國立政治大學應用數學系

碩士學位論文

Constructing Tropical Curves of Degree Two and Three with Tropical Lines

熱帶直線建構二次及三次熱帶曲線之研究

碩士班學生：江泰緯 撰

指導教授：蔡炎龍 博士

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Abstract

In this thesis, we develop an algorithm to recover tropical polynomials from plane tropical curves of degree two and three. We use tropical lines to approach a given tropical curve. Furthermore, we also give another algorithm to recover tropical polynomials from a (maximal) Newton subdivision of degree two and three.

中文摘要

在這篇論文裡，我們找到了一個方法來反推出對應到某個熱帶曲線的熱帶多項式。在給定一個二次或三次的熱帶曲線之後，我們利用熱帶直線來找出此熱帶曲線的多項式。再來，若給定一個二次或三次的牛頓細分 (Newton subdivision)，我們也能找出能對應到它的熱帶多項式。

Contents

Chapter 1

Introduction

Tropical geometry is a relatively new area in mathematics. Roughly speaking, tropical geometry is the geometry base on the tropical semiring. Tropical semiring is first developed in the 1980s by Imre Simon [6], a mathematician and computer scientist from Brazil.

Tropical geometry becomes more popular after some important applications in the fields such as the classical enumerative geometry and the algebraic geometry. (We refer to [1], [4], [8] for details.)

In tropical geometry, we usually works on the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with addition and multiplication defined by:

$$x \oplus y = \max\{x, y\},$$
$$x \odot y = x + y,$$

which is also called "max-plus" algebra. The additive identity is $0_{\mathbb{T}} = -\infty$, while the multiplicative identity is $1_{\mathbb{T}} = 0$. Observe that such a structure is not a ring, since not all elements have tropical additive inverses. For example, there is no solution in $\mathbb{T}$ for the equation $x \oplus 3 = 2$.

What we usually deal with in tropical algebraic geometry is convex piecewise linear functions.

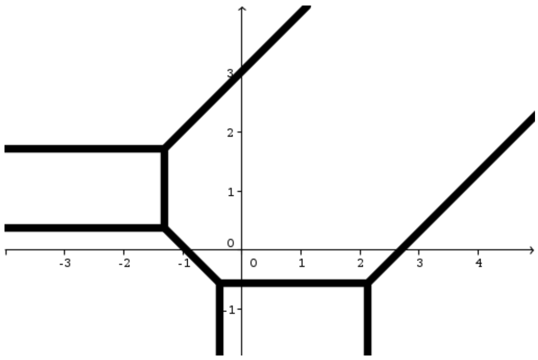


Figure 1.1:

For basic tropical geometry, one can see [1], [5], and [8]. In [1], Andrea Gathmann give an introduction about tropical algebraic geometry, including the construction of tropical curves, and the tropical version of some well-known theorem, e.g. Bézout theorem. The main references of this thesis is [1], [2], and [8].

In Section 2.2, we introduce the definitions of tropical curves. In Section 2.3, we study the tropical factorization, and define an equivalence relation so that we may have an one-to-one correspondonce between tropical polynomials and tropical curves. In Chapter 3, we give an algorithm to recover polynomials from the given tropical curves. In Chapter 4, we give a similar algorithm to recover polynomials from Newton subdivisions.

Chapter 2

Tropical Algebraic Geometry

2.1 Tropical polynomials

Definition 2.1.1 (Tropical Semiring)**.** Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. The Tropical semiring $(\mathbb{T}, \oplus, \odot)$ is an algebraic structure with two binary operations defined as followings:

$$\begin{cases} a \oplus b = \max\{a, b\}, \\ a \odot b = a + b, \end{cases}$$

where $\oplus$ is called *tropical addition* and $\odot$ is called *tropical multiplication.*

Definition 2.1.2. A polynomial $g(x) \in \mathbb{T}[x_1, x_2, \ldots, x_n]$ is called a tropical polynomial.

Example 2.1.3 (A tropical polynomial in one variable)**.**

$$\begin{aligned} g(x) &= 3 \odot x^{\odot 3} \oplus 2 \odot x^{\odot 2} \oplus 1 \odot x \oplus 0 \\ &= \max\{3x + 3, 2x + 2, x + 1, 0\} \end{aligned}$$

Example 2.1.4 (A tropical polynomial in two variables)**.**

$$\begin{aligned} g(x, y) &= (-2) \odot x \oplus (-3) \odot y \oplus 0 \\ &= \max\{x - 2, y - 3, 0\} \end{aligned}$$

2.2 Tropical curves

For a complex plane curve C, we restrict it to the open subset $(\mathbb{C}^*)^2$ of the (affine or projective) plane and then map it to the real plane by the map

$$\begin{aligned} Log : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ z = (z_1, z_2) &\mapsto (x_1, x_2) := (\log|z_1|, \log|z_2|). \end{aligned}$$

The image $A = Log(C \cap (\mathbb{C}^*)^2)$ is called the *amoeba* of the given curve C.

Example 2.2.1. $C = \{z \in \mathbb{C}^2 | e^{-2}z_1 + e^{-3}z_2 = 1\}$

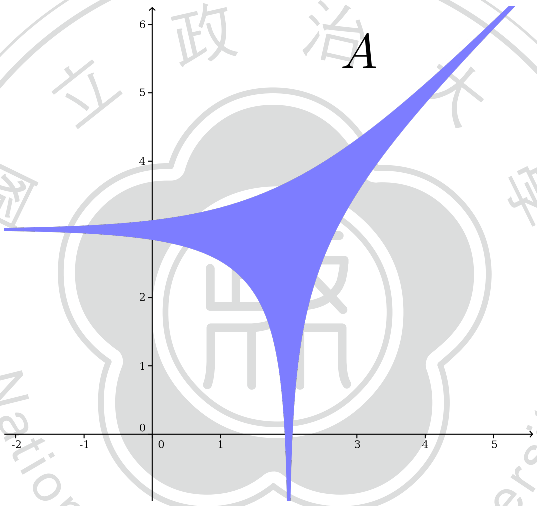


Figure 2.1: Amoeba of $C = \{z \in \mathbb{C}^2 \,|\, e^{-2}z_1 + e^{-3}z_2 = 1\}$

In fact, the shape of the picture above (that also explains the name "amoeba") can easily be explained. The curve C above contains exactly one point whose z_1-coordinate is zero, namely $(0, e^3)$. As $\log 0 = -\infty$, a small neighborhood of this point is mapped by Log to the tentacle of the amoeba A pointing to the left. Similarly, a neighborhood of $(e^2, 0)$ is mapped by Log to the tentacle pointing down, and points of the form $(z, e^3 - ez)$ with $|z| \to \infty$ to the tentacle pointing to the upper right.

Now, to make the amoeba into a combinatorial object, we consider the maps

$$\begin{aligned} Log_t : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ (z_1, z_2) &\mapsto (-\log_t |z_1|, -\log_t |z_2|) = \left(-\frac{\log |z_1|}{\log t}, -\frac{\log |z_2|}{\log t} \right). \end{aligned}$$

and the family of curves $C_t = \{z \in \mathbb{C}^2 \,|\, t^2 z_1 + t^3 z_2 = 1\}$ for small $t \in \mathbb{R}$. This family has the property that C_t passes through $(0, t^{-3})$ and $(t^{-2}, 0)$ for all t, and hence all $Log_t(C_t \cap (\mathbb{C}^*)^2)$ have their horizontal and vertical tentacles at $z_2 = 3$ and $z_1 = 2$, respectively. That is why we consider the family C_t instead of the original curve C. So if we now take the limit as $t \to 0$, we shrink the width of the amoeba to zero but keep its position in the plane, and this "zero-width amoeba" is called the *tropical curve* determined by the family C_t. In Figure 2.2, The tropical curve Γ is usually called a *tropical line*.

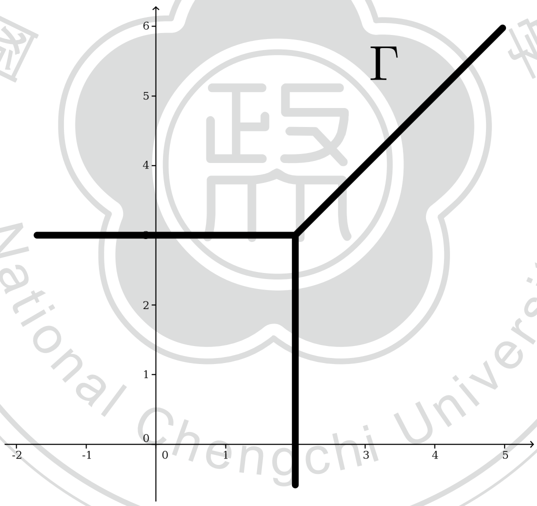


Figure 2.2: The tropical curve corresponding to the amoeba in Figure 2.1

There is an elegant way to hide the limiting process by replacing the ground field $\mathbb{C}$ by the field of *Puiseux series*.

Definition 2.2.2. A formal power series of the form $\sum_{q \in \mathbb{Q}} a_q t^q$, $a_q \in \mathbb{C}$ satisfying:

(i) the set $\{q \in \mathbb{Q} | a_q \neq 0\}$ is bounded below,

(ii) the denominators of $q \in \{q \in \mathbb{Q} | a_q \neq 0\}$ is a finite set

is called a *Puiseux series* or a fractional power series. A field K of Puiseux series is a collection of Puiseux series.

Definition 2.2.3. For $a = \sum_{q \in \mathbb{Q}} a_q t^q \in K, a \neq 0$, we may define the *valuation* of a by the map $\mathrm{val}(a) = \inf\{q \in \mathbb{Q} | a_q \neq 0\}$.

Remark 2.2.4. The infimum of the set $\{q \in \mathbb{Q} | a_q \neq 0\}$ is actually a minimum. i.e. $\mathrm{val}(a) = \inf\{q \in \mathbb{Q} | a_q \neq 0\} = \min\{q \in \mathbb{Q} | a_q \neq 0\}$.

Example 2.2.5. Let

$$a = 1 + t^{1/6} + t^{2/6} + t^{3/6} + \ldots + t^{k/6} + \ldots,$$
$$b = 1 + t^{1/2} + t^{1/3} + t^{1/4} + \ldots + t^{1/k} + \ldots,$$

and

$$c = 1 + t^{-1/6} + t^{-2/6} + t^{-3/6} + \ldots + t^{-k/6} + \ldots.$$

a is a Puiseux series, while b and c is not, since the set of denominators of $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{k}, \ldots\}$ is not finite, and $\{0, -\frac{1}{6}, -\frac{2}{6}, -\frac{3}{6}, \ldots, -\frac{k}{6}, \ldots\}$ is not bounded below.

It is easy to see that $\mathbb{C} \subset K$, so we can consider a curve C in $\mathbb{C}^2$ to be a curve in K^2, for example,

$$C = \{z \in K^2 \,|\, t^2 z_1 + t^3 z_2 = 1\}$$

For $t \to 0$, we have

$$a \approx a_{\mathrm{val}\, a}\, t^{\mathrm{val}\, a}.$$

So applying the map $\log_t$ to a, we get for a small t

$$\log_t |a| \approx \log_t |a_{\mathrm{val}\, a} t^{\mathrm{val}\, a}| = \mathrm{val}\, a + \log_t |a_{\mathrm{val}\, a}| \approx \mathrm{val}\, a.$$

Therefore, the process of applying the map Log_t and taking the limit for $t \to 0$ correspond to the map

$$\begin{aligned} \mathrm{Val} : (K^*)^2 &\to \mathbb{Q}^2 \\ (z_1, z_2) &\mapsto (x_1, x_2) := (-\mathrm{val}\, z_1, -\mathrm{val}\, z_2). \end{aligned}$$

Using this observation we can now give our first definition of plane tropical curves.

Definition 2.2.6. A plane tropical curve is a subset of $\mathbb{R}^2$ of the form $A = \text{Val}(C \cap (K^*)^2)$, where C is a plane algebraic curve in K^2 . (Strictly speaking we should take the closure of $\text{Val}(C \cap (K^*)^2)$ in $\mathbb{R}^2$ since the image of the valuation map Val is by definition contained in $\mathbb{Q}^2$)

Note that this definition is now purely algebraic and does not involve any limit taking processes.

Example 2.2.7. For the example above, $C = \{(z_1, z_2) \in K^2 \mid t^2 z_1 + t^3 z_2 = 1\}$. If $(z_1, z_2) \in C \cap (K^*)^2$, then $\text{Val}(z_1, z_2)$ can give three kind of result:

- If $\operatorname{val} z_1 > -2$, then the valuation of $z_2 = t^{-3} - t^{-1} z_1$ is -3 since all exponent of t in $t^{-1} z_1$ are bigger then -3. Hence these points map precisely to the left edge of the tropical curve determined by C.
- If $\operatorname{val} z_2 > -3$, then the valuation of $z_1 = t^{-2} - t z_1$ is -2 since all exponent of t in $t z_1$ are bigger then -2. Hence these points map precisely to the bottom edge of the tropical curve determined by C.
- If $\operatorname{val} z_1 \leq -2$ and $\operatorname{val} z_2 \leq -3$, then the equation $t^2 z_1 + t^3 z_2 = 1$ shows that the leading terms of $t^2 z_1$ and $t^3 z_2$ must have the same valuation, i.e. that $\operatorname{val} z_1 = \operatorname{val} z_2 + 1$. This leads to the upper right edge of the tropical curve determined by C.

