

# 行政院國家科學委員會專題研究計畫 成果報告

## 拋物型方程在 time scale 上的極大值原理 研究成果報告(精簡版)

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中文摘要： 在本計劃中，我探討反應擴散方程系統  
 $u_t = \Delta u_{xx} - \frac{2uv}{\beta + u}$ ,  
 $v_t = v_{xx} + \frac{uv}{\beta + u}$ .

此數學模型建立於 1989 年，藉以描述 acidic nitrate 和 ferroin 的化學反應，其中  $\beta$  是一個正常數， $u$  和  $v$  分別表示 ferroin 和 acidic nitrate 的濃度， $\Delta$  表示 ferroin 和 acidic nitrate 的擴散係數比值。我們證明了此系統之行進波解的存在性，並得到存在性的最佳結果。

中文關鍵詞： 反應擴散方程系統, 行進波, 存在性

英文摘要：

英文關鍵詞：

THE EXISTENCE OF TRAVELING WAVE FRONTS  
FOR A REACTION-DIFFUSION SYSTEM MODELLING  
THE ACIDIC NITRATE-FERROIN REACTION

By

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**Abstract.** We investigate the existence of traveling wave solutions to the one-dimensional reaction-diffusion system  $u_t = \delta u_{xx} - 2uv/(\beta + u)$ ,  $v_t = v_{xx} + uv/(\beta + u)$ , which describes the acidic nitrate-ferroin reaction. Here  $\beta$  is a positive constant,  $u$  and  $v$  represent the concentrations of the ferroin and acidic nitrate respectively, and  $\delta$  denotes the ratio of the diffusion rates. We show that this system has a unique, up to translation, traveling wave solution with speed  $c$  iff  $c \geq 2/\sqrt{\beta + 1}$ .

**1. Introduction.** Traveling waves of liquid-phase chemical reactions have received a lot of attention due to their thermodynamical and biological importance; see, for example, [1, 3, 8]. One of the examples comes from the acidic nitrate-ferroin reaction [9, 10, 6]. In the investigation of traveling waves, one of the important issues is how wave velocity depends on the initial concentrations of the reactants in the acidic nitrate-ferroin reaction. For this, Pota et al. [9, 10, 6] have established a two-variable reaction-diffusion model, from which a simple formula for wave velocity is derived. In addition, the theoretical values obtained from this formula are in good agreement with the experimental ones.

Our aim in this paper is to provide a rigorous proof for the existence of traveling wave fronts to the reaction-diffusion system proposed in [9, 10, 6]. Let  $u$  and  $v$  represent the concentrations of the ferroin and acidic nitrate respectively, and  $\delta$  denote the ratio of the diffusion rates of the ferroin and acidic nitrate. Then a dimensionless reaction-diffusion system describing the acidic nitrate-ferroin reaction reads [9, 7]

$$\begin{aligned} u_t &= \delta u_{xx} - \frac{2uv}{\beta + u}, \\ v_t &= v_{xx} + \frac{uv}{\beta + u}, \end{aligned} \tag{1.1}$$

where  $\beta$  is a positive constant.

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By a traveling wave solution of system (1.1), we mean a solution of system (1.1) of the form

$$(u(x, t), v(x, t)) = (U(z), V(z)),$$

with  $z = x - ct$ , which connects the stable steady state  $(0, 1/2)$  at  $z = -\infty$  to the unstable steady state  $(1, 0)$  at  $z = \infty$  (see [7, 4]). Here the wave speed  $c$  is a constant to be determined and the wave profile  $(U, V)$  is a pair of nonnegative functions satisfying the system

$$\delta U'' + cU' - \frac{2UV}{\beta + U} = 0, \quad (1.2a)$$

$$V'' + cV' + \frac{UV}{\beta + U} = 0, \quad (1.2b)$$

on  $\mathbb{R}$ , together with the boundary conditions

$$(U, V)(-\infty) = (0, 1/2), \quad (U, V)(+\infty) = (1, 0), \quad (1.3)$$

where the prime indicates differentiation with respect to  $z$ . Consequently, to show the existence and uniqueness of traveling wave solution to system (1.1) is equivalent to show the existence and uniqueness of nonnegative solution to system (1.2)-(1.3).

The existence and uniqueness of traveling wave solution to system (1.1) has been studied by the author in [4]. For the case  $0 < \delta \leq 1$ , it was shown that there exists  $c_{min} > 0$  such that system (1.1) has a unique, up to translation, traveling wave solution with speed  $c$  iff  $c \geq c_{min}$ . Moreover, if  $\delta = 1$ , then  $c_{min} = 2/\sqrt{\beta + 1}$ ; if  $0 < \delta < 1$ , then an estimate for  $c_{min}$  is given by

$$2/\sqrt{\beta + 1} \leq c_{min} \leq (2/\sqrt{\beta})[(\sqrt{2} + 1)/\sqrt{\delta + 2(\sqrt{2} + 1)}].$$

For the case  $\delta > 1$ , it was shown that system (1.1) has a unique, up to translation, traveling wave solution with speed  $c$  if  $c \geq 2/\sqrt{\beta}$ . On the other hand, system (1.1) has no traveling wave solution with speed  $c$  if  $c < 2/\sqrt{\beta + 1}$ .

Since the above result is incomplete, the author in the present paper continue to study this problem. Employing the idea proposed by [2], we acquire an optimal result, which is stated in the following theorem.

**THEOREM 1.1.** For  $\delta > 0$ , the system (1.1) has a unique, up to translation, traveling wave solution with speed  $c$  iff  $c \geq c^* := 2/\sqrt{\beta + 1}$ .

In view of [4], we know that the nonnegative solution of system (1.2)-(1.3), if it exists, is unique up to a translation. Besides, system (1.2)-(1.3) has no nonnegative solution if  $c < c^*$ , and the admissible speed set,

$$S := \{c \in \mathbb{R} \mid \text{system (1.2)-(1.3) admits a nonnegative solution} \},$$

is closed. Therefore, to prove Theorem 1.1, it suffices to show that  $(c^*, \infty) \subset S$ . Hereafter, let  $c > c^*$  be given. We will show that system (1.2)-(1.3) has a nonnegative solution.

We make an outline of our approach. By imposing appropriate boundary conditions, we first consider system (1.2) in a finite interval  $[-l, l]$ , and show that there exists a solution, denoted by  $(U_l, V_l)$ , to the boundary value problem via the Schauder fixed point theorem. With the help of super- and sub-solutions, we get an upper bound and

lower bound for this solution. Besides, applying an a priori estimate for derivative of solutions to the second-order linear equations, we show that, for a given  $l_0 > 0$ , the derivatives of  $U_l$  and  $V_l$  are uniformly bounded in  $[-l_0, l_0]$  for all  $l > l_0$ . Then we can use Ascoli-Arzelà theorem and a diagonal process to get the solution of system (1.2)-(1.3) by passing to the limit  $l \rightarrow \infty$ . We remark that system (1.2) has no maximum principle. Here we only need to apply the maximum principle for a single equation to compare the super/sub-solutions with the solutions.

