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不需估計共變異結構特徵組的新平滑性檢定
研究成果報告(精簡版)

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中文摘要： 假設檢定在經濟、財務或是計量分析中都是相當重要的課題。一般而言，研究者通常會先選擇使用綜合性的一致性檢定 (omnibus consistent test)。儘管這些綜合性檢定在理論上可以檢測任何背離虛無假設的情況，然而，藉由頻譜分析 (spectral analysis)，文獻上已知綜合性檢定會對「高頻(high frequency)」對立假設缺乏檢定力。為了避免此問題，

已有許多學者提出各式相對應的平滑性檢定 (smooth tests)，而且其良好的理論和實證性質都也已被揭露。基本上，現行的平滑性檢定架構主要是建立在資料序列間的共變異結構 (covariance structure)所對應的特徵組 (eigen-pairs)上。

然而，文獻上僅有極少數的資料序列所對應的特徵組有明確的型式可供分析使用。為了改善此一缺失，在此計畫中，

我提出了一個不需要資料序列間的共變異結構即可建立統計量的新平滑性檢定架構。在給定任何基本函數 (basis functions)下，我們可以有相對應於資料序列的傅立葉表現式 (Fourier representation) 及傅立葉係數 (Fourier coefficients)。基本上，這個新的平滑性檢定架構就是建立在這些傅立葉係數標準化後的主成分 (normalized principal components)上。在這樣的架構下，我們進而提出兩個由資料所驅策 (data-driven) 的新平滑性檢定。

簡而言之，與目前文獻上常用的平滑性檢定相比，這個計畫提出了一個具一般化且容易執行的新平滑性檢定架構來改善綜合性檢定所可能面臨之缺乏檢定力的問題。我們的模擬結果也支持這樣的方法。

中文關鍵詞： 綜合性檢定、 平滑性檢定、 傅立葉表現式、 傅立葉係數、 標準化後的主成分、 特徵組、 資料驅策

英文摘要： As is well known,hypothesis testing is an important task in economic, financial or econometric analysis.Generally speaking,considering an omnibus consistent test is usually the first attempt for many researchers when there is no particular preferred alternative to the null hypothesis.Despite the capability of an omnibus test to detect deviations in all directions, it is also well known that the omnibus tests may lack power against the so-called ‘‘high frequency ‘ ‘ alternatives after invoking spectral analysis. To avoid this‘‘power deficiency ‘ ‘ problem in the omnibus tests, smooth

tests are then proposed in various issues hypothesis testing problems, and good theoretical properties and empirical evidence of smooth tests have been documented by many researchers. In the existing smooth tests framework, the eigen-pairs of the (asymptotic) covariance structure of the process plays the central role. However, only for some few processes, we may have explicit forms for the corresponding eigen-pairs. In order to remedy this deficiency, this project aims to propose more general and more implementable smooth-type test. In this project,we propose a new smooth-type test approach without any knowledge of the covariance structure of the process in forming the associated test statistic. Given any basis functions, we have Fourier representations for the process and the corresponding Fourier coefficients.In essence, this new smooth-type test approach is established by using the normalized principal components for these Fourier coefficients. Moreover, two associated data-driven smooth-type tests are also proposed. In sum, other than the conventional smooth tests,
this project provide a new, general and easy-to-implement approach to increasing the
testing powers of the omnibus tests. The simulation results support this approach.

英文關鍵詞： basis functions, data-driven tests, eigen-pairs, Fourier coefficient, Fourier representation, normalized principal components omnibus tests, power deficiency, smooth tests

1 Introduction

As is well known, besides estimating the model consistently, hypothesis testing is another important task in economic, financial or econometric analysis; for example, there are the goodness-of-fit tests of Anderson and Darling (1952), the tests for martingale difference by Durlauf (1991), Dominguze and Lobato (2003) and Escanciano and Velasco (2006), and the general model specification tests of Newey (1985), Tauchen (1985), Bierens (1982), and Bierens and Ploberger (1997); to mention only a few. Some of these tests are designed for detecting some specified alternatives to the null hypothesis, they are known as "directional tests". The directional tests are not consistent because they are developed to focus their testing powers in the direction of some alternatives of interest. Another class of tests is the so-called "omnibus tests". Such tests are designed to against all alternatives and thus are consistent. In the literature, the typical omnibus tests are either Kolmogorov-Smirnov (KS) or Cramér-von Mises (CvM) types of tests.

