, the slower it grows. The population distribution in Taiwan approximates lognormal distribution; this is consistent with Eeckhout (2004).

Reference

Authur WB (2000), Increasing Returns and Path Dependence in the Economy, The University of Michigan Press, Ann Arbor.

Axtell RL (2001), Zipf’s Distribution of U.S. Firm Sizes, Science, 293:1818-1820.

Beckman M (1958), City Hierarchies and Distribution of City Size, Economic Development and Cultural Change, VI:243-248.

Chen, H.P. accepted (2003), Path-Dependent Processes and the Emergence of the Rank Size Rule, The Annals of Regional Science, forthcoming.

Chung F., Handjani S., Jungreis D. (2003), Generalizations of Polya’s Urn Problem, Annals of Combinatorics, 7:141-154.

Eaton, Jonathan and Eckstein, Zvi. (1997), Cities and Growth: Theory and Evidence from France and Japan, Regional Science and Urban Economics, 27(4 –5), pp. 443–74.

Fujita M, Krugman P, Venables A (1999), The Spatial Economy, Cambridge, MA: MIT Press.

Gabaix X (1999), Zip’s Law and the Growth of Cities, The American Economic Review, 89(2), 129-132.

Gibrat, Robert (1931), Les ine´galite´s e´conomiques; applications: aux ine´galite´s des richesses, a` la concentration des entreprises, aux populations des villes, aux statistiques des familles, etc., d’une loi nouvelle, la loi de l’effet proportionnel. Paris: Librairie du Recueil Sirey.

Glaeser, Edward L., Scheinkman, Jose A. and Shleifer, Andrei (1995), Economic Growth in a Cross-Section of Cities, Journal of Monetary Economics, *36*(1), pp. 117–43.

Dumais, Guy; Ellison, Glenn and Glaeser, Edward L. (2002), Geographic Concentration as a Dynamic Process, The Review of Economics and Statistics, *84*(2), pp. 193–204.

Hoover EM (1954), The Concept of a System of Cities: A Comment on Rutledge

Vining's Paper. Economic and Cultural Change, III:196-198.

Ioannides, Yannis M. and Overman, Henry G. (2003), Zipf's Law for Cities: An Empirical Examination, Regional Science and Urban Economics, *33*(2), pp. 127–37.

Jan Eeckhout (2004), Gibrat's Law for(ALL) Cities, AER Vol.94 no.5 1429-1451.

Losch A (1954), The Economics of Location, Yale University Press.

Lotti F, Santarelli E, Vivarelli M (2003), Does Gibrat's Law Hold Among Young, Small Firms, Journal of Evolutionary Economics, 13:213-235.

Marcus M (1969), A Note on the Determinants of the Growth of firms and Gibrat's Law, The Canadian Journal of Economics, 2(4): 580-589.

McCloughan P (1995), Simulation of Concentration Development from Modified Gibrat Growth-Entry-Exit Processes, The Journal of Industrial Economics, 43(4): 405-433.

Okuyama K, Takayasu M, Takayasu H (1999), Zipf's Law in Income Distribution of Companies, Physica A, 269:125-131.

Scheinkman JA, Woodford M (1994), Self-Organized Criticality and Economic Fluctuations, The American Economic Review, 84(2):417-421.

Simon H, Bonini CP (1958), The Size Distribution of Business Firms, American Economic Review, 607-617.

Simon H (1955), On a Class of Skew Distribution Function, Biometrika, XLII: 425-440.

Solomon S, Richmond P (2001), Power Laws of Wealth, Market Order Volumes and Market Returns, Physica A: Statistical Mechanics and its Applications, 299(1-2):188-197.

Solomon S, Blank A (2000), Power laws in cities population, financial markets and internet sites (scaling in systems with a variable number of components), Physica A, 287:279-288.

Sutton, John (1997), Gibrat's Legacy, Journal of Economic Literature, 1997, 35(1), pp.40-59.