The remaining parts of this paper are organized as follows. In Sec. 2, we construct the super- and sub-solutions. In Sec. 3, we derive some a priori estimates for solutions to the second-order linear equation which will be used in section 4. In Sec. 4, we consider system (1.2) in a finite interval  $[-l, l]$ . Then Sec. 5 is devoted to the proof of our main result by passing to the limit  $l \rightarrow \infty$ .

**2. Construction of super- and sub-solutions.** In this section, we construct super- and sub-solutions which will be used in section 4. For simplicity, we set

$$p(s) := s^2 - cs + \frac{1}{\beta + 1}.$$

Since  $c > c^*$ , the equation  $p(s) = 0$  has two positive roots  $\lambda$  and  $\lambda + d$ , where

$$\lambda = \frac{1}{2} \cdot \left( c - \sqrt{c^2 - \frac{4}{\beta + 1}} \right) \quad \text{and} \quad d = \sqrt{c^2 - \frac{4}{\beta + 1}}.$$

In addition,  $p(s) < 0$  when  $s \in (\lambda, \lambda + d)$ .

LEMMA 2.1. The function  $V^+(z) := e^{-\lambda z}$  satisfies the equation

$$(V^+)'' + c(V^+) + \frac{1}{\beta + 1}V^+ = 0, \tag{2.1}$$

for all  $z \in \mathbb{R}$ .

*Proof.* Since  $p(\lambda) = 0$ , it follows that

$$(V^+)'' + c(V^+) + \frac{1}{\beta + 1}V^+ = p(\lambda)V^+ = 0, \forall z \in \mathbb{R}.$$

□

Select  $0 < \gamma < \min\{c/\delta, \lambda\}$ . Then  $c - \delta\gamma > 0$  and  $\gamma - \lambda < 0$ . Since  $e^{(\gamma - \lambda)z} \rightarrow 0$  as  $z \rightarrow \infty$ , there exists  $z_0 > 0$  such that

$$e^{(\gamma - \lambda)z} \leq \frac{\beta + 1}{2}\gamma(c - \delta\gamma), \forall z \geq z_0,$$

which yields

$$(c - \delta\gamma)\gamma e^{-\gamma z} \geq \frac{2}{\beta + 1}V^+(z), \forall z \geq z_0. \tag{2.2}$$

Set  $M = e^{\gamma z_0}$ . Then  $M > 1$  since  $\gamma, z_0 > 0$ .

LEMMA 2.2. The function  $U^-(z) := \max\{0, 1 - Me^{-\gamma z}\}$  satisfies the inequality

$$\delta(U^-)'' + c(U^-)' - \frac{2U^-V^+}{\beta + U^-} \geq 0, \quad (2.3)$$

for all  $z \neq z_0$ .

*Proof.* For  $z < z_0$ , the inequality (2.3) holds immediately since  $U^- \equiv 0$  in  $(-\infty, z_0)$ . For  $z > z_0$ ,  $U^-(z) = 1 - Me^{-\gamma z}$  and  $0 < U^- < 1$ . So we have

$$\frac{1}{\beta + 1} \geq \frac{U^-}{\beta + U^-}. \quad (2.4)$$

A simple computation, together with (2.2), (2.4), and the fact that  $M > 1$ , yields

$$\delta(U^-)'' + c(U^-)' = M\gamma(c - \delta\gamma)e^{-\gamma z} \geq \frac{2}{\beta + 1}V^+(z) \geq \frac{2U^-V^+}{\beta + U^-}.$$

Hence (2.3) holds.  $\square$

Choose  $0 < \eta < \min\{\gamma, d\}$ . Then  $0 < \eta < \gamma$  and  $p(\lambda + \eta) < 0$ . Select

$$K > \max\{M, -M/[(\beta + 1)p(\lambda + \eta)]\}.$$

Set  $z_1 = \ln K/\eta$ . Then  $z_1 > z_0 > 0$  since  $z_0 = \ln M/\gamma$ ,  $K > M > 1$ , and  $\eta < \gamma$ .

LEMMA 2.3. The function  $V^-(z) := \max\{0, V^+(z) - Ke^{-(\lambda+\eta)z}\}$  satisfies the inequality

$$(V^-)'' + c(V^-)' + \frac{U^-V^-}{\beta + U^-} \geq 0, \quad (2.5)$$

for all  $z \neq z_1$ .

*Proof.* For  $z < z_1$ , the inequality (2.5) holds immediately since  $V^- \equiv 0$  in  $(-\infty, z_1)$ . For  $z > z_1$ ,  $V^- = V^+ - Ke^{-(\lambda+\eta)z}$  and  $U^- = 1 - Me^{-\gamma z}$ . Then a simple computation gives

$$(V^-)' = (V^+) + K(\lambda + \eta)e^{-(\lambda+\eta)z},$$

and

$$(V^-)'' = (V^+)'' - K(\lambda + \eta)^2e^{-(\lambda+\eta)z}.$$

Noting that

$$\frac{U^-}{\beta + U^-} = \frac{1}{\beta + 1} - \frac{\beta Me^{-\gamma z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})},$$

it follows that

$$\begin{aligned} \frac{U^-V^-}{\beta + U^-} &= \left( \frac{1}{\beta + 1} - \frac{\beta Me^{-\gamma z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})} \right) (V^+ - Ke^{-(\lambda+\eta)z}) \\ &= \frac{1}{\beta + 1}V^+ - \frac{1}{\beta + 1}Ke^{-(\lambda+\eta)z} \\ &\quad - \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})} + \frac{K\beta Me^{-(\lambda+\gamma+\eta)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})} \\ &\geq \frac{1}{\beta + 1}V^+ - \frac{1}{\beta + 1}Ke^{-(\lambda+\eta)z} - \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta + 1)(\beta + 1 - Me^{-\gamma z})}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 & (V^-)'' + c(V^-)' + \frac{U-V^-}{\beta+U^-} \\
 \geq & (V^+)'' - K(\lambda+\eta)^2 e^{-(\lambda+\eta)z} + c(V^+) + cK(\lambda+\eta)e^{-(\lambda+\eta)z} \\
 & + \frac{1}{\beta+1}V^+ - \frac{1}{\beta+1}Ke^{-(\lambda+\eta)z} - \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta+1)(\beta+1-Me^{-\gamma z})} \\
 = & -Ke^{-(\lambda+\eta)z}p(\lambda+\eta) - \frac{\beta Me^{-(\lambda+\gamma)z}}{(\beta+1)(\beta+1-Me^{-\gamma z})} \\
 & \text{(by (2.1) and definition of } p) \\
 \geq & \frac{M}{\beta+1}e^{-(\lambda+\eta)z} - \frac{M}{\beta+1}e^{-(\lambda+\gamma)z} \\
 & \text{(since } K > -M/[(\beta+1)p(\lambda+\eta)] \text{ and } 1 - Me^{-\gamma z} > 0) \\
 = & \frac{M}{\beta+1}e^{-(\lambda+\eta)z} [1 - e^{-(\gamma-\eta)z}] \\
 \geq & 0. \quad \text{(since } \gamma - \eta > 0 \text{ and } z > z_1 > 0)
 \end{aligned}$$

The proof of this lemma is therefore completed.  $\square$

**3. Some auxiliary lemmas.** In this section, we will establish some a priori estimates for solutions of the inhomogeneous linear equation

$$w''(z) + Aw'(z) + g(z)w(z) = h(z). \quad (3.1)$$

For this, we need the following lemma.