Generally speaking, considering an omnibus consistent test is usually the first attempt for many researchers in the hypothesis testing when there is no particular preferred alternative to the null hypothesis. Despite the capability of an omnibus test to detect deviations in all directions, there is an important limitation of omnibus tests however. After invoking spectral analysis to the covariance structure of the empirical process which forms the test statistics in omnibus tests, the empirical process can then be represented as the infinitely weighted sum of resulting eigenfunctions, where the weights are the corresponding eigenvalues. It immediately implies that omnibus tests may only have substantial local power against few orthogonal directions, i.e., the eigenfunctions with larger eigenvalues, since the sequence of eigenvalues is decreasing to zero. As a consequence, the omnibus tests may lack power against the so-called "high frequency" alternatives, the directions (eigenfunctions) with smaller eigenvalues. More details and discussions may refer to, e.g., Bierens and Ploberger (1997), and Janssen (2000), Ecasnciano (2009), among many others.

To avoid this "power deficiency" problem in the omnibus tests, Neyman (1937) first proposes the "smooth tests" for the problem of goodness-of-fit. After that, good theoretical properties and empirical evidence of smooth tests in various hypothesis testing problems have been documented by many researchers; e.g., Eubank and LaRiccia (1992) Delgado and Stute (2008) for goodness-of-fit tests, Delgado et al.(2005), and Escanciano and Mayoral (2010) for testing martingale difference hypothesis, Stute (1997) and Escanciano (2009) for model specification tests. In essence, the smooth tests are constructed by dropping the decreasing weights in resulted spectral representation, the so-called Karhunen-Loève (KL) expansion, of associated processes in forming the test statistics. Therefore, they have better power against "high frequency" alternatives than the corresponding omnibus tests. Smooth test thus can be viewed as a compromise between omnibus and directional tests.

In the existing smooth tests framework, it is obvious that the eigen-pairs of the (asymptotic) covariance structure of the process plays the central role. Only when the process is standard (weighted) Wiener process, standard (weighted) Brownian bridge, or Brownian sheet, we may have explicit forms for these eigen-pairs. That is, for a general stochastic process, Gaussian process without knowledge of covariance kernel in conventional model specification tests for example, the methods for estimating these eigen-pairs are further needed; see e.g., William and Seeger (2000, 2001), Carrasco et al (2007), and Escanciano (2009). These estimation methods, however, are sometimes not easy to implement, since there are infinitely many, or at least as many as sample size, eigen-pairs to be estimated. In order to remedy this deficiency, this project aims to propose more general and more implementable smooth-type test.

In this project, we propose a new smooth-type test approach without directly estimating the eigen-pairs for the covariance structure of the process in forming the associated test statistic. Given any basis functions, we have Fourier representations for the process and the corresponding Fourier coefficients. It is then easy to compute the eigen-pairs and the principal components for any finitely many Fourier coefficients. The new smooth-type test is established by using the resulting normalized principal components. Moreover, from a practical viewpoint, two associated data-driven smooth-type tests are also proposed. Other than the existing smooth tests, there are some good features of this proposed approach. First, the conventional smooth test is just a special case of the proposed. Second, no knowledge of covariance structure of the process is needed. Third, the implementation of the proposed test is easy since it is not hard to find the principal components for any given Fourier coefficients. Finally, because it is valid for any basis functions, the proposed approach can provide information about the underlying data-generating process in different and more general ways.

This report proceeds as follows. We propose a new smooth test in Section 2. The asymptotic properties of the proposed test and a new data-driven test are discussed in Section 3. Section 4 reports simulation results. Section 5 concludes this report.

2 The Proposed approach

Given any basis functions $b_j(t)$, $j = 1, 2, \ldots,$ for $L^2[a,b]$, Fourier analysis represents $H(t,\omega)$ as

$$H(t,\omega) = \lim_{J\to\infty} \sum_{j=1}^{J} \phi_j(\omega) b_j(t), \tag{1}$$

where $\phi_j(\omega)$ is the so-called Fourier coefficient associated with b_j. Moreover, Parseval's Theorem gives that

$$\int_a^b H^2(t,\omega)dt = \lim_{J\to\infty} \sum_{j=1}^{J} \phi_j^2(\omega). \tag{2}$$