So we can get the same graph by this definition.

Let $C \subset K^2$ be a plane algebraic curve given by the polynomial equation

$$C = \left\{ (z_1, z_2) \in K^2 \mid f(z_1, z_2) := \sum_{i,j \in \mathbb{N}} a_{ij} z_1^i z_2^j = 0 \right\}$$

for some $a_{ij} \in K$ of which only finitely many are nonzero. Note that the valuation of a summand of $f(z_1, z_2)$ is

$$\operatorname{val}(a_{ij} z_1^i z_2^j) = \operatorname{val} a_{ij} + i \operatorname{val} z_1 + j \operatorname{val} z_2.$$

Now if (z_1, z_2) is a point of C then all these summands add up to zero. In particular, the lowest valuation of these summands must occur at least twice since otherwise the corresponding terms in the sum could not cancel. For the corresponding point $(x_1, x_2) = \mathrm{Val}(z_1, z_2) = (-\mathrm{val}\, z_1, -\mathrm{val}\, z_2)$ of the tropical curve, this obviously means that in the expression

$$g(x_1, x_2) := \max\{ix_1 + jx_2 - \mathrm{val}\, a_{ij} \,|\, (i, j) \in \mathbb{N}^2 \text{ with } a_{ij} \neq 0\} \tag{2.1}$$

the maximum is taken on at least twice. It follows that the tropical curve determined by C is contained in the "corner locus" of this convex piecewise linear function g, i.e. in the locus where g is not diffrentiable.

Theorem 2.2.8 (Kapranov)**.** The closure of the amoeba $A \subset \mathbb{R}^2$ coincides with the corner locus of the convex piecewise linear function g.

Remark 2.2.9. Kapranov's theorem shows that the tropical curve determined by C is precisely the corner locus of g.

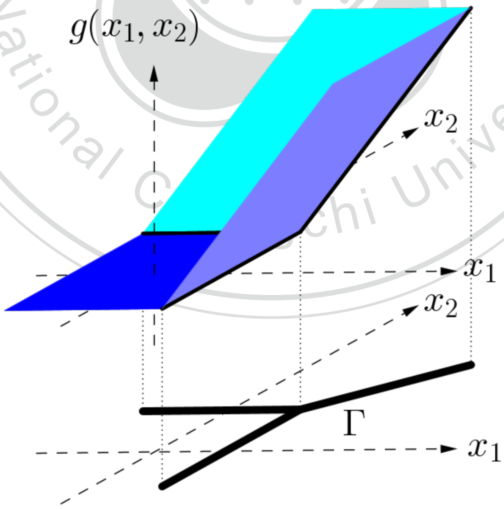


Figure 2.3: A tropical curve as the corner locus of a convex piecewise linear function

Example 2.2.10. Let us consider the curve $C = \{z \in K^2 \,|\, t^2 z_1 + t^3 z_2 - 1 = 0\}$ again. The corresponding convex piecewise linear function is

$$g(x_1, x_2) = \max\{x_1 - 2, x_2 - 3, 0\}$$

Figure 2.3 shows that the relation between tropical curve and the convex piecewise linear function g.

Now, with the two tropical operations defined in Section 2.1, we can rewrite g as

$$g(x_1, x_2) = (-2) \odot x_1 \oplus (-3) \odot x_2 \oplus 0$$

which is of the form of tropical polynomials.

So we can also rewrite our convex piecewise linear function (2.1) above as

$$g(x_1, x_2) = \bigoplus_{i,j \in \mathbb{N}} (-\operatorname{val} a_{ij}) \odot x_1^{\odot i} \odot x_2^{\odot j}$$

Therefore, we can give an alternative definition of plane tropical curves that does not involve the somewhat complicated field of Puiseux series any more:

Definition 2.2.11. A plane tropical curve is a subset of $\mathbb{R}^2$ that is the corner locus of a rational tropical polynomial.

From examples above, we may observe a simple proposition:

Proposition 2.2.12. Suppose

$$g(x_1, x_2) = a \odot x \oplus b \odot y \oplus 0,$$

where $a, b \in \mathbb{Q}$, and let Γ be the corner locus of g. Then the coordinate of the vertex of Γ is $(-a, -b)$.

Remark 2.2.13. Let $g(x_1, x_2) = \bigoplus_{i,j \in \mathbb{N}} a_{ij} \odot x_1^{\odot i} \odot x_2^{\odot j} = \max\{ix_1 + jx_2 + a_{ij} \,|\, (i,j) \in \mathbb{N}^2, a_{i,j} \in \mathbb{Q}\}$. Each term of $g(x_1, x_2)$ corresponds to a plane $g(x_1, x_2) = ix + jy + a_{ij}$.

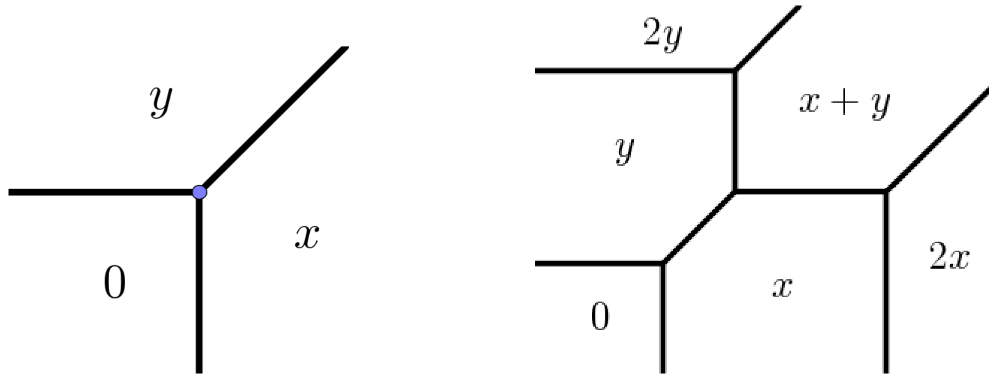


Figure 2.4: The correspondence between coefficients and planes

Example 2.2.14. There is a special case of plane tropical curves. If the tropical polynomial g is the maximum of linear functions without constant terms, i.e.

$$g(x_1, x_2) = \bigoplus_i x_1^{\odot a_1^{(i)}} x_2^{\odot a_2^{(i)}} = \max\{a_1^{(i)} x_1 + a_2^{(i)} x_2 \mid i = 1, \ldots, n\} \tag{2.2}$$

for some $a^{(i)} = (a_1^{(i)}, a_2^{(i)}) \in \mathbb{N}^2$, then, since for each $i, j = 1, \ldots, n, i \neq j$,

$$a_1^{(i)} x_1 + a_2^{(i)} x_2 = a_1^{(j)} x_1 + a_2^{(j)} x_2$$

is a line passing through the origin, the corner locus of g is a cone.

We have seen that a tropical curve is a graph in $\mathbb{R}^2$ whose edges are line segments. Let us consider Γ locally around a vertex $V \in \Gamma$. For simplicity we shift coordinates so that V is the origin in $\mathbb{R}^2$ and thus Γ becomes a cone locally around V. Then Γ is locally the corner locus of a tropical polynomial of the form (2.2)

Example 2.2.15. For convenient, we consider the tropical polynomial

$$g(x_1, x_2) = \max\{2x + 3y, 4x + y, 3x, x, 2y, 2x + y\}.$$

to be an example, and let

$$a^{(1)} = (2, 3), a^{(2)} = (4, 1), a^{(3)} = (3, 0), a^{(4)} = (1, 0), a^{(5)} = (0, 2), a^{(6)} = (2, 1).$$

Let Δ be the convex hull of the points $a^{(i)}$ and Γ be the tropical curve of g. We may discover that $a^{(6)}$ is irrelevant for the tropical curve Γ, since

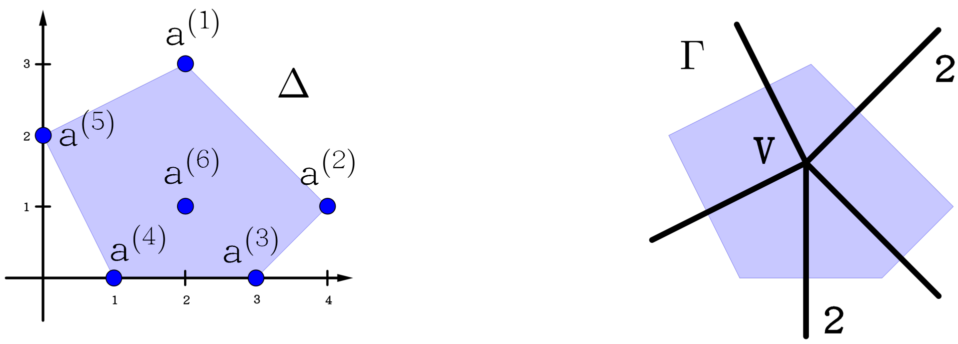


Figure 2.5: A local picture of a tropical curve

- for $x_1 > 0$, we have $a_1^{(2)}x_1 + a_2^{(2)}x_2 = 4x_1 + x_2 > 2x_1 + x_2 = a_1^{(6)}x_1 + a_2^{(6)}x_2$;
- for $x_2 > 0$, we have $a_1^{(1)}x_1 + a_2^{(1)}x_2 = 2x_1 + 3x_2 > 2x_1 + x_2 = a_1^{(6)}x_1 + a_2^{(6)}x_2$;
- for $x_1 < 0$, $x_2 < 0$, we have $a_1^{(4)}x_1 + a_2^{(4)}x_2 = x_1 > 2x_1 + x_2 = a_1^{(6)}x_1 + a_2^{(6)}x_2$.

Hence g and Γ remains the same if we drop this term.

In fact, it is impossible for any point $a^{(i)}$ which is not a vertex of Δ that the expression $a_1^{(i)}x_1 + a_2^{(i)}x_2$ is strictly bigger than all the other $a_1^{(j)}x_1 + a_2^{(j)}x_2$ for some $x_1, x_2 \in \mathbb{R}$.

It is now easy to see that the corner locus of g consists precisely of those points where

$$g(x_1, x_2) = a_1^{(i)}x_1 + a_2^{(i)}x_2 = a_1^{(j)}x_1 + a_2^{(j)}x_2$$

for two adjacent vertices $a^{(i)}$ and $a^{(j)}$ of Δ. For instance, if $g(x_1, x_2) = a_1^{(1)}x_1 + a_2^{(1)}x_2 = a_1^{(2)}x_1 + a_2^{(2)}x_2$ for some $x_1, x_2 \in \mathbb{R}$, then we have $x_1 = x_2, x_1 > 0, x_2 > 0$, i.e. the half-ray starting from the origin and pointing in the direction $(1, 1)$, which is the outward normal of the edge joining $a^{(1)}$ and $a^{(2)}$. By the same way, we will get the other four half-rays shown in Figure 2.5 on the right. The tropical curve Γ is simply the union of all these half-rays around V.

Remark 2.2.16. In particular, all edges of Γ have rational slopes, since each $a^{(i)}$ is in $\mathbb{N}^2$.

There is one more important condition on the edges of Γ around V, which is called the *balancing condition*.

If $a^{(1)}, \dots, a^{(n)}$ are the vertices of Δ in clockwise direction, then an outward normal vector of the edge joining $a^{(i)}$ and $a^{(i+1)}$ (where we set $a^{(n+1)} := a^{(1)}$) is $v^{(i)} := (a_2^{(i)} - a_2^{(i+1)}, a_1^{(i+1)} - a_1^{(i)})$ for $i = 1, \dots, n$. In particular, it follows that

$$\sum_{i=1}^{n} v^{(i)} = 0. \tag{2.3}$$

Let $u^{(i)}$ be the primitive integral vector in the direction of $v^{(i)}$ and $w^{(i)} \in \mathbb{N}_{>0}$ such that $v^{(i)} = w^{(i)} \cdot u^{(i)}$. We call $w^{(i)}$ the *weight* of the corresponding edge of Γ. Thus, we may consider Γ to be a weighted graph and rewrite (2.3) as

$$\sum_{i=1}^{n} w^{(i)} \cdot u^{(i)} = 0, \tag{2.4}$$

which states that the weighted sum of the primitive integral vectors of the edges around every vertex of Γ is 0.

Example 2.2.17. Let us continue the Example 2.2.15. The edges of Γ pointing upper-right and pointing down have weight 2 (since $v^{(1)} = (2, 2) = 2 \cdot (1, 1)$ and $v^{(3)} = (0, -2) = 2 \cdot (0, -1)$), whereas all other edges have weight 1. Then the balancing condition around the vertex V reads

$$2 \cdot (1, 1) + (1, -1) + 2 \cdot (0, -1) + (-2, -1) + (-1, 2) = (0, 0)$$

in this example.