LEMMA 3.1. Let  $A$  be a positive constant and let  $g$  be a continuous function on  $[a, b]$ . Suppose that  $\phi_1$  and  $\phi_2$  are the unique solutions of the second-order linear equation

$$\mathcal{L}[y] := y'' - Ay' + g(z)y = 0$$

on  $[a, b]$  such that

$$\phi_1(a) = 0, \quad \phi_1'(a) = 1 \quad (3.2)$$

and

$$\phi_2(b) = 0, \quad \phi_2'(b) = -1. \quad (3.3)$$

Then we have the following estimates for  $\phi_1$  and  $\phi_2$ :

$$|\phi_1(z)| + |\phi_1'(z)| \leq e^{(K_1+A+1)(b-a)} \quad (3.4)$$

and

$$|\phi_2(z)| + |\phi_2'(z)| \leq e^{(K_1+A+1)(b-a)}, \quad (3.5)$$

for all  $z \in (a, b)$ , where  $K_1 = \|g\|_{C[a,b]}$ . If, in addition,  $g \leq 0$ , then the Wronskian of  $\phi_1$  and  $\phi_2$ , denoted by  $W(\phi_1, \phi_2)$ , can be estimated by

$$|W(\phi_1, \phi_2)(z)| \geq \frac{1}{A}(e^{A(b-a)} - 1) > 0, \quad (3.6)$$

for all  $z \in (a, b)$ .

*Proof.* To prove (3.4) and (3.5), we rewrite the equation  $\mathcal{L}[y] = 0$  as a first-order system

$$Y' = B(z)Y, \quad (3.7)$$

where

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix} \text{ and } B(z) = \begin{bmatrix} 0 & 1 \\ -g(z) & A \end{bmatrix}.$$

Consider  $z \in (a, b)$ . It is easy to see that any solution  $Y$  of (3.7) satisfies the integral equation

$$Y(z) = Y(a) + \int_a^z B(\tau)Y(\tau)d\tau, \quad (3.8)$$

and therefore,

$$\|Y(z)\| \leq \|Y(a)\| + (K_1 + A + 1) \int_a^z \|Y(\tau)\|d\tau,$$

where  $\|\cdot\|$  denotes the absolute norm and we have use the fact that  $\|B(\cdot)\| = \max\{|g(\cdot)|, A+1\} \leq K_1 + A + 1$ . Hence one can easily deduce that

$$\|Y(z)\| \leq \|Y(a)\|e^{(K_1+A+1)(z-a)}. \quad (3.9)$$

Replacing  $a$  by  $b$  in (3.8) and arguing as above, we also get that

$$\|Y(z)\| \leq \|Y(b)\|e^{(K_1+A+1)(b-z)}. \quad (3.10)$$

Now if we choose  $Y = [\phi_1 \ \phi_1']^T$ , then  $\|Y(z)\| = |\phi_1(z)| + |\phi_1'(z)|$  and  $\|Y(a)\| = 1$ . Therefore (3.4) follows immediately from (3.9). By a similar way, we get (3.5) by selecting  $Y = [\phi_2 \ \phi_2']^T$  in (3.10).

Now we prove (3.6). Applying the Abel's formula and noting that  $W(\phi_1, \phi_2)(b) = -\phi_1(b)$ , we get that

$$W(\phi_1, \phi_2)(z) = -\phi_1(b)e^{A(b-z)}. \quad (3.11)$$

To estimate  $\phi_1(b)$ , we introduce the function

$$\rho(z) = \frac{1}{A}(e^{A(z-a)} - 1),$$

which is the unique solution of the second-order linear equation  $\rho'' - A\rho' = 0$  on  $[a, b]$  such that

$$\rho(a) = 0, \quad \rho'(a) = 1.$$

For  $z \in (a, b)$ , noting that  $\rho \geq 0$  and  $g \leq 0$ , we see that the function  $\psi := \phi_1 - \rho$  satisfies

$$\psi'' - A\psi' + g\psi = -g\rho \geq 0$$

and

$$\psi(a) = \psi'(a) = 0.$$

Then it follows from [11, Theorem 13 of Chapter 1] that  $\psi \geq 0$  and hence  $\phi_1 \geq \rho$  on  $[a, b]$ . In particular,

$$\phi_1(b) \geq \rho(b) = \frac{1}{A}(e^{A(b-a)} - 1) > 0. \quad (3.12)$$

Combining (3.11) and (3.12), we finally obtain that

$$|W(\phi_1, \phi_2)(z)| = \phi_1(b)e^{A(b-z)} \geq \frac{1}{A}(e^{A(b-a)} - 1) > 0.$$



The proof of this Lemma is therefore completed.  $\square$

Indeed, we get the following two lemmas for a priori estimates.

LEMMA 3.2. Let  $A$  be a positive constant and let  $g$  and  $h$  be continuous functions on  $[a, b]$ . Suppose that  $w \in C([a, b]) \cap C^2((a, b))$  satisfies the differential equation (3.1) in  $(a, b)$  and  $w(a) = w(b) = 0$ . If

$$-K_1 \leq g \leq 0 \text{ and } |h| \leq K_2 \text{ on } [a, b],$$

for some constants  $K_1, K_2$ , then there exists a positive constant  $K_3$ , depending only on  $A, K_1$ , and the length of the interval  $[a, b]$ , such that

$$\|w\|_{C([a, b])} \leq K_2 K_3. \quad (3.13)$$

*Proof.* First, we note that both  $w'(a+) := \lim_{z \rightarrow a+} w'(z)$  and  $w'(b-) := \lim_{z \rightarrow b-} w'(z)$  exist. To see this, let  $\bar{z} \in (a, b)$  be fixed. Integrating equation (3.1) from  $\bar{z}$  to  $z$  and rearranging the resulting equation, we obtain that

$$w'(z) = w'(\bar{z}) - A(w(z) - w(\bar{z})) - \int_{\bar{z}}^z g(\tau)w(\tau)d\tau + \int_{\bar{z}}^z h(\tau)d\tau,$$

for any  $z \in (a, b)$ . Since  $w, g$ , and  $h$  are right-continuous at the point  $a$ , the right-hand limit on the right-hand side of the above equation exists at  $a$ . Hence  $w'(a+)$  exists. Arguing as above, we get  $w'(b-)$  exists.