Given J, let $\mathbf{\Phi}_J(\omega) = [\phi_1(\omega)\phi_2(\omega)\cdots\phi_J(\omega)]'$, and $\lambda_1, \lambda_2, \ldots, \lambda_J$ and $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_J$ respectively be the associated eigenvalues and eigenvectors for the variance-covariance matrix of $\mathbf{\Phi}_J(\omega)$, $\mathrm{Var}(\mathbf{\Phi}_J)$ say, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_J > 0$. The we have

$$\mathrm{Var}(\mathbf{\Phi}_J) = \boldsymbol{U}\mathbf{\Lambda}\boldsymbol{U}', \quad \boldsymbol{U}'\boldsymbol{U} = I_J, \tag{3}$$

where $\boldsymbol{U} = [\boldsymbol{u}_1\boldsymbol{u}_2\cdots\boldsymbol{u}_J]$ a $J \times J$ matrix, $\mathbf{\Lambda}$ is a $J \times J$ diagonal matrix with entries λ_i, $i = 1, \ldots, J$, along the principal diagonal, and I_J is a $J \times J$ identity matrix. Accordingly, we may further represent $\mathbf{\Phi}_J$ as

$$\mathbf{\Phi}_J(\omega) = \left[\boldsymbol{U}\mathbf{\Lambda}^{1/2}\mathbf{\Phi}_J^*(\omega)\right],$$

where

$$\mathbf{\Phi}_J^*(\omega) = \mathbf{\Lambda}^{-1/2}\boldsymbol{U}'\mathbf{\Phi}_J(\omega) := [\phi_1^*(\omega)\phi_2^*(\omega)\cdots\phi_J^*(\omega)]' \tag{4}$$

is a vector of J normalized principal components for the Fourier coefficients associated with basis functions $\{b_j(t)\}$. Note that these principal components $\{\phi_j^*(\omega)\}$ are independent random variables with zero mean and unity variance. Besides, since λ_j are ranked in decreasing order, $\phi_1^*(\omega)$ is the most informative component which accounts for the most variations of resulted Fourier coefficients, $\phi_2^*(\omega)$ is the second one, and so on.

Given these J normalized principal components for the resulted Fourier coefficients

$\{\phi_j(\omega)\}$, we further have

$$\sum_{j=1}^{J}\phi_j^2(\omega) = \mathbf{\Phi}'_J\mathbf{\Phi}_J = \left[\boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\mathbf{\Phi}^*_J(\omega)\right]'\left[\boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\mathbf{\Phi}^*_J(\omega)\right]$$
$$= \mathbf{\Phi}^*_J(\omega)'\boldsymbol{\Lambda}^{1/2}\boldsymbol{U}'\boldsymbol{U}\boldsymbol{\Lambda}^{1/2}\mathbf{\Phi}^*_J(\omega) = \mathbf{\Phi}^*_J(\omega)'\boldsymbol{\Lambda}\mathbf{\Phi}^*_J(\omega)$$
$$= \sum_{j=1}^{J}\lambda_j\phi_j^{*2}(\omega), \tag{5}$$

a weighted sum of independent random variables $\phi_j^{*2}(\omega)$, $j = 1, 2, \cdots, J$. As a consequence, we may represent (1) and (2), respectively, as,

$$H(t,\omega) = \lim_{J\to\infty}\sum_{j=1}^{J}\phi_j(\omega)b_j(t) = \lim_{J\to\infty}\sum_{j=1}^{J}\sqrt{\lambda_j}\phi_j^*(\omega)b_j(t), \tag{6}$$
$$\int_a^b H^2(t,\omega)dt = \lim_{J\to\infty}\sum_{j=1}^{J}\phi_j^2(\omega) = \lim_{J\to\infty}\sum_{j=1}^{J}\lambda_j\phi_j^{*2}(\omega). \tag{7}$$

Note that if $H(t,\omega)$ is Gaussian, then for $j = 1, 2, \cdots$, $\phi_j^*(\omega)$ are independent standard Gaussian random variables and $\phi_j^{*2}(\omega)$ are independent chi-squared random variables with degree of freedom one.