Remark 2.2.18. In this thesis, we will usually label the edges with their corresponding weights unless these weights are 1.

Definition 2.2.19. The *(toric) degree* of a plane tropical curve Γ is a collection D of integral vectors such that: a positive multiple of an integral vector $u \in D$ if and only if there exists an end (i.e. an unbounded edge) of Γ which is in the direction of u. In such case, we include mu into D, where m is the sum of multiplicities of all such ends.

Example 2.2.20. Again in the Example 2.2.15, the degree of the plane tropical curve Γ is $\{2(1,1),(1,-1),2(0,-1),(-2,-1),(-1,2)\}$.

Definition 2.2.21. If the degree of a plane tropical curve Γ is $\{(-d,0),(0,-d),(d,d)\}$, then Γ is called a plane tropical curve *of degree d*.

Definition 2.2.22. A plane tropical curve of degree d is a weighted graph Γ in $\mathbb{R}^2$ such that

(a) every (bounded) edge of Γ is a line segment with rational slope;

(b) Γ has d ends each in the direction $(-1,0),(0,-1),(1,1)$ (where an end of weight w counts w times);

(c) at every vertex V of Γ the balancing condition holds: the weighted sum of the primitive integral vectors of the edges around V is zero.

Remark 2.2.23. Strictly speaking, we have only explained above why a plane tropical curve in the sense of Definition 2.2.11 gives rise to a curve in the sense of Definition 2.2.22. One can show that the converse holds as well; according to Andreas Gathmann [1], a proof can be found in [3] or [7] chapter 5.

Remark 2.2.24. With this definition it has now become a combinatorial problem to find all types of plane tropical curves of a given degree.

In fact, the construction given in Example 2.2.15 globalizes well. Assume that Γ is the tropical curve given as the corner locus of the tropical polynomial

$$g(x_1,x_2)=\max\{a_1^{(i)}x_1+a_2^{(i)}x_2+b^{(i)}\mid i=1,\dots,n\}$$

If g is the tropicalization of a polynomial of degree d, then the $a^{(i)}$ are all integer points in the triangle $\Delta_d:=\{(a_1,a_2)\in\mathbb{N}^2\mid a_1+a_2\le d\}$. Consider two terms $i,j\in\{1,\dots,n\}$ with $a^{(i)}\neq a^{(j)}$. If there is a point $(x_1,x_2)\in\mathbb{R}^2$ such that

$$g(x_1,x_2)=a_1^{(i)}x_1+a_2^{(i)}x_2+b^{(i)}=a_1^{(j)}x_1+a_2^{(j)}x_2+b^{(j)},$$

then we draw a straight line in Δ_d through the points $a^{(i)}$ and $a^{(j)}$. In this way, we obtain a subdivision of Δ_d whose edges correspond to the edges of Γ and whose 2-dimensional cells correspond to the vertices of Γ.

Figure 2.6: Tropical lines are "dual" to Δ_1

Definition 2.2.25. The subdivision obtained by the construction above is usually called *Newton subdivision* corresponding to Γ.

Definition 2.2.26. A plane tropical curve is called *smooth* if it is of degree d and its Newton subdivision is maximal (i.e. consists of d^2 triangles of area $\dfrac{1}{2}$ each).

Example 2.2.27. Figure 2.7 shows all types of smooth plane tropical curves of degree two.

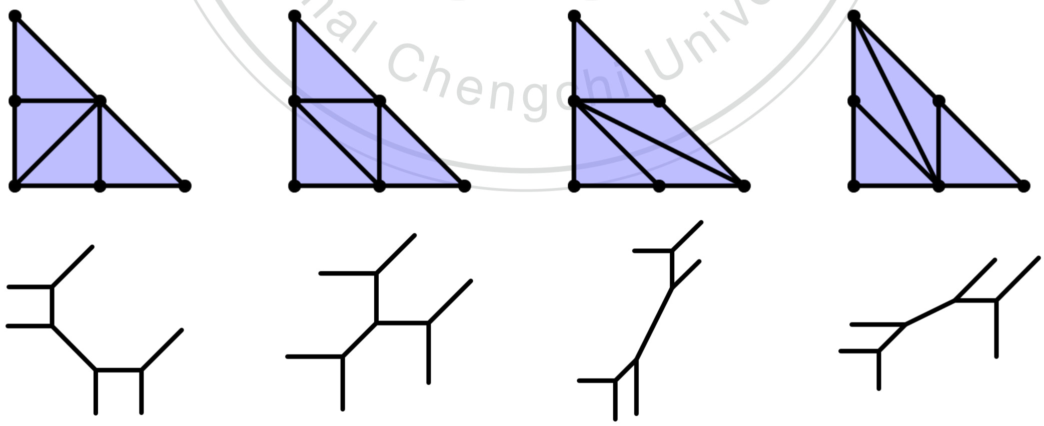


Figure 2.7: The four types of (smooth) tropical plane conic

Although it is a quite convenient way to draw a tropical curve by drawing its Newton subdivision first, there still have some problems. For example, not every subdivision gives rise to a type of tropical curves.

Example 2.2.28. Here is an example of subdivisions that is not induced by a tropical curve. In Figure 2.8 on the left, we see that the edge E_1 should meet E_2 at

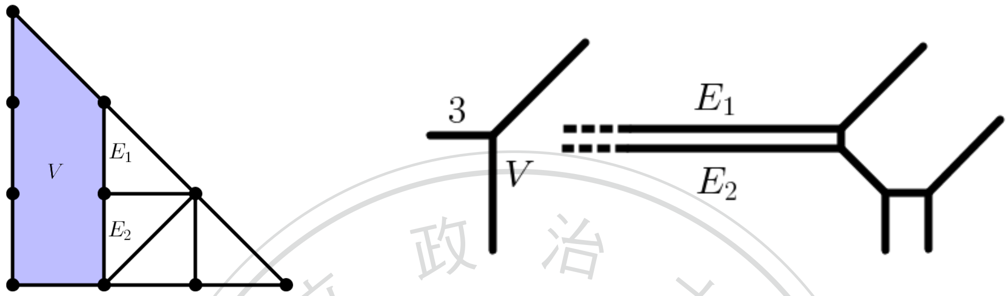


Figure 2.8: A subdivision that is not induced by a tropical curve

the vertex V, which is impossible (see Figure 2.8 on the right), since E_1 is parallel to E_2.

Definition 2.2.29. A subdivision that corresponds to a tropical curve is usually called a *regular polyhedral subdivision.*

Proposition 2.2.30. If a Newton subdivision is maximal, then it must be a regular polyhedral subdivision.

2.3 Tropical factorization

Definition 2.3.1. Let g be a tropical polynomial. If a tropical curve Γ is a corner locus of g, then we say that Γ is a tropical curve of g, and denote it by $\mathcal{T}(g)$.

Theorem 2.3.2. Let g_1, g_2 be two tropical polynomials. We have

$$\mathcal{T}(g_1 \odot g_2) = \mathcal{T}(g_1) \cup \mathcal{T}(g_2).$$

i.e. The tropical curve of $g_1 \odot g_2$ is exactly the union of the tropical curves of g_1 and g_2. In particular, the union of two plane tropical curves of degree d_1 and d_2 is always a plane tropical curve of degree $d_1 + d_2$

Example 2.3.3. Let $g_1(x, y) = (-3) \odot x \oplus (-1) \odot y \oplus 0$ and $g_2(x, y) = 1 \odot x \oplus 1 \odot y \oplus 0$. We have

$$g_1(x, y) \odot g_2(x, y) = (-2) \odot x^{\odot 2} \oplus (x \odot y) \oplus y^{\odot 2} \oplus 1 \odot x \oplus 1 \odot y \oplus 0.$$

In Figure 2.9, we see that $\mathcal{T}(g_1 \odot g_2)$ is indeed the union of $\mathcal{T}(g_1)$ and $\mathcal{T}(g_2)$.

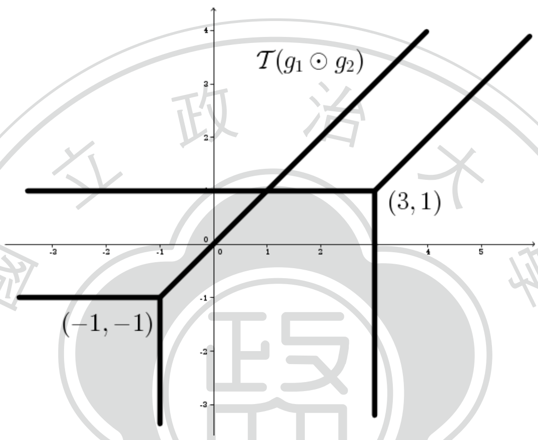


Figure 2.9: $\mathcal{T}(g_1 \odot g_2) = \mathcal{T}(g_1) \cup \mathcal{T}(g_2)$

Corollary 2.3.4. Let Γ be a tropical curve of degree ≥ 2. If Γ is an union of two tropical curves Γ_1 and Γ_2 of degree lower than Γ, then there exists two tropical polynomials g_1 and g_2 with $\Gamma_1 = \mathcal{T}(g_1)$ and $\Gamma_2 = \mathcal{T}(g_2)$, resp, such that $\Gamma = \mathcal{T}(g_1 \odot g_2)$

Example 2.3.5. Let $g(x, y) = x \oplus y \oplus 0$. We now consider the tropical square of this polynomial

$$\begin{aligned} g(x, y) \odot g(x, y) &= x^{\odot 2} \oplus (x \odot y) \oplus y^{\odot 2} \oplus x \oplus y \oplus 0 \\ &= \max\{2x, x + y, 2y, x, y, 0\}, \end{aligned}$$

then the tropical curve determined by this polynomial is still the same as g (but with weight 2). But as piecewise linear maps the function $g(x, y) \odot g(x, y)$ is the same as

$$\max\{2x, 2y, 0\} = x^{\odot 2} \oplus y^{\odot 2} \oplus 0,$$

and this tropical polynomial cannot be written as a product of two linear tropical polynomials.

From Example 2.3.5, we know that the reducibility of tropical polynomials (of degree 2) and of plane tropical curves may not be the same.

Definition 2.3.6. Two tropical polynomials are said to be equivalent($\sim$) if their tropical curves are the same.

It is easy to see that this equivalence is an equivalence relation. Hence we may define the equivalence class of a tropical polynomial g with respect to $\sim$, and denote it by $\overline{g}$.

Now, we may introduce the definition of *maximal coefficients* of a tropical polynomial.

Definition 2.3.7. A coefficient a_{ij} of a tropical polynomial $g(x, y)$ is a maximal coefficient if for any $b \in \mathbb{Q}$ with $b > a_{ij}$, the tropical polynomial $h(x, y)$ formed by replacing a_{ij} with b is not equivalent to $g(x, y)$.

Definition 2.3.8. A tropical polynomial is said to be *maximally represented* if all its coefficients are maximal coefficients.

Remark 2.3.9. If $g(x, y)$ is a tropical polynomial that $\mathcal{T}(g)$ is smooth, then g must be maximally represented.

For any tropical polynomial $g(x, y)$, the maximally represented polynomial of g may be unique, but not for the equivalence class $\overline{g}$, see the following example.

Example 2.3.10. Let us consider the following three polynomials:

$$g_1(x, y) = 1 \odot x \oplus 1 \odot y \oplus 0,$$

$$g_2(x, y) = 6 \odot x \oplus 6 \odot y \oplus 5,$$

and

$$g_3(x, y) = 1 \odot x^{\odot 2} \oplus 1 \odot (x \odot y) \oplus x.$$

In Figure 2.10, we see that g_1, g_2 and g_3 are not the same piecewise linear functions, but with the same corner locus. Since tropical lines are smooth, these three polynomials are all the maximally represented polynomials.

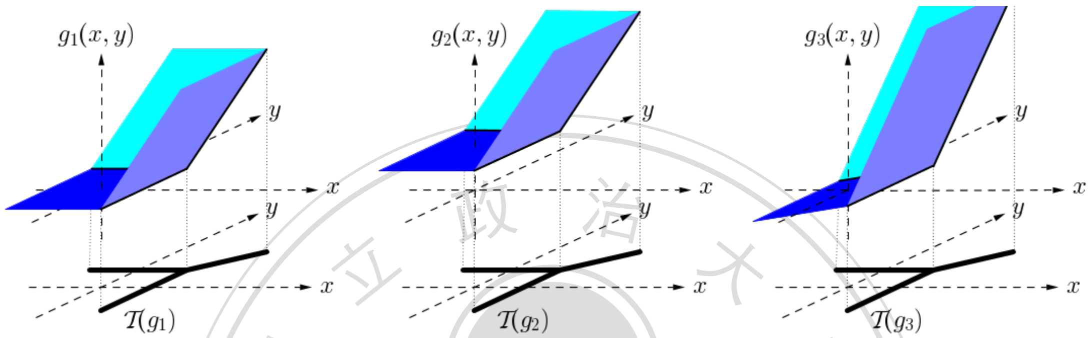


Figure 2.10:

Although the maximally represented polynomials of $\overline{g}$ are not unique, we discover the relation between them,

$$g_2(x, y) = 5 \odot g_1(x, y)$$

and

$$g_3(x, y) = x \odot g_1(x, y),$$

which would lead us to the following proposition.