Let  $\phi_1$  and  $\phi_2$  be as in Lemma 3.1. Consider any point  $z \in (a, b)$ . Multiplying (3.1) by  $\phi_1$ , integrating the resulting inequality from  $a$  to  $z$ , and then using integration by parts, we get that

$$\phi_1(z)w'(z) - (\phi_1'(z) - A\phi_1(z))w(z) + w(a) = \int_a^z \phi_1(\tau)h(\tau)d\tau, \quad (3.14)$$

where we have used (3.2) and the fact that  $\mathcal{L}[\phi_1] = 0$  and  $w'(a+)$  exists. Similarly, multiplying (3.1) by  $\phi_2$ , integrating the resulting inequality from  $z$  to  $b$ , and then using integration by parts, we deduce that

$$-\phi_2(z)w'(z) + (\phi_2'(z) - A\phi_2(z))w(z) + w(b) = \int_z^b \phi_2(\tau)h(\tau)d\tau, \quad (3.15)$$

where we have used (3.3) and the fact that  $\mathcal{L}[\phi_2] = 0$  and  $w'(b-)$  exists. Note that  $w(a) = w(b) = 0$ . Multiplying (3.14) and (3.15) by  $\phi_2$  and  $\phi_1$  respectively and then summing up, we finally get that

$$W(\phi_1, \phi_2)(z)w(z) = \phi_2(z) \int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1(z) \int_z^b \phi_2(\tau)h(\tau)d\tau,$$

which gives that

$$w(z) = \frac{\phi_2(z) \int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1(z) \int_z^b \phi_2(\tau)h(\tau)d\tau}{W(\phi_1, \phi_2)(z)}.$$

This, together with the assumption that  $|h| \leq K_2$ , implies that

$$|w(z)| \leq K_2 \cdot \frac{|\phi_2(z)| \int_a^z |\phi_1(\tau)|d\tau + |\phi_1(z)| \int_z^b |\phi_2(\tau)|d\tau}{|W(\phi_1, \phi_2)(z)|},$$

Finally, using (3.4), (3.5), and (3.6), one can easily deduce from the above inequality that (3.13) holds.  $\square$

LEMMA 3.3. Let  $A$ ,  $g$ , and  $h$  be as in Lemma 3.2. Suppose that  $w \in C([a, b]) \cap C^2((a, b))$  satisfies (3.1) in  $(a, b)$ . If  $\|w\|_{C([a, b])} \leq K_0$  for some constant  $K_0$ , then there exists a positive constant  $K_4$ , depending only on  $A$ ,  $K_0$ ,  $K_1$ ,  $K_2$ , and the length of the interval  $[a, b]$ , such that

$$\|w'\|_{C([a, b])} \leq K_4. \quad (3.16)$$

*Proof.* Let  $\phi_1$  and  $\phi_2$  be as in Lemma 3.1. Consider any point  $z \in (a, b)$ . Multiplying (3.14) and (3.15) by  $\phi_2'$  and  $\phi_1'$  respectively and then summing up, we deduce that

$$\begin{aligned} & W(\phi_1, \phi_2)(z)(w'(z) + Aw(z)) + \phi_2'(z)w(a) + \phi_1'(z)w(b) \\ &= \phi_2'(z) \int_a^z \phi_1(\tau)h(\tau)d\tau + \phi_1'(z) \int_z^b \phi_2(\tau)h(\tau)d\tau, \end{aligned}$$

which yields

$$w'(z) = \frac{\phi_2'(z)[\int_a^z \phi_1(\tau)h(\tau)d\tau - w(a)] + \phi_1'(z)[\int_z^b \phi_2(\tau)h(\tau)d\tau - w(b)]}{W(\phi_1, \phi_2)(z)} - Aw(z).$$

This, together with the assumptions that  $|h| \leq K_2$  and  $|w| \leq K_0$ , implies that

$$|w'(z)| \leq \frac{|\phi_2'(z)|[K_2 \int_a^b \phi_1(\tau)|d\tau + K_0] + |\phi_1'(z)|[K_2 \int_a^b \phi_2(\tau)|d\tau + K_0]}{|W(\phi_1, \phi_2)(z)|} + AK_0.$$

Using (3.4), (3.5), and (3.6), one can easily deduce from the above inequality that (3.16) holds.  $\square$

**4. The system in a finite interval  $[-l, l]$ .** In this section, we consider the system

$$\delta U'' + cU' - \frac{2UV}{\beta + U} = 0 \quad \text{in } (-l, l), \quad (4.1a)$$

$$V'' + cV' + \frac{UV}{\beta + U} = 0 \quad \text{in } (-l, l), \quad (4.1b)$$

together with the boundary conditions

$$(U, V)(-l) = (U^-, V^-)(-l), \quad (U, V)(l) = (U^-, V^-)(l). \quad (4.2)$$

We will apply the Schauder fixed point theorem to show the existence of solutions of (4.1)-(4.2). For readers' convenience, we state the theorem in the following.

LEMMA 4.1. Let  $E$  be a closed convex set in a Banach space and let  $T : E \rightarrow E$  be a continuous mapping such that  $TE$  is precompact. Then  $T$  has a fixed point.

Let  $l > z_1$ . For convenience, we set  $I_l := [-l, l]$ ,  $X := C(I_l) \times C(I_l)$ , and

$$E := \{(U, V) \in X \mid U^- \leq U \leq U^+ \equiv 1 \text{ and } V^- \leq V \leq V^+ \text{ in } I_l\}.$$

It is easy to verify that  $E$  is a closed convex set in the Banach space  $X$  equipped with the norm  $\|(f_1, f_2)\|_X = \|f_1\|_{C(I_l)} + \|f_2\|_{C(I_l)}$ . Since  $U^-$  and  $V^-$  are nonnegative, it follows that  $U \geq 0$  and  $V \geq 0$  for any  $(U, V) \in E$ .

LEMMA 4.2. For a given  $(U_0, V_0) \in E$ , there exists a unique solution to the boundary value problem

$$\delta U'' + cU' - \frac{2UV_0}{\beta + U} = 0 \quad \text{in } (-l, l), \quad (4.3a)$$

$$V'' + cV' + \frac{U_0V_0}{\beta + U_0} = 0 \quad \text{in } (-l, l), \quad (4.3b)$$

$$(U, V)(-l) = (U^-, V^-)(-l), \quad (U, V)(l) = (U^-, V^-)(l). \quad (4.3c)$$

Moreover, this solution  $(U, V)$  satisfies  $U > 0$ ,  $V > 0$ , and  $U' > 0$  in  $(-l, l)$ .

*Proof.* Note that system (4.3) is not a coupled system, so that we can deal with the existence and uniqueness of  $U$  and  $V$  separately. Since  $l > z_1 > z_0 > 0 > -l$ , definition of  $U^-$  and  $V^-$  implies that  $U^-(-l) = V^-(-l) = 0$ ,  $U^-(l) > 0$ , and  $V^-(l) > 0$ .

Since the equation for  $V$  is a inhomogeneous linear equation, the existence and uniqueness of  $V$  can be easily obtained by [5, Theorem 3.1 of Chapter 12]. Moreover, since  $V'' + cV' \leq 0$  in  $(-l, l)$  and  $V(\pm l) \geq 0$ , it follows from the maximum principle that  $V > 0$  in  $(-l, l)$ .