2.1 New smooth-type test approach

In the hypothesis testing framework, if $H_n(t,\omega)$ is the resulting empirical process such that $H_n(t,\omega)$ converges in distribution to $H(t,\omega)$ under the null, that is,

$$H_n(\cdot) \Longrightarrow H(\cdot) \quad \text{under } H_o,$$

then the L_2-norm of $H(t,\omega)$ in (7) is the asymptotic distribution of the so-called Cramér-von Mises (CvM) test statistic, $\int_a^b H_n(t,\omega)^2dt$ say. As is well known, the conventional CvM tests suffer the problem of power deficiency since the diminishing weights λ_j in (7) indicates that the "high-frequency" alternatives, the directions of $\phi_j^{*2}(\omega)$ with larger j, can not be detected by the test. Therefore, we may drop the decreasing weights λ_j, $j = 1, \ldots$, to improve testing power. As a consequence, based on proper sample counterpart for ϕ_j^*, $\phi_{n,j}^*$ say, we propose a new smooth-type test as

$$T_{n,J} = \sum_{j=1}^{J}\phi_{n,j}^*(\omega). \tag{8}$$

2.2 Data-driven smooth-type tests

Based on statistic $T_{n,J}$ constructed in (8), we may propose two possible data-driven tests. Given the upper bound $\bar{J}$ depending on the sample size, the first one introduces some penalty function, $P(n, J, q)$ say, as in typical model selection framework, that is

$$T^1_{n,\tilde{J}} = \sum_{j=1}^{\tilde{J}} \phi^{*2}_{n,j}, \tag{9}$$

where $\tilde{J} = \arg\min_{1\le J\le \bar{J}} \sum_{j=1}^{J} \phi^{*2}_{n,j} - P(n, J, q)$, and

$$P(n, J, q) = \begin{cases} 2J \log n, & \text{if } \max_{1\le j\le \bar{J}} |\phi^*_{n,j}| \le \sqrt{q \log n}, \\ 2J, & \text{otherwise,} \end{cases}$$

q is some fixed number; see e.g., Inglot and Ledwina (2006), Ecsanciano and Lobato (2009) and Escanciano and Mayoral (2010).

Beaised, following Bierens (1990), we adopt another data-driven method that takes a particular component $(\phi^*_{n,j})^2$ as the test statistic. Specifically, let $\hat{j}$ be a number between one and $\bar{J}$ such that

$$(\phi^*_{n,\hat{j}})^2 = \max\left((\phi^*_{n,j})^2,\ 1 \le j \le \bar{J}\right).$$

The second proposed data-driven test is

$$T^2_{n,\tilde{j}} = (\phi^*_{n,\tilde{j}})^2, \tag{10}$$

where for some pre-specified numbers $\gamma > 0$ and $0 < \rho < 1$,

$$\tilde{j} = \begin{cases} j_o, & \text{if } (\phi^*_{n,\hat{j}})^2 - (\phi^*_{n,j_o})^2 \le \gamma n^\rho; \\ \hat{j}, & \text{otherwise,} \end{cases} \tag{11}$$

with j_o randomly chosen from a given set (e.g., $\{1, 2, \ldots, \bar{J}\}$).

3 Asymptotic Properties

The result below follows easily from the maintained assumption that H_n converges weakly to a Gaussian process H with mean zero. Let $\mathbf{\Phi}_{n,J} = [\phi_{n,1}\phi_{n,2}\cdots\phi_{n,J}]'$ and it corresponding normalized principal components $\mathbf{\Phi}^*_{n,J} = [\phi^*_{n,1}\phi^*_{n,2}\cdots\phi^*_{n,J}]'$, then we have

Lemma 3.1 *For a given J, $\mathbf{\Phi}_{n,J} \xrightarrow{d} \mathbf{\Phi}_J$, as $n \to \infty$, where $\mathbf{\Phi}_J$ is a vector of J normal random variables with mean zero.*

It is then not difficult to see that the normalized principal components of $\mathrm{var}(\boldsymbol{\Phi}_{n,J})$ also converges in distribution to those of $\mathrm{var}(\boldsymbol{\Phi}_J)$. Therefore, it follows that

Lemma 3.2 *For a given J,*

$$\boldsymbol{\Phi}^*_{n,J} \xrightarrow{d} \boldsymbol{\Phi}^*_J \sim \mathcal{N}(0,\ \boldsymbol{I}_J),\ \text{as } n \to \infty.$$

The distribution of $J_{n,J}$ is an immediate consequence of Lemma 3.2.