Proposition 2.3.11. Let $g(x, y)$ be a tropical polynomial. Then we have

$$\mathcal{T}(g) = \mathcal{T}(g \odot (a \odot x^{\odot b} \odot y^{\odot c}))$$

where $a \in \mathbb{Q}, b, c \in \mathbb{N}$.

With Proposition 2.3.11, we have the "uniqueness" of the maximally represented polynomial of $\overline{g}$.

Chapter 3

Recovering Tropical Polynomials from Tropical curves

In this chapter, we will introduce algorithms to recover the tropical polynomial from a given tropical curve.

3.1 Tropical curves of degree two

Theorem 3.1.1. Let Γ be a smooth plane tropical curve of degree two. Then Γ can be represented as a corner locus of the tropical polynomial which is a product of two linear tropical polynomials plus a certain tropical polynomial.

Before the proof of Theorem 3.1.1, We first consider a tropical curve Γ locally around a vertex $V \in \Gamma$ in the following two cases.

Example 3.1.2. For convenience, we let $g(x,y) = (-5) \odot x \oplus (-4) \odot y \oplus 0$. If we add the coefficient of the constant term by -2, then we have

$$\begin{aligned} h(x,y) &:= (-5) \odot x \oplus (-4) \odot y \oplus (-2) \\ &= (-2) \odot ((-3) \odot x \oplus (-2) \odot y \oplus 0) \\ &\sim (-3) \odot x \oplus (-2) \odot y \oplus 0, \end{aligned}$$

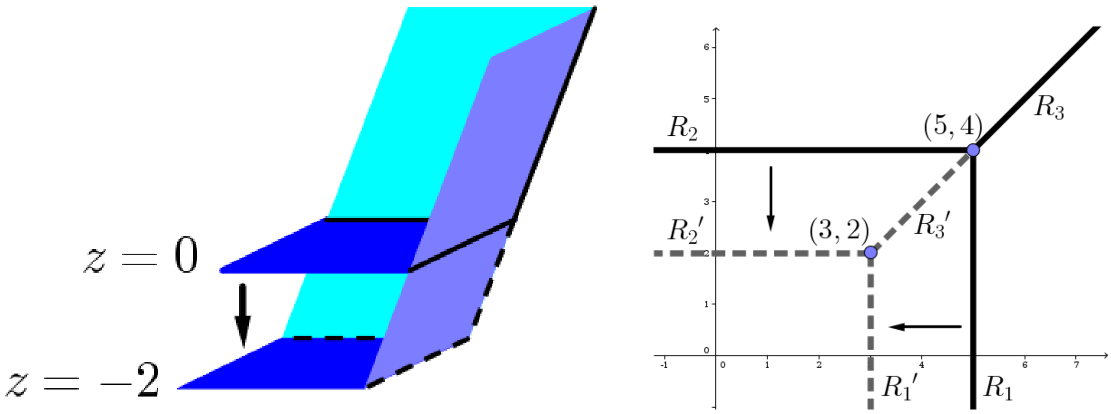


Figure 3.1: The tropical line shift to right by 2

which means that the tropical line at $(5,4)$ shift to the tropical line at $(3,2)$.

From the construction of Newton subdivisions in Section 2.2, we know the ray R_1 shown in Figure 3.1 is determined by $g(x,y) = x - 5 = 0$, where

$$\begin{aligned} g(x,y) = x - 5 = 0 \;&\Leftrightarrow\; \max\{x-5, y-4, 0\} = x - 5 = 0 \\ &\Leftrightarrow\; y - 4 \le x - 5 = 0 \\ &\Leftrightarrow\; x = 5,\; y \le 4. \end{aligned}$$

If we add the coefficient of constant term by -2, then the ray R_1 moves to $R_1' = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) = x - 5 = -2\}$, where

$$\begin{aligned} h(x,y) = x - 5 = -2 \;&\Leftrightarrow\; \max\{x-5, y-4, -2\} = x - 5 = -2 \\ &\Leftrightarrow\; y - 4 \le x - 5 = -2 \\ &\Leftrightarrow\; x = 3,\; y \le 2. \end{aligned}$$

There are similar results of R_2 and R_3, which move to R_2' and R_3', respectly.

Example 3.1.3. Let $g(x,y) = (x \odot y) \oplus x \oplus y \oplus 0$. The corresponding tropical curve is in Figure 3.2.

Since a tropical curve $\mathcal{T}(g)$, for $g(x,y) = \max\{a_1^{(i)}x + a_2^{(i)}y + b^{(i)} \,|\, i = 1, \ldots, n\}$, is the union of all these rays

$$\{(x,y) \in \mathbb{R}^2 \,|\, g(x,y) = a_1^{(i)}x + a_2^{(i)}y + b^{(i)} = a_1^{(j)}x + a_2^{(j)}y + b^{(j)}\}$$

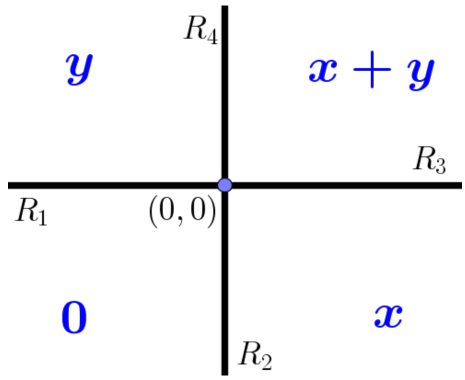


Figure 3.2: The tropical curve of $g(x, y) = (x \odot y) \oplus x \oplus y \oplus 0$

where $i, j = 1, \ldots, n$, $i \neq j$. In this example, the rays is determined by these four planes: $z = 0$, $z = x$, $z = y$, and $z = x+y$. So if we add a negative number, e.g. -2, to the coefficient of constant term of g, we will have a similar result in Example 3.1.2 that the rays R_1 and R_2 move to R_1' and R_2', respectly. Furthermore, we have a new edge E_1 determined by $h_1(x, y) = x = y$, where $h_1(x, y) := (x \odot y) \oplus x \oplus y \oplus (-2)$. Explicitly,

$$\begin{aligned} h_1(x, y) = x = y \quad &\Leftrightarrow \quad \max\{x + y, x, y, -2\} = x = y \\ &\Leftrightarrow \quad x + y \leq x = y, \;\; -2 \leq x = y \\ &\Leftrightarrow \quad x = y, \;\; -2 \leq x, y \leq 0, \end{aligned}$$

which implies the line segment E_1 shown in Figure 3.3 on the left.

Now if we add a positive number, e.g. 3, to the coefficient of constant term of g, then we will have the result shown in Figure 3.3 on the right. The edge E_2 is determined by $h_2(x, y) = x+y = 3$, where $h_2(x, y) := (x \odot y) \oplus x \oplus y \oplus 3$. Explicitly,

$$\begin{aligned} h_2(x, y) = x + y = 3 \quad &\Leftrightarrow \quad \max\{x + y, x, y, 3\} = x + y = 3 \\ &\Leftrightarrow \quad x \leq x + y = 3, \;\; y \leq x + y = 3 \\ &\Leftrightarrow \quad x + y = 3, \;\; 0 \leq x, y \leq 3, \end{aligned}$$

which implies the line segment E_2.

Remark 3.1.4. One may observe that the "weight" of E_1 is just the added number $|-2| = 2$, where $(0, 0) - (-2, -2) = (2, 2) = 2(1, 1)$. And the "weight" of E_2 is just

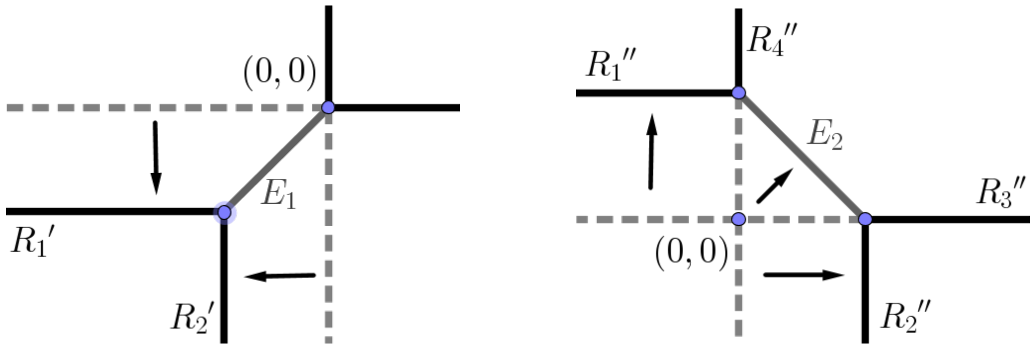


Figure 3.3: The effect of adding numbers to the coefficient of constant term

the number 3, where $(3,0)-(0,3)=(3,-3)=3(1,-1)$.

In Example 3.1.3, we show the effect on tropical curves that tuning a coefficient of the corresponding tropical polynomials.

Let us beginning the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Since there are just four types of smooth plane tropical curves of degree two, we will prove this theorem by cases.

case 1.

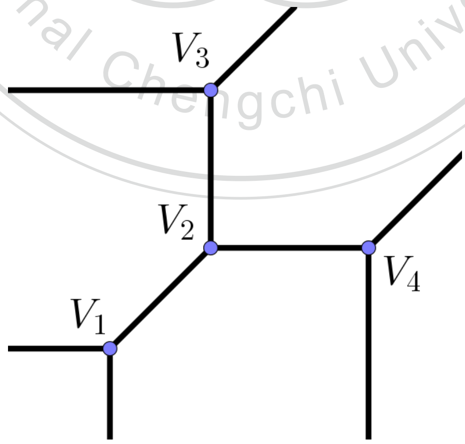


Figure 3.4:

Let $V_i=(v_1^{(i)},v_2^{(i)})$, $i=1,2,3,4$, and g be the corresponding tropical polynomial of this tropical curve. In this case, we may observe that the local graphs around V_1, V_3, and V_4 are locally tropical lines. If we "push" the vertex V_4 to V_2 (which actually means that adding a positive number to the coefficient of the $x^{\odot 2}$-term of g so that such tropical line would shift to V_2), the resulting curve would become an union of two tropical lines, i.e. the tropical line at V_1 and the tropical line at V_3.

The number we should add to $x^{\odot 2}$-term of g is $c_1 = v_1^{(4)} - v_1^{(2)}$ in this case, so that the vertex V_4 would move to V_2, and the curve becomes the union of two tropical lines.

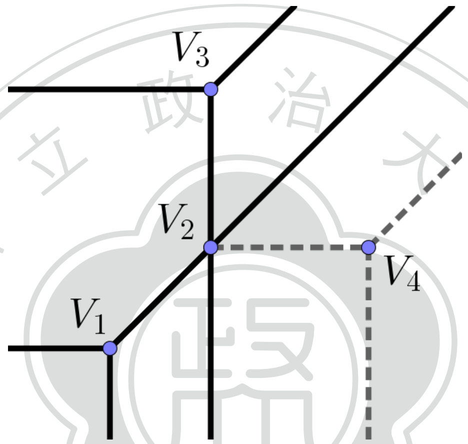


Figure 3.5:

Conversely, if we substrct the $x^{\odot 2}$-term of the tropical polynomial of this union by c_1, then we will get the polynomial that corresponds to the original tropical curve. The polynomial of this union is

$$\begin{aligned} G(x,y) &:= ((-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0) \odot ((-v_1^{(3)}) \odot x \oplus (-v_2^{(3)}) \odot y \oplus 0) \\ &= ((-v_1^{(1)}) \odot (-v_1^{(3)})) \odot x^{\odot 2} \oplus ((-v_2^{(1)}) \odot (-v_1^{(3)})) \odot (x \odot y) \\ &\quad \oplus ((-v_2^{(1)}) \odot (-v_2^{(3)})) \odot y^{\odot 2} \oplus (-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0. \end{aligned}$$

Next, we do the substraction to the $x^{\odot 2}$-term by adding the number to the other terms. For example, if we want to substract the x-term of $3 \odot x \oplus 4 \odot y \oplus 0$ by 2, then due to the equivalence $\sim$, we may write the substraction as

$$\begin{aligned}(3\odot x\oplus 4\odot y\oplus 0)\oplus(6\odot y\oplus 2) &= 3\odot x\oplus 6\odot y\oplus 2\\ &= 2\odot(1\odot x\oplus 4\odot y\oplus 0)\\ &\sim 1\odot x\oplus 4\odot y\oplus 0.\end{aligned}$$

By this way, we add the coefficients of all the other terms but $x^{\odot 2}$-term by c_1, then we have

$$\begin{aligned}g(x,y)=G(x,y) \quad\oplus\quad & (((-v_2^{(1)})\odot(-v_1^{(3)})\odot c_1)\odot(x\odot y)\\ &\oplus((-v_2^{(1)})\odot(-v_2^{(3)})\odot c_1)\odot y^{\odot 2}\\ &\oplus((-v_1^{(1)})\odot c_1)\odot x\\ &\oplus((-v_2^{(1)})\odot c_1)\odot y\oplus c_1).\end{aligned}$$

case 2.