Now we treat the existence and uniqueness of  $U$ . For this, we first consider the initial value problem

$$\delta U'' + cU' - \frac{2UV_0}{\beta + U} = 0, \quad (4.4a)$$

$$U(-l) = (U^-)(-l), \quad U'(-l) = m, \quad (4.4b)$$

where  $m$  is a constant. By the existence and uniqueness theorem, for each  $m$  the initial value problem (4.4) has a unique local solution  $U(z, m)$  and this solution can be continued as long as  $U + \beta > 0$ . When  $m = 0$ ,  $U(z, 0) \equiv 0$  due to the uniqueness. For any fixed  $m < 0$ , since  $U(-l, m) = (U^-)(-l) = 0$  and  $U'(-l, m) = m < 0$ , it follows that there exists  $\delta > 0$  such that  $U(z, m) < 0$  for all  $z \in (-l, -l + \delta]$ . On the other hand, one can easily deduce from (4.4a) that

$$\left( e^{cz/\delta} U' \right)' = \frac{2}{\delta} \frac{UV_0}{\beta + U} e^{cz/\delta},$$

where the prime denotes differentiation with respect to  $z$ . Then an integration of the above equation gives

$$e^{cz/\delta} U'(z, m) = me^{-cl/\delta} + \int_{-l}^z \frac{2}{\delta} \frac{U(\tau, m)V_0(\tau)}{\beta + U(\tau, m)} e^{c\tau/\delta} d\tau, \quad (4.5)$$

which implies that  $U(z, m) < 0$  and  $U'(z, m) < 0$  as long as  $U(z, m)$  exists for  $z > 0$ . For any fixed  $m > 0$ , we can use a similar argument as that for  $m < 0$  to discover that  $U'(z, m) > 0$  and  $U(z, m) > 0$  as long as  $U$  exists for  $z > 0$ , so that the solution can be continued to the interval  $I_l$ . Note that  $U^-(l) > 0$  due to  $l > z_1 > z_0$  and definition of  $U^-$ . From the above discussion, we see that  $U(l, m) = U^-(l)$  unless  $m > 0$ .

Next, we show that there exists  $m^* > 0$  such that  $U(l, m^*) = U^-(l)$  by using the shooting method. To this end, we consider  $m > 0$ . For  $z \in (-l, l]$ , recalling that  $U(z, m) > 0$  and  $V_0(z) \geq 0$  in  $(-l, l]$ , we deduce from (4.5) that

$$U'(z, m) \geq me^{-c(z+l)/\delta}.$$

Then an integration of the above equation from  $-l$  to  $l$  gives

$$U(l, m) \geq \frac{m\delta}{c}(1 - e^{-2cl/\delta}) > U^-(l),$$

if  $m$  is sufficiently large. Note that  $U(l, 0) < U^-(l)$  since  $U(z, 0) \equiv 0$  and  $U^-(l) > 0$ . Since  $U(z, m)$  is a continuous function of  $m$  for  $m \geq 0$ , there exists  $m^* > 0$  such that  $U(l, m^*) = U^-(l)$ .

Finally, set  $U(z) := U(z, m^*)$ . Then  $U$  is a solution of (4.3a) with  $U(-l) = (U^-)(-l)$  and  $U(l) = (U^-)(l)$ . This gives the existence of  $U$ . In addition, we see from the above discussion that  $U > 0$  and  $U' > 0$  in  $(-l, l)$ . Applying the maximum principle, we can easily get the uniqueness of  $U$ . Hence we complete the proof of this lemma.  $\square$

Now we define the mapping  $T : E \rightarrow X$  by

$$T(U_0, V_0) = (U, V), \quad \forall (U_0, V_0) \in E,$$

where  $(U, V)$  is the unique solution of the boundary value problem (4.3). It is obvious that any fixed point of  $T$  is a solution of the problem (4.1)-(4.2).

LEMMA 4.3.  $TE \subseteq E$ .

*Proof.* For a given  $(U_0, V_0) \in E$ , let

$$(U, V) := T(U_0, V_0).$$

We claim that  $V^- \leq V \leq V^+$  on  $I_l$ . Observing that  $0 \leq U^- \leq U_0 \leq U^+ \equiv 1$  and  $0 \leq V^- \leq V_0 \leq V^+$ , we get that

$$\frac{U^-V^-}{\beta + U^-} \leq \frac{U_0V_0}{\beta + U_0} \leq \frac{V^+}{\beta + 1},$$

so that

$$V'' + cV' + \frac{U^-V^-}{\beta + U^-} \leq 0, \quad (4.6)$$

and

$$V'' + cV' + \frac{V^+}{\beta + 1} \geq 0, \quad (4.7)$$

for all  $z$  in  $(-l, l)$ . Set  $w_1 = V - V^-$ . Since  $V^- = 0$  and  $V \geq 0$  in  $[-l, z_1]$ , it follows that

$$w \geq 0, \quad \forall z \in [-l, z_1]. \quad (4.8)$$

From (4.3c), we know that  $w_1(l) = 0$ . In addition, (2.5) and (4.6) give that  $w_1''(z) + cw_1'(z) \leq 0$  for all  $z \in (z_1, l)$ . It follows from the maximum principle that  $w_1 \geq 0$  in  $[z_1, l]$ . This, together with (4.8), implies that  $V^- \leq V$  in  $I_l$ . With a similar argument, we also get that  $V \leq V^+$  in  $I_l$ .

Now we show that  $U^- \leq U$  in  $I_l$ . Since  $U^- \equiv 0$  in  $[-l, z_0]$  and  $U \geq 0$  in  $[-l, z_0]$ , it follows that

$$U \geq U^- \text{ in } [-l, z_0]. \quad (4.9)$$

Hence it remains to show that  $U \geq U^-$  in  $(z_0, l]$ . Since  $V_0 \leq V^+$ , it follows that

$$\frac{2UV_0}{\beta + U} \leq \frac{2UV^+}{\beta + U},$$

so that

$$\delta U'' + cU' - \frac{2UV^+}{\beta + U} \leq 0 \text{ in } (z_0, l). \quad (4.10)$$

For simplicity, we set  $\psi(\xi) = \xi/(\beta + \xi)$ . Then (2.3) and (4.10) imply that the function  $w_2 := U - U^-$  satisfies  $\delta w_2'' + cw_2' - q(z)w_2 \leq 0$  in  $(z_0, l)$ , where

$$q(z) = \begin{cases} 2V^+(z) \cdot \frac{\psi(U(z)) - \psi(U^-(z))}{U(z) - U^-(z)}, & \text{if } U(z) \neq U^-(z), \\ 2V^+(z) \cdot \psi'(U(z)), & \text{if } U(z) = U^-(z). \end{cases}$$

By means of the mean-value theorem, we see that  $q$  is nonnegative in  $(z_0, l)$ . In addition, from (4.9) and (4.3c), we know that  $w_2(z_0) \geq 0$  and  $w_2(l) = 0$ . Hence the maximum principle asserts that  $w_2 \geq 0$  in  $[z_0, l]$ . Hence  $U^- \leq U$  in  $[z_0, l]$ .