Theorem 3.3 *For a given J,*

$$T_{n,J} = \left(\boldsymbol{\Phi}^*_{n,J}\right)'\left(\boldsymbol{\Phi}^*_{n,J}\right) \xrightarrow{d} (\boldsymbol{\Phi}^*_J)'(\boldsymbol{\Phi}^*_J) \sim \chi^2(J),\ \text{as } n \to \infty.$$

The result below establishes the null distribution of the data-driven test (11):

Theorem 3.4 *With $\tilde{j}$ determined by (11),*

$$T^2_{n,\tilde{j}} \xrightarrow{d} \chi^2(1),$$

where $T^2_{n,\tilde{j}}$ is given by (10).

In contrast with the data-driven method considered by Escanciano and Mayoral (2010), the finite-sample size of this test ought to be more accurate because the statistic includes only one principal component and its null distribution is $\chi^2(1)$. Without including more components in the test, this test may not be powerful enough to detect the deviation from the null in other directions.

4 Simulations

In the simulations we consider testing the martingale difference hypothesis and testing the linearity of model specification. As the conventional smooth test is available only for the former, the proposed test is compared with that of Escanciano and Mayoral (2010), $T^{EM}_{n,J}$ say, in this case but not otherwise. As benchmarks, we also compute the CvM and KS tests based on a wild bootstrap procedure in all experiments. In our simulations, we consider three sample sizes $n = 100, 200, 300$. For the proposed smooth tests, we simulate $T_{n,J}$ for $J = 1, 2, 3, 4, 5$ and the proposed data-driven tests, $T^1_{n,\tilde{J}}$ and $T^2_{n,\tilde{j}}$ with j_o randomly drawn from $\{1, 2, 3, 4, 5\}$, $\rho = 0.5$, and $(\gamma, \bar{J}) = (0.8, 5), (0.7, 8), (0.6, 11)$ for $n = 100, 200$ and 300, respectively. All nominal sizes are 5%. The number of Monte Carlo replications is 3000; the number of bootstraps is 500.

4.1 Testing the Martingale Difference Hypothesis

For testing the martingale difference hypothesis, we follow the simulations in Escanciano and Mayoral (2010). Letting u_t be i.i.d. $\mathcal{N}(0,1)$, we consider three different data generating processes (DGPs) for size simulations.

(1) IID: $y_t = u_t$.

(2) GARCH: $y_t = \sigma_t u_t$, with $\sigma_t^2 = 0.001 + 0.01 y_{t-1}^2 + 0.90\sigma_{t-1}^2$.

(3) SV (Stochastic Volatility): $y_t = \exp(\sigma_t) u_t$, where $\sigma_t = 0.936\sigma_{t-1} + 0.32 v_t$, and v_t are also i.i.d. $\mathcal{N}(0,1)$ and $\{u_t\}$ and $\{v_t\}$ are mutually independent.

The conventional CvM and KS tests are:

$$\begin{aligned} \mathrm{CvM}_n &= \frac{1}{n}\sum_{i=2}^{n}\left[\frac{1}{\hat{\sigma}_n\sqrt{n}}\sum_{t=1}^{n} y_t \mathbf{1}(y_{t-1} \le y_{i-1})\right]^2, \\ \mathrm{KS}_n &= \max_{i=2,\cdots,n}\left|\frac{1}{\hat{\sigma}_n\sqrt{n}}\sum_{t=1}^{n} y_t \mathbf{1}(y_{t-1} \le y_{i-1})\right|. \end{aligned}$$

Under suitable regularity conditions,

$$H_n(\xi) \Rightarrow \mathbf{W}(\tau^2(\xi)),$$

where $\mathbf{W}$ is the standard Wiener process, $\tau^2(\xi) := \sigma^{-2}\mathbb{E}[y_t^2 \mathbf{1}(y_{t-1} \le \xi)]$, and $\sigma^2 := \mathbb{E}[y_t^2]$. It is well known that the eigen-pairs associated with the covariance kernel of $\mathbf{W}$ are:

$$\begin{aligned} \lambda_j^{\varepsilon} &= \frac{1}{(j-1/2)^2\pi^2}, \\ \phi_j^{\varepsilon}(t) &= \sqrt{2}\sin((j-1/2)\pi t), \quad t \in [0,1],\ j = 1, 2, \ldots \end{aligned}$$

The empirical sizes are summarized in Table 1. As expected, the empirical sizes of the bootstrapped CvM and KS tests are very close to the nominal size 5% in all cases. The smooth tests of Escanciano and Mayoral (2010), $T_{n,J}^{EM}$, and the proposed smooth tests, $T_{n,J}$, perform reasonably well in most cases but are under-sized when the DGPs is SV. It can also be seen that the data-driven test, $T_{n,\tilde{J}}^{1}$, of Escanciano and Mayoral (2010) is severely over-sized, yet the proposed data-driven test, $T_{n,\tilde{J}}^{2}$, has very accurate sizes.