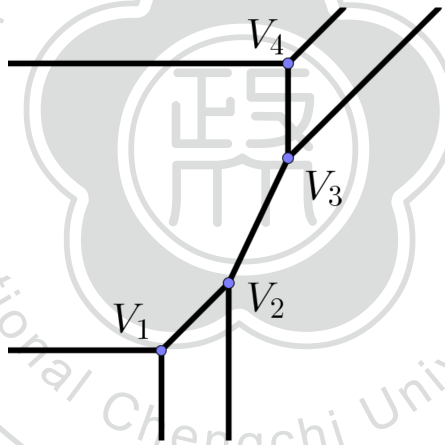


Figure 3.6:

Let $V_i=(v_1^{(i)},v_2^{(i)})$, $i=1,2,3,4$, and g be the corresponding tropical polynomial of this tropical curve. In this case, we may observe that the local graphs around V_1 and V_4 are locally tropical lines. From our experience, if we add $c_2=v_1^{(3)}-v_1^{(2)}$ to the coefficient of $x^{\odot 2}$-term of

$$\begin{aligned}G(x,y) &:= ((-v_1^{(1)})\odot x\oplus(-v_2^{(1)})\odot y\oplus 0)\odot((-v_1^{(4)})\odot x\oplus(-v_2^{(4)})\odot y\oplus 0)\\ &= ((-v_1^{(1)})\odot(-v_1^{(4)}))\odot x^{\odot 2}\oplus((-v_2^{(1)})\odot(-v_1^{(4)}))\odot(x\odot y)\\ &\quad\oplus((-v_2^{(1)})\odot(-v_2^{(4)}))\odot y^{\odot 2}\oplus(-v_1^{(1)})\odot x\oplus(-v_2^{(1)})\odot y\oplus 0,\end{aligned}$$

then we will get the polynomial corresponding to the original tropical curve, i.e.

$$g(x,y) = h(x,y) \oplus (((-v_1^{(1)}) \odot (-v_1^{(4)}) \odot c_2) \odot x^{\odot 2}).$$

case 3.

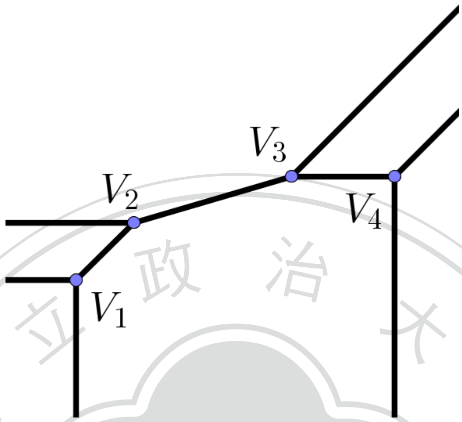


Figure 3.7:

Let $V_i = (v_1^{(i)}, v_2^{(i)})$, $i = 1, 2, 3, 4$, and g be the corresponding tropical polynomial of this tropical curve.

This case is similar to case 2. We first compute the polynomial of the union of the tropical curve at V_1 and the tropical curve at V_4,

$$\begin{aligned} G(x,y) &:= ((-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0) \odot ((-v_1^{(4)}) \odot x \oplus (-v_2^{(4)}) \odot y \oplus 0) \\ &= ((-v_1^{(1)}) \odot (-v_1^{(4)})) \odot x^{\odot 2} \oplus ((-v_1^{(1)}) \odot (-v_2^{(4)})) \odot (x \odot y) \\ &\quad \oplus ((-v_2^{(1)}) \odot (-v_2^{(4)})) \odot y^{\odot 2} \oplus (-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0. \end{aligned}$$

Second, we add $c_3 = v_2^{(3)} - v_2^{(2)}$ to the coefficient of the $y^{\odot 2}$-term of $G(x,y)$, then we will have the desired polynomial

$$g(x,y) = G(x,y) \oplus (((-v_2^{(1)}) \odot (-v_2^{(4)}) \odot c_3) \odot y^{\odot 2}).$$

case 4.

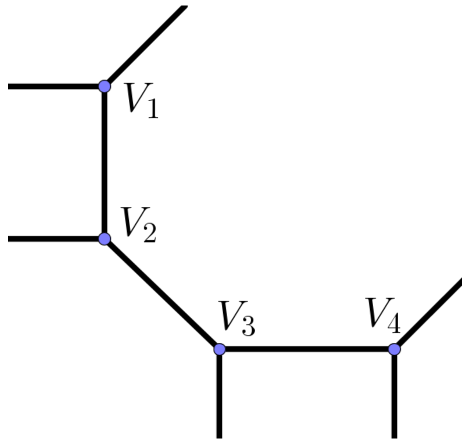


Figure 3.8:

Let $V_i = (v_1^{(i)}, v_2^{(i)})$, $i = 1, 2, 3, 4$, and g be the corresponding tropical polynomial of this tropical curve.

This case is also similar to case 2. We first compute the polynomial of the union of the tropical curve at V_1 and the tropical curve at V_4,

$$\begin{aligned} G(x,y) &:= ((-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0) \odot ((-v_1^{(4)}) \odot x \oplus (-v_2^{(4)}) \odot y \oplus 0) \\ &= ((-v_1^{(1)}) \odot (-v_1^{(4)})) \odot x^{\odot 2} \oplus ((-v_1^{(1)}) \odot (-v_2^{(4)})) \odot (x \odot y) \\ &\quad \oplus ((-v_2^{(1)}) \odot (-v_2^{(4)})) \odot y^{\odot 2} \oplus (-v_1^{(1)}) \odot x \oplus (-v_2^{(4)}) \odot y \oplus 0. \end{aligned}$$

Second, we add $c_4 = v_1^{(3)} - v_1^{(2)}$ to the coefficient of the constant term of $G(x,y)$, then we will have the desired polynomial

$$g(x,y) = G(x,y) \oplus c_4.$$

By these four cases, the proof is completed.

$\square$

Algorithm 3.1.5. Here we give an algorithm to recover the polynomial of a given smooth plane tropical curve of degree two.

Now, given a smooth plane tropical curve of degree two, denoted by Γ.

1. Choose two vertices V_1 and V_2 of Γ that are locally tropical lines, and let V_3 and V_4 be the last two vertices.

2. Compute the product of the polynomials of these two tropical lines, and denote it by $G(x, y)$.

3. Let V_5 be the intersection point of these two tropical lines. Let v_1 be the vector from V_5 to V_3, and v_2 be the vector from V_5 to V_4. Then we compute the vector $v = v_1 + v_2$.

4. Write $v = w \cdot u$, where $u = (u_1, u_2)$ is a primitive integral vector, and w is a positive rational weight. For example, $(-2, 0) = 2 \cdot (-1, 0)$ and $(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \cdot (1, 1)$.

5. (1) If $v = (-1, -1)$, then we substract the coefficient of constant term of $G(x, y)$ by w;

   (2) if $v = (1, 1)$, then we add the coefficient of constant term of $G(x, y)$ by w;

   (3) if $v = (-1, 0)$, then we add the coefficient of $x^{\odot 2}$-term of $G(x, y)$ by w;

   (4) if $v = (1, 0)$, then we substract the coefficient of $x^{\odot 2}$-term of $G(x, y)$ by w;

   (5) if $v = (0, -1)$, then we add the coefficient of $y^{\odot 2}$-term of $G(x, y)$ by w;

   (6) if $v = (0, 1)$, then we substract the coefficient of $y^{\odot 2}$-term of $G(x, y)$ by w;

After these five steps, we will get the desired polynomial corresponding to Γ.

Example 3.1.6. Given a plane tropical curve of degree 2 as in Figure 3.9.

With some observations, we may discover that the graphs locally around $(1, 1)$, $(2, 5)$, and $(4, 2)$ are tropical lines. Then we first compute the polynomial of the union of the tropical line at $(1, 1)$ and the tropical line at $(2, 5)$, i.e.

$$\begin{aligned} h(x, y) &:= ((-1) \odot x \oplus (-1) \odot y \oplus 0) \odot ((-2) \odot x \oplus (-5) \odot y \oplus 0) \\ &= (-3) \odot x^{\odot 2} \oplus (-3) \odot x \odot y \oplus (-6) \odot y^{\odot 2} \oplus (-1) \odot x \oplus (-1) \odot y \oplus 0. \end{aligned}$$

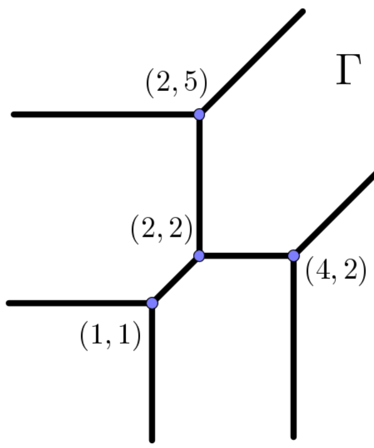


Figure 3.9: An example of smooth plane tropical curves of degree 2

Next, the vector stated in Algorithm 3.1.5 is

$$v := [(2,2) - (2,2)] + [(4,2) - (2,2)] = (2,0) = 2 \cdot (1,0).$$

So, by Algorithm 3.1.5, we substract the coefficient of $x^{\odot 2}$-term of $h(x,y)$ by 2. Thus, the desired polynomial is

$$(-5) \odot x^{\odot 2} \oplus (-3) \odot x \odot y \oplus (-6) \odot y^{\odot 2} \oplus (-1) \odot x \oplus (-1) \odot y \oplus 0$$

Example 3.1.7. Given a plane tropical curve of degree 2 as in Figure 3.10.

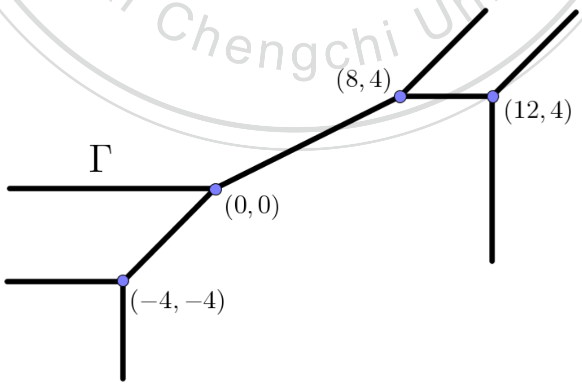


Figure 3.10: An example of smooth plane tropical curves of degree 2

By the similar way in Example 3.1.6, we first compute the polynomial of the union of the tropical line at $(-4,-4)$ and the tropical line at $(12,4)$, i.e.

$$\begin{aligned} h(x,y) &:= (4 \odot x \oplus 4 \odot y \oplus 0) \odot ((-12) \odot x \oplus (-4) \odot y \oplus 0) \\ &= (-8) \odot x^{\odot 2} \oplus x \odot y \oplus y^{\odot 2} \oplus 4 \odot x \oplus 4 \odot y \oplus 0. \end{aligned}$$

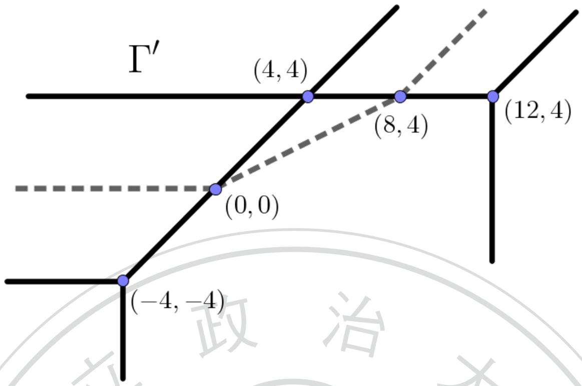


Figure 3.11: The union of $\mathcal{T}(4 \odot x \oplus 4 \odot y \oplus 0)$ and $\mathcal{T}((-12) \odot x \oplus (-4) \odot y \oplus 0)$

Next, the vector stated in Algorithm 3.1.5 is

$$v := [(0,0) - (4,4)] + [(8,4) - (4,4)] = (0,-4) = 4 \cdot (0,-1).$$

So, by Algorithm 3.1.5, we add the coefficient of $y^{\odot 2}$-term of $h(x,y)$ by 4. Thus, the desired polynomial is

$$(-8) \odot x^{\odot 2} \oplus x \odot y \oplus 4 \odot y^{\odot 2} \oplus 4 \odot x \oplus 4 \odot y \oplus 0.$$

3.2 Tropical curves of degree three

Definition 3.2.1. A maximal Newton subdivision of degree three is said to be *normal* if it is not one of the types shown in Figure 3.12.

Proposition 3.2.2. Let Δ be a Newton subdivision of degree three which is maximal. If Δ is normal, then there must be a maximal Newton subdivision of degree two as a subgraph of Δ.