Finally, we show that  $U \leq U^+$  in  $I_l$ . Since  $U^+ \equiv 1$  and  $V_0 \geq 0$ , we see that  $U^+$  satisfies

$$\delta(U^+)'' + c(U^+) - \frac{2U^+V_0}{\beta + U^+} \leq 0 \text{ in } (-l, l),$$

and  $U^+(\pm l) = 1 \geq U^-(\pm l) = U(\pm l)$ . By a similar argument as the proof for  $U^- \leq U$  in  $[z_0, l]$ , we get that  $U \leq U^+$  in  $I_l$ .  $\square$

LEMMA 4.4.  $T$  is a continuous mapping.

*Proof.* For given  $(U_0, V_0)$  and  $(\tilde{U}_0, \tilde{V}_0)$  in  $E$ , let

$$(U, V) = T(U_0, V_0) \text{ and } (\tilde{U}, \tilde{V}) = T(\tilde{U}_0, \tilde{V}_0). \quad (4.11)$$

It is easy to see that  $w_1 := U - \tilde{U}$  satisfies  $w_1(-l) = w_1(l) = 0$  and

$$w_1'' + \frac{c}{\delta}w_1' + g(z)w_1 = h_1(z).$$

Here,

$$g(z) = \begin{cases} -\frac{2V_0(z)}{\delta} \cdot \frac{\psi(U(z)) - \psi(\tilde{U}(z))}{U(z) - \tilde{U}(z)}, & \text{if } U(z) \neq \tilde{U}(z), \\ -\frac{2V_0(z)}{\delta} \cdot \psi'(U(z)), & \text{if } U(z) = \tilde{U}(z), \end{cases}$$

and

$$h_1(z) = -\frac{2}{\delta} \cdot \psi(\tilde{U}(z))(V_0(z) - \tilde{V}_0(z)),$$

where  $\psi$  is given in the proof of Lemma 4.3. Noting that  $0 \leq U, \tilde{U} \leq 1$  and  $0 \leq \psi'(\xi) \leq 1/\beta$  for  $0 \leq \xi \leq 1$ , we can apply the mean-value theorem to deduce that

$$0 \leq \frac{\psi(U(z)) - \psi(\tilde{U}(z))}{U(z) - \tilde{U}(z)} \leq \frac{1}{\beta}, \text{ for } U(z) \neq \tilde{U}(z). \quad (4.12)$$

Together with the fact that  $\delta > 0$ ,  $0 \leq V_0 \leq V^+$ , and  $0 \leq \psi'(U(z)) \leq 1/\beta$ , we find that  $-K_1 \leq g \leq 0$  with

$$K_1 = \frac{2}{\delta\beta} \cdot \|V^+\|_{C(I_l)}.$$

In addition, since  $0 < \psi(\tilde{U}) \leq 1$ , it is easy to see that

$$|h_1| \leq \frac{2}{\delta} \cdot \|V_0 - \tilde{V}_0\|_{C(I_l)}.$$

Then Lemma 3.2 asserts that there exists a positive constant  $C_1$ , depending only on  $\delta$ ,  $c$ ,  $K_1$  and  $l$ , such that

$$\|w_1\|_{C(I_l)} \leq \frac{2C_1}{\delta} \cdot \|V_0 - \tilde{V}_0\|_{C(I_l)},$$

which, together with definition of  $w_1$ , implies that

$$\|U - \tilde{U}\|_{C(I_l)} \leq \frac{2C_1}{\delta} \cdot \|V_0 - \tilde{V}_0\|_{C(I_l)}. \quad (4.13)$$

Set  $w_2 = V - \tilde{V}$ . Then  $w_2$  satisfies  $w_2(-l) = w_2(l) = 0$  and

$$w_2'' + cw_2' = h_2(z),$$

where

$$h_2 = \psi(\tilde{U}_0)\tilde{V}_0 - \psi(U_0)V_0.$$

It is easy to see that

$$h_2 = \tilde{V}_0 \left( \psi(\tilde{U}_0) - \psi(U_0) \right) + \psi(U_0)(\tilde{V}_0 - V_0). \quad (4.14)$$

Since  $0 \leq U_0, \tilde{U}_0 \leq 1$ , we can apply the mean-value theorem to get that

$$|\psi(\tilde{U}_0) - \psi(U_0)| \leq \frac{1}{\beta} |\tilde{U}_0 - U_0|,$$

and therefore,

$$\left| \psi(\tilde{U}_0) - \psi(U_0) \right| \leq \frac{1}{\beta} \|\tilde{U}_0 - U_0\|_{C(I_l)}.$$

Together with the fact that

$$|\tilde{V}_0| \leq \|V^+\|_{C(I_l)}, \quad |\psi(U_0)| \leq 1, \quad \text{and} \quad |V_0 - \tilde{V}_0| \leq \|V_0 - \tilde{V}_0\|_{C(I_l)},$$

we deduce from (4.14) that

$$|h_2| \leq \frac{1}{\beta} \cdot \|V^+\|_{C(I_l)} \|U_0 - \tilde{U}_0\|_{C(I_l)} + \|V_0 - \tilde{V}_0\|_{C(I_l)}.$$

Then Lemma 3.2 asserts that there exists a positive constant  $C_2$ , depending only on  $c$ , and  $l$ , such that

$$\|w_2\|_{C(I_l)} \leq \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)} \|U_0 - \tilde{U}_0\|_{C(I_l)} + C_2 \|V_0 - \tilde{V}_0\|_{C(I_l)},$$

which, together with definition of  $w_2$ , implies that

$$\|V - \tilde{V}\|_{C(I_l)} \leq \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)} \|U_0 - \tilde{U}_0\|_{C(I_l)} + C_2 \|V_0 - \tilde{V}_0\|_{C(I_l)}. \quad (4.15)$$

Using (4.11), (4.13), (4.15), and definition of the norm  $\|\cdot\|_X$ , we obtain that

$$\begin{aligned}
 & \|T(U_0, V_0) - T(\tilde{U}_0, \tilde{V}_0)\|_X \\
 &= \|(U, V) - (\tilde{U}, \tilde{V})\|_X \\
 &= \|U - \tilde{U}\|_{C(I_l)} + \|V - \tilde{V}\|_{C(I_l)} \\
 &\leq \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)} \|U_0 - \tilde{U}_0\|_{C(I_l)} + \left(\frac{2C_1}{\delta} + C_2\right) \cdot \|V_0 - \tilde{V}_0\|_{C(I_l)} \\
 &\leq C_3(\|U_0 - \tilde{U}_0\|_{C(I_l)} + \|V_0 - \tilde{V}_0\|_{C(I_l)}) \\
 &= C_3\|(U_0, V_0) - (\tilde{U}_0, \tilde{V}_0)\|_X,
 \end{aligned} \tag{4.16}$$

where  $C_3 = \frac{C_2}{\beta} \cdot \|V^+\|_{C(I_l)} + \frac{2C_1}{\delta} + C_2$ .