For power simulations we consider the following DGPs: Let u_t be i.i.d. $\mathcal{N}(0,1)$.

(4) NLMA (Nonlinear Moving Average): $y_t = u_{t-1}u_{t-2}(u_{t-2} + u_t + 1)$.

(5) BIL (Bilinear): $y_t = u_t + 0.15 u_{t-1} y_{t-1} + 0.05 u_{t-1} y_{t-2}$.

(6) TAR-1 (Threshold AR):

$$y_t = \begin{cases} -0.5y_{t-1} + u_t, & \text{if } y_{t-1} \geq 1, \\ 0.4y_{t-1} + u_t, & \text{otherwise.} \end{cases}$$

(7) Exp-AR (Exponential AR): $y_t = 0.6y_{t-1} \exp(-0.5y_{t-1}^2) + u_t$.

The empirical powers are summarized in Table 2. Clearly, KS_n and CvM_n have quite different power performance. While KS_n has no power in all cases but BIL, CvM_n has no power against NLMA and has high power against TAR-1. Compared with CvM_n, the conventional smooth tests $T_{n,J}^{EM}$ have better empirical powers under NLMA and Exp-AR (except for $T_{n,1}$) and have comparable powers under BIL and TAR-1. It is interesting to observe that the proposed smooth tests, $T_{n,J}$, dominate the conventional smooth tests in most cases. These results are encouraging, as they suggest that deviations from the null may be detected without the knowledge of the covariance kernel of the limiting process. As for the proposed data-driven smooth test $T_{n,\tilde{j}}^2$, its powers are, in general, lower than those of $T_{n,J}$, and the power loss may be quite significant. This is so because $T_{n,\tilde{j}}^2$ focuses only on a particular direction but $T_{n,J}$ have power against several different directions. Note that we did not simulate $T_{n,\tilde{J}}^1$ here, due to its severe size distortion.

4.2 Testing Linear Model Specification

We now consider testing the hypothesis of a correct linear model specification:

$$H_0 : \mathbb{P}\left(\mathbb{E}\left[y_t|x_t\right] = x_t\theta_o\right) = 1 \text{ for some } \theta_o \in \mathbf{\Theta} \subset \mathbb{R}.$$

Following Lee et al. (1993), we generate four DGPs for power simulations with u_t i.i.d. $\mathcal{N}(0, 1)$ and $y_0 = 0$.

(1) NLAR (Nonlinear AutoRegressive): $y_t = 0.7|y_{t-1}|/[|y_{t-1}| + 2] + u_t$.

(2) STAR (Smooth Transition AutoRegressive): $y_t = 0.6\Phi(y_{t-1})y_{t-1} + u_t$, where $\Phi(\cdot)$ denotes the standard normal distribution function.

(3) Threshold AutoRegressive (TAR-2):

$$y_t = \begin{cases} 0.9y_{t-1} + u_t, & \text{if } |y_{t-1}| < 1, \\ -0.3y_{t-1} + u_t, & \text{otherwise.} \end{cases}$$

(4) Sign autoregressive (SGN): $y_t = \text{sgn}(y_{t-1}) + u_t$, where

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The CvM and KS test statistics in this simulations are computed as:

$$\text{CvM}_n = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(y_t - x_t\hat{\theta}_n\right)\mathbf{1}(x_t \le x_i)\right|^2,$$

$$\text{KS}_n = \max_{i=1,\ldots,n}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(y_t - x_t\hat{\theta}_n\right)\mathbf{1}(x_t \le x_i)\right|,$$

where $\hat{\theta}_n$ is the OLS estimator. As the KL expansion is not available in this case, we do not consider the conventional smooth tests in these simulations.

The empirical powers are summarized in Table 3. Compared with CvM_n and KS_n, $T_{n,J}$ with $J = 2, 3, 4, 5$ perform significantly better under TAR-2 and SGN but are less powerful under NLAR; none of these tests have power advantage under STAR. Similar to the previous power simulations, the data-driven test $T^2_{n,\tilde{j}}$ has lower powers than the proposed smooth tests in general. Yet, $T^2_{n,\tilde{j}}$ still outperforms CvM_n and KS_n under TAR-2 and SGN. Note that a smooth test may have very low power for some J (for example, $J_{n,1}$ under TAR-2 and SGN), the data-driven test always has some reasonable power.