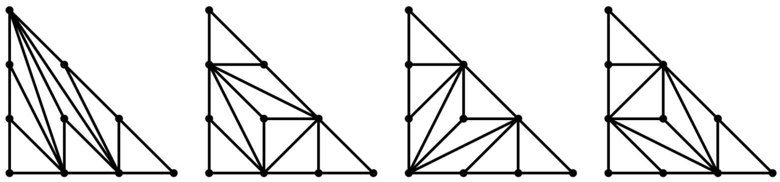


Figure 3.12: The special four types

Proof. There are only the four types of Newton subdivisions shown in Figure 3.12 that does not have a maximal subdivision of degree two as a subgraph, we may check in Appendix A for all types of maximal subdivisions of degree three.

□

Theorem 3.2.3. Let Γ be a smooth plane tropical curve of degree three. Then Γ can be represented as a corner locus of the tropical polynomial which is a product of three linear tropical polynomials plus a certain tropical polynomial.

We leave the proof to the end of Section 4.2.

Algorithm 3.2.4. Here we give an algorithm to recover the polynomial of a given smooth plane tropical curve of degree three.

Now, given a smooth plane tropical curve of degree three, denoted by Γ. Let Δ be the Newton subdivision corresponding to Γ.

case 1. Δ is normal.

1. Up to isomorphic, we may just consider the six types shown in Figure 3.13.

2. Let Δ_2 be the subdivision of degree two which is a subgraph of Δ. Let Δ_1 be a subdivision of degree one which is also a subgraph of Δ but not a subgraph of Δ_2. Let Γ_1 and Γ_2 be the local graph of Γ corresponding to Δ_1 and Δ_2, respectly.

3. Compute the corresponding polynomials of Γ_1 and Γ_2, and then compute their product, we denote the product by $G(x, y)$.

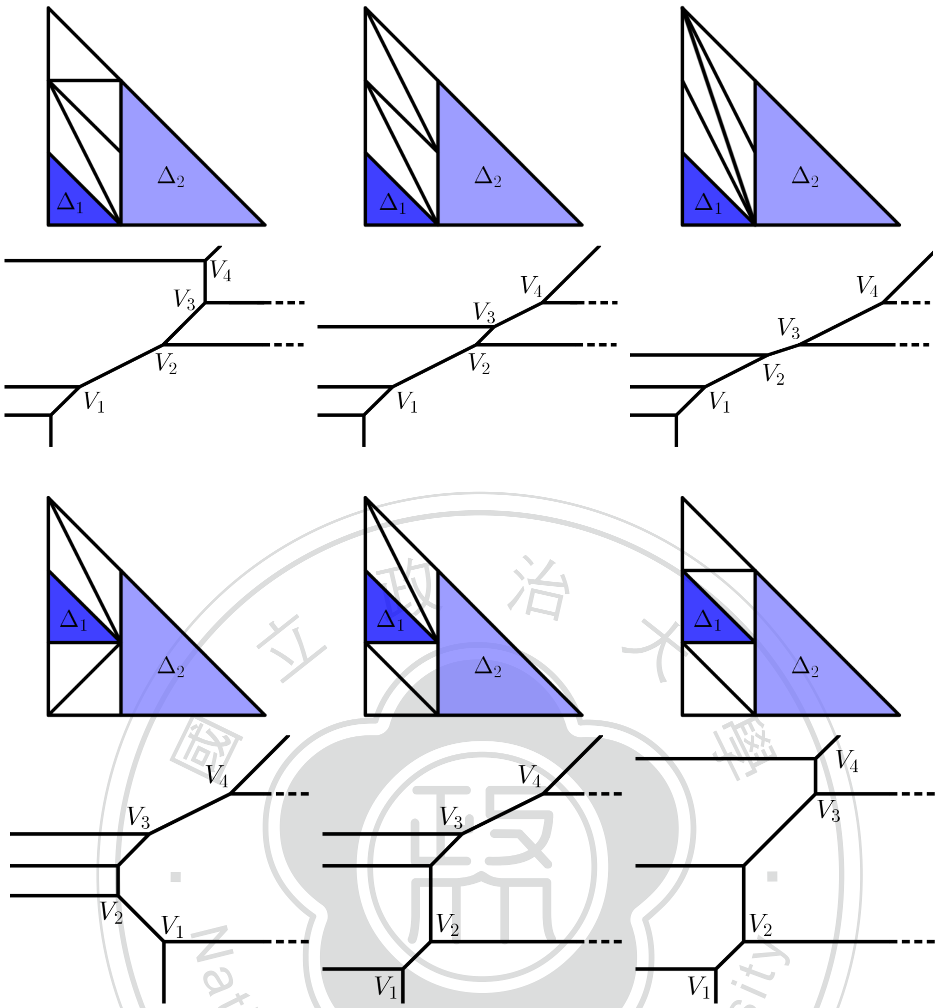


Figure 3.13:

4. Let $V_i = (v_1^{(i)}, v_2^{(i)})$, $i = 1, 2, 3, 4$.

   (1) For the first type in Figure 3.13, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(3)}$. We add the coefficient of the $y^{\odot 2}$-term of $G(x, y)$ by c_1, and add the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by $c_1 - c_2$.

   (2) For the second type, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(3)}$. We add the coefficient of the $y^{\odot 2}$-term of $G(x, y)$ by c_1, and add the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by $c_1 + c_2$.

   (3) For the third type, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(2)}$. We add the

coefficient of the $y^{\odot 2}$-term of $G(x, y)$ by c_1, and add the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by $c_1 + c_2$.

(4) For the forth type, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(3)}$. We add the coefficient of the constant term of $G(x, y)$ by c_1, and add the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by c_2.

(5) For the fifth type, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(3)}$. We substract the coefficient of the constant term of $G(x, y)$ by c_1, and add the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by c_2.

(6) For the sixth type, let $c_1 = v_2^{(2)} - v_2^{(1)}$, and $c_2 = v_2^{(4)} - v_2^{(3)}$. We substract the coefficient of the constant term of $G(x, y)$ by c_1, and substract the coefficient of the $y^{\odot 3}$-term of $G(x, y)$ by c_2.

After these steps, we will get the desired polynomial corresponding to Γ.

case 2. Δ is not normal.

1. Up to isomorphic, we may just consider the two types shown in Figure 3.14.

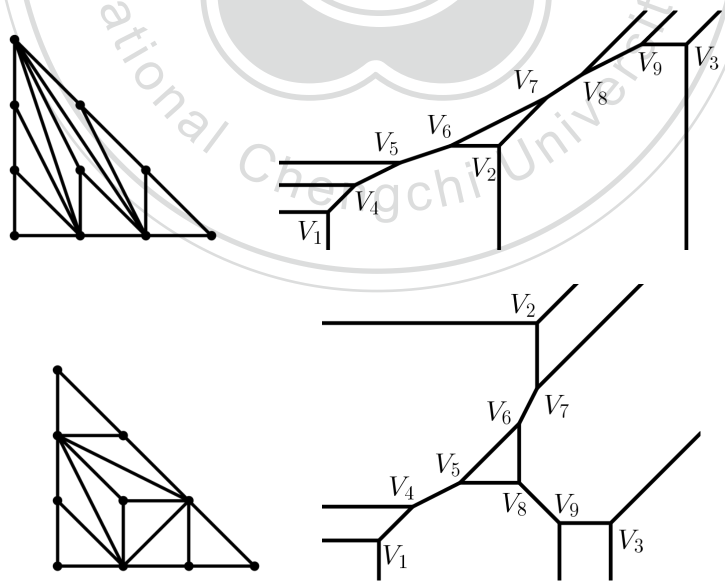


Figure 3.14:

2. For the first type in Figure 3.14,

   (1) we observe that the graphs locally around V_1, V_2, and V_3 are tropical lines. Let Γ_1, Γ_2, and Γ_3 be the tropical lines whose vertices are at V_1, V_2, and V_3, respectly.

   (2) Find out the tropical polynomials of Γ_1, Γ_2, and Γ_3. Then, compute their product, and denote it by $G(x, y)$.

   (3) Let $c_1 = v_2^{(2)} - v_2^{(4)}$, $c_2 = v_1^{(2)} - v_2^{(2)} - v_1^{(9)} + v_2^{(9)}$, and $c_3 = c_1 + c_2 + \frac{1}{2}(v_2^{(7)} + v_2^{(8)}) - v_2^{(5)}$. Next, we add the coefficients of the $y^{\odot 2}$-term, the $x \odot y^{\odot 2}$-term, and the $y^{\odot 3}$-term of $G(x, y)$ by c_1, c_2, and c_3, respectly. And the resulting polynomial is the desired tropical polynomial.

3. For the second type in Figure 3.14,

   (1) let $w_1 = v_2^{(5)} - v_2^{(4)}$, $w_2 = v_2^{(7)} - v_2^{(6)}$, and $w_3 = v_2^{(8)} - v_2^{(9)}$. Let $c_1 = \frac{1}{2}(w_1 + w_2 + w_3) - w_2$, $c_2 = \frac{1}{2}(w_1 + w_2 + w_3) - w_3$, and $c_3 = \frac{1}{2}(w_1 + w_2 + w_3) - w_1$.

   (2) Let $V_1' = (v_1^{(1)} + c_1, v_2^{(1)})$, $V_2' = (v_1^{(2)} - c_2, v_2^{(2)} - c_2)$, and $V_3' = (v_1^{(3)}, v_2^{(3)} + c_3)$. Let g_1, g_2, and g_3 be the tropical polynomials corresponding to the tropical lines at V_1', V_2', and V_3'.

   (3) Compute $G(x, y) = g_1(x, y) \odot g_2(x, y) \odot g_3(x, y)$.

   (4) We add the coefficients of the x-term, the $y^{\odot 2}$-term, and the $x^{\odot 2} \odot y$-term of $G(x, y)$ by c_1, c_2, and c_3, respectly. And the resulting polynomial is the desired tropical polynomial.

Example 3.2.5. Let us consider the curve in Figure 3.15:

We see in Figure 3.16 that the corresponding subdivision is normal and is the forth type in Figure 3.13, so we compute the polynomials of Γ_1 and Γ_2, and denote them by g_1 and g_2, respectly. Next, we compute $G(x, y) = g_1(x, y) \odot g_2(x, y)$.

Since Γ_1 is a tropical line at $(-2, -2)$, we have

$$g_1(x, y) = 2 \odot x \oplus 2 \odot y \oplus 0.$$

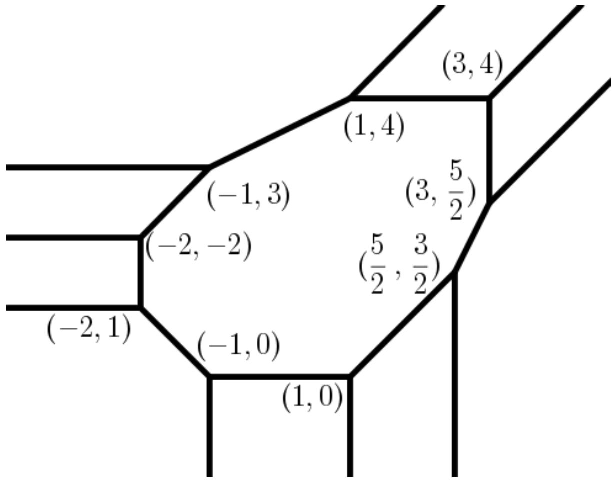


Figure 3.15:

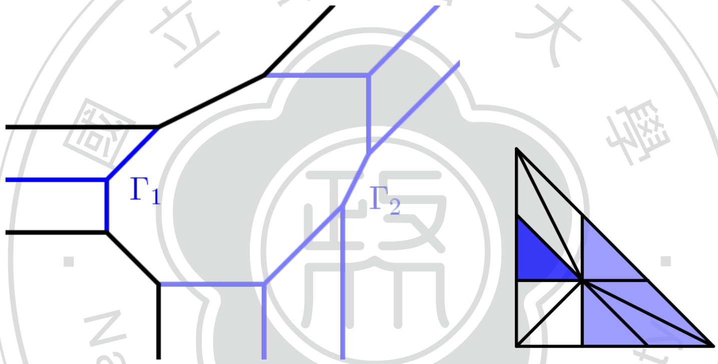


Figure 3.16:

By Algorithm 3.1.5, we have

$$g_2(x,y) = (-\frac{7}{2}) \odot x^{\odot 2} \oplus (-3) \odot x \odot y \oplus (-4) \odot y^{\odot 2} \oplus (-1) \odot x \oplus y \oplus 0.$$

Next, we have the product

$$\begin{aligned} G(x,y) &= g_1(x,y) \odot g_2(x,y) \\ &= (-\frac{3}{2}) \odot x^{\odot 3} \oplus (-1) \odot x^{\odot 2} \odot y \oplus (-1) \odot x \odot y^{\odot 2} \oplus (-2) \odot y^{\odot 3} \\ &\quad \oplus 1 \odot x^{\odot 2} \oplus 2 \odot x \odot y \oplus 2 \odot y^{\odot 2} \oplus 2 \odot x \oplus 2 \odot y \oplus 0. \end{aligned}$$

Let $c_1 = 1 - 0 = 1$ and $c_2 = 4 - 3 = 1$. So we add the coefficients of constant term and the $y^{\odot 3}$-term of $G(x,y)$ by c_1 and c_2, respectly. Thus, we have the desired polynomial

$$
\begin{aligned}
&(-\frac{3}{2}) \odot x^{\odot 3} \oplus (-1) \odot x^{\odot 2} \odot y \oplus (-1) \odot x \odot y^{\odot 2} \oplus (-1) \odot y^{\odot 3} \\
&\oplus 1 \odot x^{\odot 2} \oplus 2 \odot x \odot y \oplus 2 \odot y^{\odot 2} \oplus 2 \odot x \oplus 2 \odot y \oplus 1.
\end{aligned}
$$

Chapter 4

Recovering Tropical Polynomials from Newton Subdivisions

4.1 Newton subdivisions of degree two

Lemma 4.1.1. Each (maximal) Newton subdivision of degree two has two subdivisions of degree one as subgraphs.