For a given  $\epsilon > 0$ , we choose  $0 < \delta < \epsilon/C_3$ . Then, by (4.16), we have

$$\|T(U_0, V_0) - T(\tilde{U}_0, \tilde{V}_0)\|_X < \epsilon,$$

for any  $(U_0, V_0), (\tilde{U}_0, \tilde{V}_0) \in E$  such that  $\|(U_0, V_0) - (\tilde{U}_0, \tilde{V}_0)\|_X < \delta$ . This shows that  $T$  is a continuous mapping. Hence the proof of this lemma is completed.  $\square$

LEMMA 4.5.  $T$  is precompact.

*Proof.* For a given sequence  $\{(U_{0,n}, V_{0,n})\}_{n \in \mathbb{N}}$  in  $E$ , let  $(U_n, V_n) = T(U_{0,n}, V_{0,n})$ . Since  $U^-$  and  $U^+$  are bounded in  $I_l$ , and  $U^- \geq 0$ , we can easily see from definition of the set  $E$  and Lemma 4.3 that the sequences

$$\{U_{0,n}\}, \{V_{0,n}\}, \{U_n\}, \{V_n\}, \left\{\frac{2U_n V_{0,n}}{\beta + U_n}\right\}, \text{ and } \left\{\frac{2U_{0,n} V_{0,n}}{\beta + U_{0,n}}\right\}$$

are uniformly bounded in  $I_l$ . Then, by Lemma 3.3, it follows that the sequences

$$\{U'_n\} \text{ and } \{V'_n\},$$

are also uniformly bounded in  $I_l$ . Therefore, we can use Arzela-Ascoli theorem to get a subsequence  $\{(U_{n_j}, V_{n_j})\}$  of  $\{(U_n, V_n)\}$  such that

$$(U_{n_j}, V_{n_j}) \rightarrow (U, V),$$

uniformly in  $I_l$  as  $j \rightarrow \infty$ , for some  $(U, V) \in E$ . Hence the set  $\overline{T(E)}$  is compact in  $E$ . So  $T$  is precompact.  $\square$

In Lemma 4.3, Lemma 4.4, and Lemma 4.5, we have proved that the mapping  $T$  satisfies all the assumptions of Lemma 4.1. Hence  $T$  has a fixed point, which is a nonnegative solution of system (4.1)-(4.2). So we have the following theorem.

LEMMA 4.6. System (4.1)-(4.2) admits a solution  $(U, V)$  on  $I_l$ . Moreover,

$$0 \leq U^- \leq U \leq 1 \text{ and } 0 \leq V^- \leq V \leq V^+ \tag{4.17}$$

on  $I_l$ .

**5. The proof of the main result.** Now we are in a position to prove the main results.

**proof of Theorem 1.1:** Let  $\{l_n\}_{n \in \mathbb{N}}$  be an increasing sequence in  $(z_1, \infty)$  such that  $l_n \rightarrow \infty$  as  $n \rightarrow \infty$  and let  $(U_n, V_n)$ ,  $n \in \mathbb{N}$ , be a solution of system (4.1)-(4.2) with  $l = l_n$ . For any fixed  $N \in \mathbb{N}$ , since the function  $V^+$  is bounded above in  $[-l_N, l_N]$ , it follows from (4.17) that the sequences

$$\{U_n\}_{n \geq N}, \{V_n\}_{n \geq N}, \text{ and } \left\{ \frac{U_n V_n}{\beta + U_n} \right\}_{n \geq N}$$

are uniformly bounded in  $[-l_N, l_N]$ . Then we can use Lemma 3.3 to infer that the sequences

$$\{U'_n\}_{n \geq N} \text{ and } \{V'_n\}_{n \geq N}$$

are also uniformly bounded in  $[-l_N, l_N]$ . Using (4.1), we can express  $U''_n$  and  $V''_n$  in terms of  $U_n, V_n, U'_n$  and  $V'_n$ . Differentiating (4.1), we can use the resulting equations to express  $U'''_n$  and  $V'''_n$  in terms of  $U_n, V_n, U'_n, V'_n, U''_n$  and  $V''_n$ . Consequently, the sequences

$$\{U''_n\}_{n \geq N}, \{V''_n\}_{n \geq N}, \{U'''_n\}_{n \geq N} \text{ and } \{V'''_n\}_{n \geq N}$$

are uniformly bounded in  $[-l_N, l_N]$ . With the aid of Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence  $\{(U_{n_j}, V_{n_j})\}$  of  $\{(U_n, V_n)\}$  such that

$$U_{n_j} \rightarrow U, U'_{n_j} \rightarrow U', U''_{n_j} \rightarrow U'',$$

and

$$V_{n_j} \rightarrow V, V'_{n_j} \rightarrow V', V''_{n_j} \rightarrow V'',$$

uniformly in any compact interval of  $\mathbb{R}$  as  $n \rightarrow \infty$ , for some functions  $U$  and  $V$  in  $C^2(\mathbb{R})$ . Then it is easy to see that  $(U, V)$  is a nonnegative solution of system (1.2) and satisfies (4.17) and  $U' \geq 0$  over  $\mathbb{R}$ . From definitions of  $U^-$  and  $V^+$ , we see that  $U^-(z) \rightarrow 1$  and  $V^+(z) \rightarrow 0$  as  $z \rightarrow \infty$ . This, together with (4.17), implies that

$$(U, V)(+\infty) = (1, 0). \quad (5.1)$$

Now it remains to show that  $(U, V)(-\infty) = (0, 1/2)$ . We divide the proof into several steps:

Step 1: We claim that

$$(U', V')(+\infty) = (0, 0). \quad (5.2)$$

Integrating equation (1.2a) from 0 to  $z$  gives that

$$\delta[U'(z) - U'(0)] + c[U(z) - U(0)] = \int_0^z \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau. \quad (5.3)$$

Since  $U(+\infty)$  exists, we see from (5.3) that  $U'(\infty)$  exists iff the improper integral

$$\int_0^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau \quad (5.4)$$

converges. Indeed, the improper integral (5.4) is convergent since otherwise it diverges to  $\infty$ . Then (5.3) gives that  $U'(\infty) = \infty$  and therefore  $U(\infty) = \infty$ , a contradiction to the fact that  $U(\infty)$  exists. Hence  $U'(\infty)$  exists. Moreover, one can easily verify that  $U'(\infty) = 0$  due to  $U(+\infty) = 1$ . Similarly, integrating equation (1.2b) from 0 to  $z$  and arguing as above, we also get  $V'(\infty) = 0$ .



Step 2: We claim that  $(U, V)(-\infty)$  exists and  $1 > U(-\infty) \geq 0$ ,  $V(-\infty) \geq 0$ . Since  $U$  is nondecreasing and  $0 \leq U \leq 1$ , it follows that  $U(-\infty)$  exists and  $0 \leq U(-\infty) \leq 1$ . Note that  $U(-\infty) \neq 1$ . Otherwise, the monotonicity of  $U$  implies that  $U \equiv 1$ , which together with (1.2a), gives that  $V \equiv 0$ , a contradiction to the fact that  $V \geq V^- > 0$  in  $(z_1, \infty)$ .