5 Concluding Remarks

In this project we propose a more operational approach to constructing smooth tests without knowing the covariance kernel of the limiting process. Our simulations confirm that the proposed smooth test has good finite-sample performance and can serve as a useful complement to the conventional omnibus tests, such as the CvM and KS tests. There are some important research directions. First, we find that the data-driven test of Escanciano and Mayoral (2010) suffers from size distortion but the proposed data-driven test is not powerful. A data-driven test with improved size and power performance would be highly desired. Second, smooth tests gain power advantage in certain directions by sacrificing test consistency. It is therefore important to construct a consistent and omnibus test that carries the spirit of smooth tests. These topics are currently being investigated.

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Table 1: Size simulations: Testing the martingale difference hypothesis.

	IID			GARCH			SV		
Test	n=100	200	300	100	200	300	100	200	300
CvM_n	5.0	5.3	4.5	5.0	4.9	4.0	4.0	4.1	5.4
KS_n	5.3	5.1	4.5	5.1	5.3	4.1	3.8	3.9	4.8
$T_{n,1}^{EM}$	4.6	5.2	4.2	4.5	4.6	4.0	2.2	2.7	3.6
$T_{n,2}^{EM}$	4.3	5.2	4.1	4.0	5.2	4.0	2.2	2.4	2.4
$T_{n,3}^{EM}$	3.5	4.7	4.3	4.0	5.4	3.7	2.0	1.7	2.5
$T_{n,4}^{EM}$	3.0	4.3	4.6	3.8	5.2	3.5	1.4	1.7	2.1
$T_{n,5}^{EM}$	3.0	4.0	4.2	3.2	4.2	3.7	1.4	1.8	1.9
$T_{n,\tilde{J}}^{1}$	7.7	11.7	12.0	7.7	12.9	11.5	4.5	6.4	6.6
$T_{n,1}$	5.4	5.2	4.5	5.2	5.1	3.9	3.5	4.1	4.9
$T_{n,2}$	4.5	5.6	4.5	4.4	4.8	4.3	3.4	3.9	3.6
$T_{n,3}$	4.3	5.0	4.1	3.7	4.8	3.6	2.1	3.3	3.5
$T_{n,4}$	3.6	4.9	4.4	3.1	4.2	2.8	2.0	2.9	3.5
$T_{n,5}$	3.1	4.6	4.3	2.4	3.9	2.5	1.8	2.9	3.1
$T_{n,\tilde{j}}^{2}$	4.9	5.9	4.1	5.3	5.2	5.1	4.2	5.0	5.5

Notes:
1. The entries are rejection frequencies in percentage; the nominal size is 5%.
2. For $T_{n,\tilde{j}}^{2}$, j_o is randomly drawn from $\{1, 2, 3, 4, 5\}$, and $\rho = 0.5$.
When $n = 100, 200, 300$, we set, $(\gamma, \bar{J}) = (0.8, 5), (0.7, 8), (0.6, 11)$, respectively.
3. For $T_{n,\tilde{J}}^{1}$, we set $\underline{m} = 3$ and $\bar{J} = 5, 8, 11$ when $n = 100, 200$ and 300, respectively.

Table 2: Power simulations: Testing the martingale difference hypothesis.