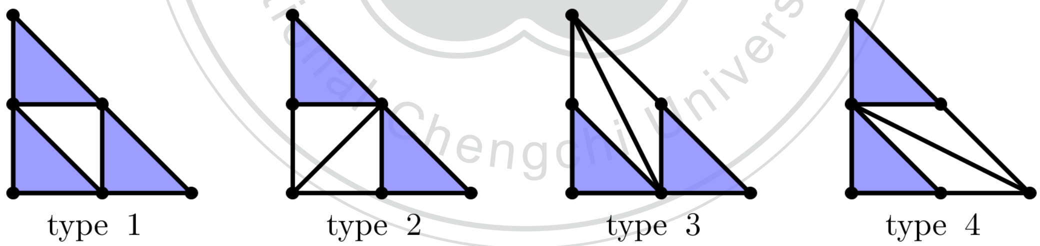


Figure 4.1: Four type of Newton subdivision of degree two

Theorem 4.1.2. For a Newton subdivision of degree two, there is a tropical polynomial corresponding to this subdivision, obtained by replacing a coefficient of a product of two tropical linear polynomials with a suitable number.

Proof. For a maximal Newton subdivision of degree two, by Proposition 2.2.30, there is a tropical curve corresponding to this subdivision. Thus, by Theorem 3.1.1, there is a polynomial obtained by replacing a coefficient of a product of two tropical linear polynomials with a suitable number. □

Example 4.1.3. Let us consider the Newton subdivision of type 4 shown in Figure 4.1.

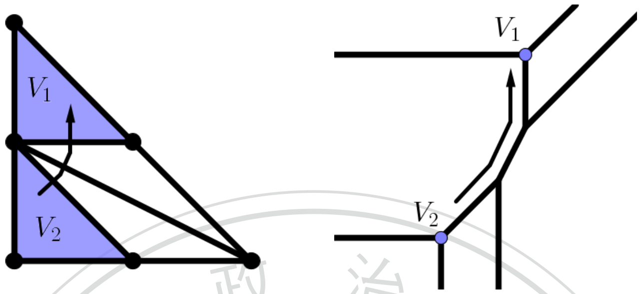


Figure 4.2: Type 4 of the Newton subdivision of degree two

In Figure 4.2, we see that the 1-cell V_1 is at the upper right of V_2, since the corresponding vertex of V_1 is at the upper right of the corresponding vertex of V_2. If the corresponding vertex of V_1 is at (a_1, a_2), and the corresponding vertex of V_2 is at (b_1, b_2), we have $a_1 > b_1$, $a_2 > b_2$, and $a_2 - a_1 > b_2 - b_1$. So we may suppose $(a_1, a_2) = (4, 7)$ and $(b_1, b_2) = (0, 0)$ for example. The product of these tropical linear polynomials is

$$\begin{aligned}&((-4) \odot x \oplus (-7) \odot y \oplus 0) \odot (x \oplus y \oplus 0)\\ &= (-4) \odot x^{\odot 2} \oplus (-4) \odot x \odot y \oplus (-7) \odot y^{\odot 2} \oplus x \oplus y \oplus 0.\end{aligned}$$

In Figure 4.3, the edge E is incident to the vertices corresponding to y-term and $2x$-term, which means that E is determined by the coefficients of y-term and $2x$-term. So we replace the coefficient of y, for instance, with 3. The resulting polynomial is

$$(-4) \odot x^{\odot 2} \oplus (-4) \odot x \odot y \oplus (-7) \odot y^{\odot 2} \oplus x \oplus 3 \odot y \oplus 0,$$

which is an example of tropical polynomials corresponding to Newton subdivisions of type 4.

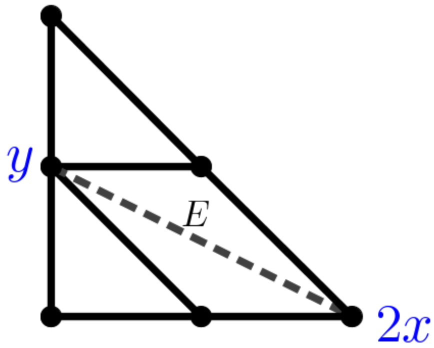


Figure 4.3: The edge E is determined by y-term and $2x$-term

4.2 Newton subdivisions of degree three

Theorem 4.2.1. For a normal subdivision Δ, there exist three tropical polynomials $g_1(x, y)$, $g_2(x, y)$, and $h(x, y)$, where g_1 corresponds to a subdivision of degree one, and g_2 corresponds to a maximal subdivision of degree two that is a subgraph of Δ, such that $g_1(x, y) \odot g_2(x, y) \oplus h(x, y)$ corresponds to Δ.

Proof. By Proposition 3.2.2, each normal subdivision has a maximal subdivision of degree two as a subgraph, so up to isomorphic, we have six types shown in Figure 4.4, where Δ_2 is a maximal subdivision of degree two.

We see in Figure 4.4 that these six types can be obtained from the following two subdivisions shown in Figure 4.5.

case 1. Let us start from the subdivision in Figure 4.5 on the left.

Let g_1 and g_2 be the corresponding tropical polynomials of Δ_1 and Δ_2, respectly.

In Figure 4.6, we see that if we add a suitable number c_1 to the coefficient of $y^{\odot 2}$-term of the tropical polynomial $G_1(x, y) := g_1(x, y) \odot g_2(x, y)$, then we get a polynomial G_2 which corresponds to the second case in Figure 4.6, where $G_2(x, y) := G_1(x, y) \oplus h_1(x, y)$, and $h_1(x, y)$ is the tropical polynomial obtained by adding c_1

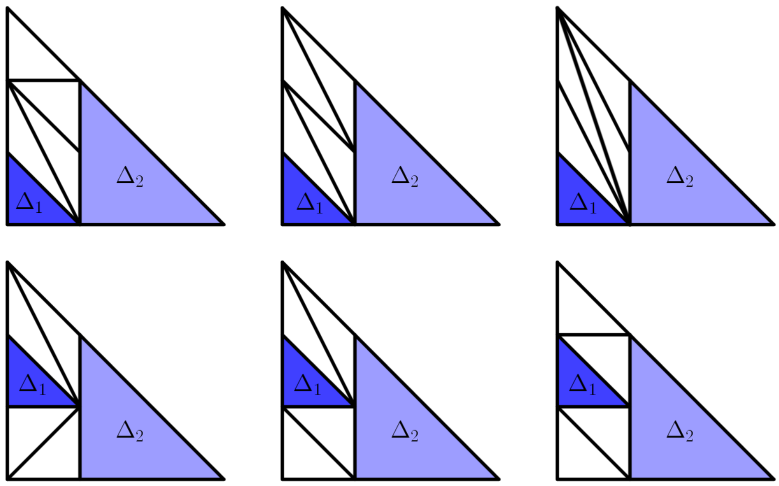


Figure 4.4: Six types of maximal subdivisions of degree three

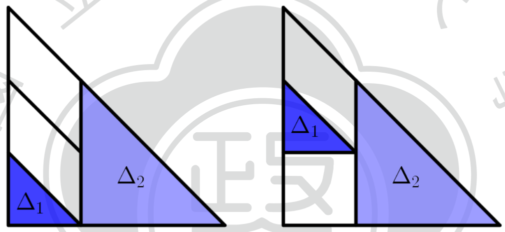


Figure 4.5:

to the coefficient of $y^{\odot 2}$-term of G_1. In the same way, if we add a siutable number c_2 to the $y^{\odot 3}$-term of G_2, then we will have a polynomial G_3 which corresponds to the third case in Figure 4.6, where $G_3(x, y) := G_2(x, y) \oplus h_2(x, y)$, and $h_2(x, y)$ is the tropical polynomial obtained by adding c_2 to the coefficient of $y^{\odot 3}$-term of G_2. On the other hand, if we add a siutable number c_3 (which is larger than c_2) to the $y^{\odot 3}$-term of G_2, then we will have a polynomial G_4 which corresponds to the forth case in Figure 4.6, where $G_4(x, y) := G_2(x, y) \oplus h_3(x, y)$, and $h_3(x, y)$ is the tropical polynomial obtained by adding c_3 to the coefficient of $y^{\odot 3}$-term of G_2.

In detail, suppose

$$g_1(x, y) = a_1 \odot x \oplus a_2 \odot y \oplus 0$$

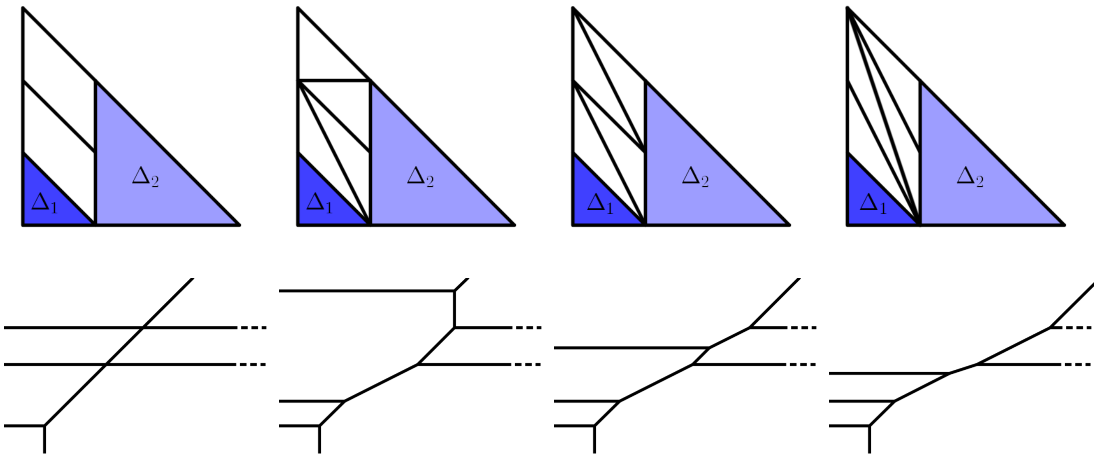


Figure 4.6: The local graph of the corresponding subdivisions

and

$$g_2(x,y) = b_1 \odot x^{\odot 2} \oplus b_2 \odot x \odot y \oplus b_3 \odot y^{\odot 2} \oplus b_4 \odot x \oplus b_5 \odot y \oplus 0.$$

Then we have

$$\begin{aligned} G_1(x,y) &= g_1(x,y) \odot g_2(x,y) \\ &= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\ &\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus (a_2 \odot b_3) \odot y^{\odot 3} \\ &\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\ &\quad \oplus (a_2 \odot b_5 \oplus b_3) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus 0. \end{aligned}$$

Choose $0 < c_1 < a_2 - b_5$, and let $h_1(x,y) = ((a_2 \odot b_5 \oplus b_3) \odot c_1) \odot y^{\odot 2}$. Then we have

$$\begin{aligned} G_2(x,y) &= G_1(x,y) \oplus h_1(x,y) \\ &= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\ &\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus (a_2 \odot b_3) \odot y^{\odot 3} \\ &\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\ &\quad \oplus ((a_2 \odot b_5 \oplus b_3) \odot c_1) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus 0. \end{aligned}$$

Let $H_1(x,y) = h_1(x,y)$, then we have $G_2(x,y) = G_1(x,y) \oplus H_1(x,y)$.

Next, choose $c_1 < c_2 < c_1 + 2b_5 - b_3$. Let

$$h_2(x,y) = ((a_2 \odot b_3) \odot c_2) \odot y^{\odot 3}.$$

Then we have

$$
\begin{aligned}
G_3(x,y) &= G_2(x,y) \oplus h_2(x,y) \\
&= G_1(x,y) \oplus h_1(x,y) \oplus h_2(x,y) \\
&= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\
&\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus ((a_2 \odot b_3) \odot c_2) \odot y^{\odot 3} \\
&\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\
&\quad \oplus ((a_2 \odot b_5 \oplus b_3) \odot c_1) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus 0.
\end{aligned}
$$

Let $H_2(x,y) = h_1(x,y) \oplus h_2(x,y)$, then we have $G_3(x,y) = G_1(x,y) \oplus H_2(x,y)$.