To show the existence of  $V(-\infty)$ , we need to claim that  $V \leq 1/2$  on  $\mathbb{R}$ . From (1.2), we deduce that

$$\delta U' + 2V' + c(U + 2V) \equiv K,$$

for some constant  $K$ . Letting  $z \rightarrow \infty$  in the above equation and using (5.1) and (5.2), we discover that  $K = c$ . Therefore,

$$\delta U' + 2V' + c(U + 2V - 1) \equiv 0. \quad (5.5)$$

Set  $W_1 := U + 2V - 1$ . Since  $U \leq 1$ , we can use (4.17) to get that

$$W_1(z) \leq 2V \leq 2V^+ \leq 2e^{-\lambda z}, \forall z \in \mathbb{R}. \quad (5.6)$$

In addition, since  $U' \geq 0$ , we can use (5.5) to deduce that

$$W_1' + cW_1 = (1 - \delta)U' \leq 0 \quad (5.7)$$

on  $\mathbb{R}$ , if  $\delta \geq 1$ . Multiplying (5.7) by the integrating factor  $e^{cz}$ , one can easily deduce that  $[e^{cz}W_1(z)]' \leq 0$ , which, together with (5.6), implies that

$$e^{cz}W_1(z) \leq e^{cz^*}W_1(z^*) \leq 2e^{(c-\lambda)z^*},$$

for any  $-\infty < z^* < z < \infty$ . Letting  $z^* \rightarrow -\infty$  in the above inequality and noting that  $e^{(c-\lambda)z^*} \rightarrow 0$  due to  $\lambda < c$ , we get  $W_1(z) \leq 0$  and therefore  $U + 2V \leq 1$  on  $\mathbb{R}$ . This, together with the fact that  $U \geq 0$ , implies that  $V \leq 1/2$  on  $\mathbb{R}$ . Now we consider the case  $\delta < 1$ . Set  $W_2 := \delta U + 2V - 1$ . Since  $U \leq 1$  and  $\delta < 1$ , it follows from (4.17) that

$$W_2(z) \leq 2V \leq 2V^+ \leq 2e^{-\lambda z}, \forall z \in \mathbb{R}.$$

In addition, since  $c > 0$ ,  $\delta < 1$ , and  $U \geq 0$ , we can use (5.5) to deduce that

$$W_2' + cW_2 = c(\delta - 1)U \leq 0$$

on  $\mathbb{R}$ . Arguing as the proof for  $W_1 \leq 0$ , we can easily get  $W_2 \leq 0$  and therefore  $\delta U + 2V \leq 1$  on  $\mathbb{R}$ . This, together with the fact that  $\delta U \geq 0$ , implies that  $V \leq 1/2$  on  $\mathbb{R}$ .

Now we claim that  $V(-\infty)$  exists and  $V(-\infty) \geq 0$ . Since  $V(\infty) = 0$  and  $V(z_1 + 1) \geq V^-(z_1 + 1) > 0$ , we can use the mean-value theorem to infer that there exists  $\xi_1 > z_1 + 1$  such that  $V'(\xi_1) < 0$ . Multiplying (1.2b) by the integrating factor  $e^{cz}$ , one can easily deduce that

$$[e^{cz}V'(z)]' = -e^{cz} \frac{UV}{\beta + U} \leq 0,$$

which implies that  $e^{cz}V'(z)$  is nonincreasing. Therefore, for  $z > \xi_1$ ,  $e^{cz}V'(z) \leq e^{c\xi_1}V'(\xi_1) < 0$ . Thus  $V' < 0$  in  $[\xi_1, \infty)$ . Let  $\xi_2 := \inf\{z \mid V' < 0 \text{ in } [z, \infty)\}$ . Then  $\xi_2 = -\infty$  or a finite number. If  $\xi_2 = -\infty$ , then  $V' < 0$  over  $\mathbb{R}$ . This, together with the fact that  $0 \leq V \leq 1/2$ , implies that  $V(-\infty)$  exists and  $V(-\infty) \geq 0$ . If  $\xi_2$  is a finite number, then  $V'(\xi_2) = 0$ , which together the monotonicity of  $e^{cz}V'(z)$ , implies that  $e^{cz}V'(z) \geq e^{c\xi_2}V'(\xi_2) = 0$  for

$z \leq \xi_2$ . Hence  $V' \geq 0$  in  $(-\infty, \xi_2]$ . This, together with the fact that  $V \geq 0$ , implies that  $V(-\infty)$  exists and  $V(-\infty) \geq 0$ .

Step 3: We claim that

$$(U', V')(-\infty) = (0, 0). \quad (5.8)$$

Integrating equation (1.2a) from  $z$  to  $\infty$  and recalling that  $U(\infty) = 1$  and  $U'(\infty) = 0$  gives that

$$-\delta U'(z) + c[1 - U(z)] = 2 \int_z^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau. \quad (5.9)$$

Since  $U \geq 0$  and  $U' \geq 0$ , equation (5.9) implies that

$$\int_z^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau \leq \frac{c}{2},$$

so that improper integral

$$\int_{-\infty}^\infty \frac{U(\tau)V(\tau)}{\beta + U(\tau)} d\tau \quad (5.10)$$

converges. Letting  $z \rightarrow -\infty$  in (5.9) and recalling the fact that  $U(-\infty)$  exists, we infer that  $U'(-\infty)$  exists. Furthermore, since  $U' \geq 0$ , it follows that  $U'(-\infty) \geq 0$ . Indeed,  $U'(-\infty) = 0$ . Otherwise,  $U'(-\infty) > 0$ , which implies  $U(-\infty) = -\infty$ , a contradiction to the fact that  $U(-\infty)$  exists. By a similar argument, we also get  $V'(-\infty) = 0$ .

Step 4: We claim that  $(U, V)(-\infty) = (0, 1/2)$ . Since both  $U(-\infty)$  and  $V(-\infty)$  exist, the convergence of the improper integral (5.10) implies that

$$U(-\infty)V(-\infty) = 0. \quad (5.11)$$

On the other hand, letting  $z \rightarrow -\infty$  in (5.5) and using (5.8), we get that

$$U(-\infty) + 2V(-\infty) = 1. \quad (5.12)$$

Recall that  $U(-\infty) \neq 1$ . Then (5.11) and (5.12) yield  $(U, V)(-\infty) = (0, 1/2)$ . This completes the proof of this theorem.  $\square$

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無研發成果推廣資料

100 年度專題研究計畫研究成果彙整表

計畫主持人：符聖珍		計畫編號：100-2115-M-004-003-					
計畫名稱：拋物型方程在 time scale 上的極大值原理							
成果項目		量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）	
		實際已達成數（被接受或已發表）	預期總達成數（含實際已達成數）	本計畫實際貢獻百分比			
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
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<p style="text-align: center;">其他成果</p> <p>(無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)</p>	無
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	成果項目	量化	名稱或內容性質簡述
科 教 處 計 畫 加 填 項 目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與(閱聽)人數	0	

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