	NLMA			BIL			TAR-1			Exp-AR		
Test	n=100	200	300	100	200	300	100	200	300	100	200	300
CvM_n	3.9	6.4	10.5	14.9	30.6	47.6	73.6	94.3	99.1	22.6	39.3	53.6
KS_n	3.4	4.6	8.1	32.6	55.1	74.0	0.1	0.0	0.0	7.1	7.6	6.7
$T_{n,1}^{EM}$	1.4	1.7	2.1	14.1	29.8	46.3	69.7	91.7	98.5	20.5	31.4	39.5
$T_{n,2}^{EM}$	6.0	15.1	24.5	12.4	24.4	39.1	75.3	96.7	99.8	42.4	78.7	94.1
$T_{n,3}^{EM}$	7.0	21.0	34.6	18.6	43.6	65.8	69.9	96.6	99.8	47.1	83.8	96.6
$T_{n,4}^{EM}$	7.7	24.8	41.3	15.7	39.0	60.5	65.8	95.9	99.7	40.4	79.0	95.0
$T_{n,5}^{EM}$	7.7	26.3	44.6	14.7	41.6	64.0	63.6	96.8	99.9	36.3	77.0	94.3
$T_{n,1}$	2.2	2.6	3.8	19.3	39.3	59.0	74.0	94.2	99.1	16.2	25.1	32.5
$T_{n,2}$	11.6	24.2	34.9	17.7	34.5	53.4	65.1	91.6	98.8	60.0	91.9	98.7
$T_{n,3}$	11.6	28.7	45.3	27.8	60.0	81.5	78.6	98.4	100.0	52.7	86.7	97.6
$T_{n,4}$	12.1	32.1	49.4	25.9	59.1	82.3	86.9	99.7	100.0	46.8	83.1	96.5
$T_{n,5}$	11.7	35.7	56.5	20.5	54.2	78.7	81.8	99.5	100.0	40.8	79.4	95.6
$T_{n,\tilde{j}}^2$	13.3	20.6	31.5	17.0	29.0	44.8	56.4	84.0	95.7	33.7	56.7	76.4

Notes:
1. The entries are rejection frequencies in percentage; the nominal size is 5%.
2. The parameters for the proposed tests are the same as those in Table 1.

Table 3: Power simulations: Testing model linearity.

Test	NLAR			STAR			TAR-2			SGN		
	n=100	200	300	100	200	300	100	200	300	100	200	300
CvM_n	34.5	63.2	82.0	28.7	60.0	82.2	10.0	15.6	24.2	18.2	24.0	35.1
KS_n	34.8	63.3	83.3	28.2	59.4	82.3	9.8	18.1	32.1	23.9	51.7	81.2
$T_{n,1}$	37.0	65.3	85.0	34.7	66.3	85.8	2.8	3.0	3.3	4.3	4.8	7.3
$T_{n,2}$	20.5	50.8	72.1	17.4	47.8	72.9	17.2	64.3	92.7	71.6	99.3	100.0
$T_{n,3}$	21.2	53.7	78.0	26.9	73.1	92.9	27.5	69.7	93.1	71.6	99.0	100.0
$T_{n,4}$	17.1	50.3	74.2	23.8	69.1	91.6	86.4	99.9	100.0	94.8	100.0	100.0
$T_{n,5}$	14.7	44.7	70.4	18.6	61.5	88.9	82.2	99.7	100.0	92.8	99.9	100.0
$T^2_{n,\tilde{j}}$	18.3	32.0	48.8	16.9	32.7	53.2	62.6	89.9	98.7	74.1	97.2	99.8

NOTE:
1. The entries are rejection frequencies in percentage; the nominal size is 5%.
2. The parameters for the proposed tests are the same as those in Table 1.

國科會補助計畫衍生研發成果推廣資料表

日期:2012/08/26

國科會補助計畫	計畫名稱: 不需估計共變異結構特徵組的新平滑性檢定
	計畫主持人: 徐士勛
	計畫編號: 100-2410-H-004-082- 學門領域: 數理與數量方法

無研發成果推廣資料

100 年度專題研究計畫研究成果彙整表

計畫主持人：徐士勛	計畫編號：100-2410-H-004-082-
計畫名稱：不需估計共變異結構特徵組的新平滑性檢定	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	1	1	50%		部分與此計畫相關的研究成果，已結合於文章’’Constructing Smooth Tests without Estimating the Eigen-pairs of the Limiting Process’’（與管中閔合著）中，目前投稿中。
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	3	3	100%		原核定清單核准 1 名博士班研究生

							與2名碩士班研究生獎助學金，已於計畫進行之初即核准變更為3名博士班研究生
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：(以 100 字為限)

部分與此計畫相關的研究成果，已結合於文章’’Constructing Smooth Tests without Estimating the Eigen-pairs of the Limiting Process’’(與管中閔合著)中，目前投稿中。

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）(以 500 字為限)

與目前文獻上常用的平滑性檢定相比,這個計畫提出了一個具一般化且容易執行的新平滑性檢定架構來改善綜合性檢定所可能面臨之缺乏檢定力的問題。除了計畫所考慮的兩種檢定外,所考慮的檢定架構可以進一步用來重新檢視已經存在的各種應用 Cramer-von Mises 型式的檢定上，相信對於這些檢定的檢定力會有所幫助。