Again, we choose $c_1 + 2b_5 - b_3 < c_3 < 2c_1 + 2b_5 - b_3$, and let

$$h_3(x,y) = ((a_2 \odot b_3) \odot c_3) \odot y^{\odot 3}.$$

Then we have

$$
\begin{aligned}
G_4(x,y) &= G_2(x,y) \oplus h_3(x,y) \\
&= G_1(x,y) \oplus h_1(x,y) \oplus h_3(x,y) \\
&= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\
&\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus ((a_2 \odot b_3) \odot c_3) \odot y^{\odot 3} \\
&\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\
&\quad \oplus ((a_2 \odot b_5 \oplus b_3) \odot c_1) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus 0.
\end{aligned}
$$

Let $H_3(x,y) = h_1(x,y) \oplus h_3(x,y)$, then we have $G_4(x,y) = G_1(x,y) \oplus H_3(x,y)$.

Thus, we complete the proof in this case.

case 2. Now, let us consider the lower three cases in Figure 4.4.

Let g_1 and g_2 be the corresponding tropical polynomials of Δ_1 and Δ_2, respectly. In Figure 4.7, we see that the three cases can be obtained by tuning the coefficients of constant term and $3y$-term of the tropical polynomial $G_1(x,y) := g_1(x,y) \odot g_2(x,y)$.

Suppose

$$g_1(x,y) = a_1 \odot x \oplus a_2 \odot y \oplus 0$$

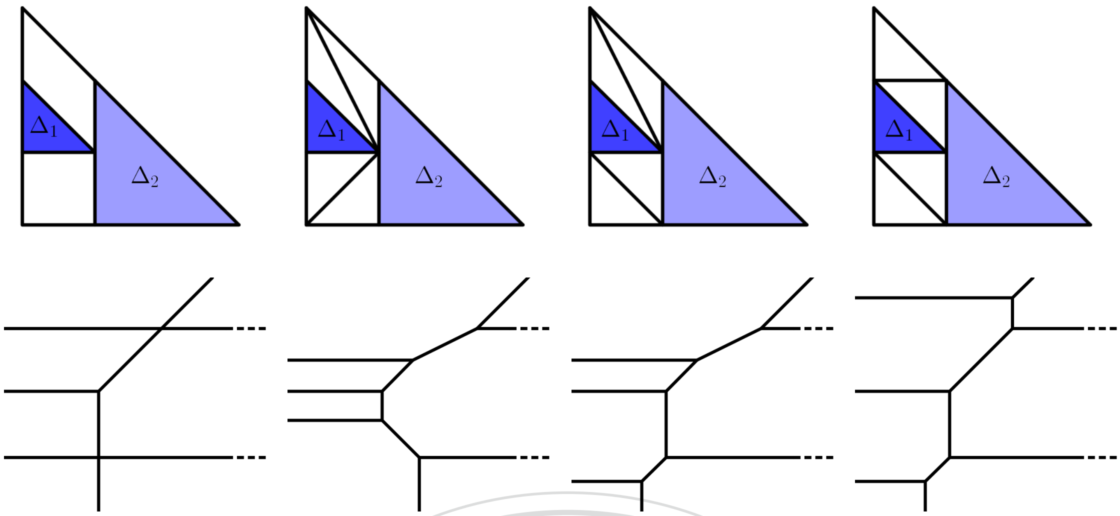


Figure 4.7: The local graph of the corresponding subdivisions

and

$$g_2(x,y) = b_1 \odot x^{\odot 2} \oplus b_2 \odot x \odot y \oplus b_3 \odot y^{\odot 2} \oplus b_4 \odot x \oplus b_5 \odot y \oplus 0,$$

then we have

$$\begin{aligned} G_1(x,y) &= g_1(x,y) \odot g_2(x,y) \\ &= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\ &\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus (a_2 \odot b_3) \odot y^{\odot 3} \\ &\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\ &\quad \oplus (a_2 \odot b_5 \oplus b_3) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus 0. \end{aligned}$$

Now, we choose $0 < c_1 < a_2 + b_5 - b_3$ and $0 < c_2 < b_5 - a_2$. Let $h_1(x,y) = ((a_2 \odot b_3) \odot c_1) \odot y^{\odot 3} \oplus c_2$. Then we have

$$\begin{aligned} G_2(x,y) &:= G_1(x,y) \oplus h_1(x,y) \\ &= (a_1 \odot b_1) \odot x^{\odot 3} \oplus (a_1 \odot b_2 \oplus a_2 \odot b_1) \odot x^{\odot 2} \odot y \\ &\quad \oplus (a_1 \odot b_3 \oplus a_2 \odot b_2) \odot x \odot y^{\odot 2} \oplus ((a_2 \odot b_3) \odot c_1) \odot y^{\odot 3} \\ &\quad \oplus (a_1 \odot b_4 \oplus b_1) \odot x^{\odot 2} \oplus (a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot x \odot y \\ &\quad \oplus (a_2 \odot b_5 \oplus b_3) \odot y^{\odot 2} \oplus (a_1 \oplus b_4) \odot x \oplus (a_2 \oplus b_5) \odot y \oplus c_2, \end{aligned}$$

which corresponds to the second case in Figure 4.7.

Next, choose $c_3 > 0$ and the same c_1 for convenience. Let

$$\begin{aligned} h_2(x,y) &= ((a_1 \odot b_1) \odot c_3) \odot x^{\odot 3} \oplus ((a_1 \odot b_2 \oplus a_2 \odot b_1) \odot c_3) \odot x^{\odot 2} \odot y \\ &\oplus ((a_1 \odot b_3 \oplus a_2 \odot b_2) \odot c_3) \odot x \odot y^{\odot 2} \oplus ((a_2 \odot b_3) \odot c_1 \odot c_3) \odot y^{\odot 3} \\ &\oplus ((a_1 \odot b_4 \oplus b_1) \odot c_3) \odot x^{\odot 2} \oplus ((a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot c_3) \odot x \odot y \\ &\oplus ((a_2 \odot b_5 \oplus b_3) \odot c_3) \odot y^{\odot 2} \oplus ((a_1 \oplus b_4) \odot c_3) \odot x \\ &\oplus ((a_2 \oplus b_5) \odot c_3) \odot y. \end{aligned}$$

Then we have

$$\begin{aligned} G_3(x,y) &:= G_1(x,y) \oplus h_2(x,y) \\ &= ((a_1 \odot b_1) \odot c_3) \odot x^{\odot 3} \oplus ((a_1 \odot b_2 \oplus a_2 \odot b_1) \odot c_3) \odot x^{\odot 2} \odot y \\ &\oplus ((a_1 \odot b_3 \oplus a_2 \odot b_2) \odot c_3) \odot x \odot y^{\odot 2} \oplus ((a_2 \odot b_3) \odot c_1 \odot c_3) \odot y^{\odot 3} \\ &\oplus ((a_1 \odot b_4 \oplus b_1) \odot c_3) \odot x^{\odot 2} \oplus ((a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot c_3) \odot x \odot y \\ &\oplus ((a_2 \odot b_5 \oplus b_3) \odot c_3) \odot y^{\odot 2} \oplus ((a_1 \oplus b_4) \odot c_3) \odot x \\ &\oplus ((a_2 \oplus b_5) \odot c_3) \odot y \oplus 0, \end{aligned}$$

which corresponds to the third case in Figure 4.7.

Next, for convenience, we choose the same c_3, and let

$$\begin{aligned} h_3(x,y) &= ((a_1 \odot b_1) \odot c_3) \odot x^{\odot 3} \oplus ((a_1 \odot b_2 \oplus a_2 \odot b_1) \odot c_3) \odot x^{\odot 2} \odot y \\ &\oplus ((a_1 \odot b_3 \oplus a_2 \odot b_2) \odot c_3) \odot x \odot y^{\odot 2} \oplus ((a_1 \odot b_4 \oplus b_1) \odot c_3) \odot x^{\odot 2} \\ &\oplus ((a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot c_3) \odot x \odot y \\ &\oplus ((a_2 \odot b_5 \oplus b_3) \odot c_3) \odot y^{\odot 2} \oplus ((a_1 \oplus b_4) \odot c_3) \odot x \\ &\oplus ((a_2 \oplus b_5) \odot c_3) \odot y. \end{aligned}$$

Then we have

$$\begin{aligned} G_4(x,y) &:= G_1(x,y) \oplus h_3(x,y) \\ &= ((a_1 \odot b_1) \odot c_3) \odot x^{\odot 3} \oplus ((a_1 \odot b_2 \oplus a_2 \odot b_1) \odot c_3) \odot x^{\odot 2} \odot y \\ &\oplus ((a_1 \odot b_3 \oplus a_2 \odot b_2) \odot c_3) \odot x \odot y^{\odot 2} \oplus (a_2 \odot b_3) \odot y^{\odot 3} \\ &\oplus ((a_1 \odot b_4 \oplus b_1) \odot c_3) \odot x^{\odot 2} \oplus ((a_1 \odot b_5 \oplus a_2 \odot b_4 \oplus b_2) \odot c_3) \odot x \odot y \\ &\oplus ((a_2 \odot b_5 \oplus b_3) \odot c_3) \odot y^{\odot 2} \oplus ((a_1 \oplus b_4) \odot c_3) \odot x \\ &\oplus ((a_2 \oplus b_5) \odot c_3) \odot y \oplus 0. \end{aligned}$$

which corresponds to the last case in Figure 4.7, and we complete the proof. $\square$

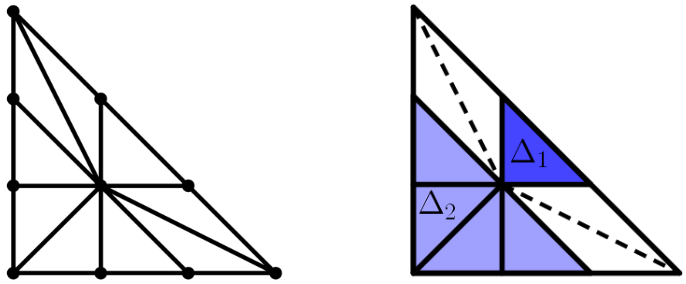


Figure 4.8:

Example 4.2.2. Let us consider the normal subdivision in the left of Figure 4.8.

Use the algorithm in Section 4.1, we may suppose the tropical polynomial of Δ_2 to be

$$\begin{aligned}&(x \oplus (-4) \odot y \oplus 0) \odot ((-4) \odot x \oplus y \oplus 0) \oplus 1\\ &= (-4) \odot x^{\odot 2} \oplus x \odot y \oplus (-4) \odot y^{\odot 2} \oplus x \oplus y \oplus 1,\end{aligned}$$

and the tropical polynomial of Δ_1 to be $(-8) \odot x \oplus (-8) \odot y \oplus 0$.

Then we have the product of Δ_1 and Δ_2 to be

$$\begin{aligned}&(-12) \odot x^{\odot 3} \oplus (-8) \odot x^{\odot 2} \odot y \oplus (-8) \odot x \odot y^{\odot 2} \oplus (-12) \odot y^{\odot 3}\\ &\oplus (-4) \odot x^{\odot 2} \oplus x \odot y \oplus (-4) \odot y^{\odot 2} \oplus x \oplus y \oplus 1.\end{aligned}$$

Next, we replace the coefficients of $x^{\odot 3}$ and $y^{\odot 3}$ by -11, then we have the desired polynomial

$$\begin{aligned}&(-11) \odot x^{\odot 3} \oplus (-8) \odot x^{\odot 2} \odot y \oplus (-8) \odot x \odot y^{\odot 2} \oplus (-11) \odot y^{\odot 3}\\ &\oplus (-4) \odot x^{\odot 2} \oplus x \odot y \oplus (-4) \odot y^{\odot 2} \oplus x \oplus y \oplus 1.\end{aligned}$$

Now, let us start the proof of Theorem 3.2.3.

Proof of Theorem 3.2.3. First, we prove the theorem for tropical curves which correspond to normal subdivisions.

In the proof of Theorem 4.2.1, we know that there are six types for normal subdivisions, up to isomorphic. So we may just prove the theorem for these six types.

Let us consider the following example first.

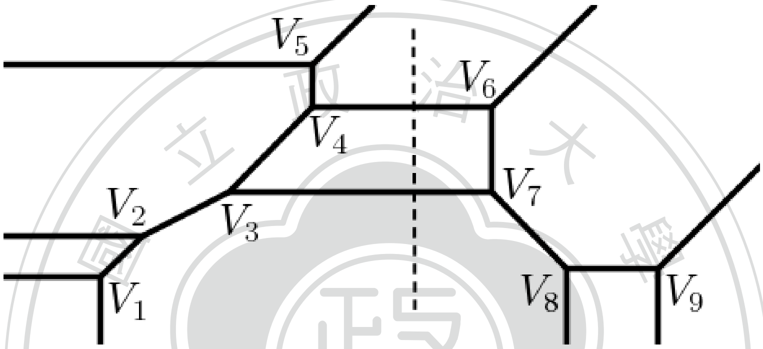


Figure 4.9:

This curve is of the type shown in Figure 4.10. The right part of this curve is just a smooth plane tropical curve of degree two, so we can use the algorithm in Section 3.1 to find out its polynomial.

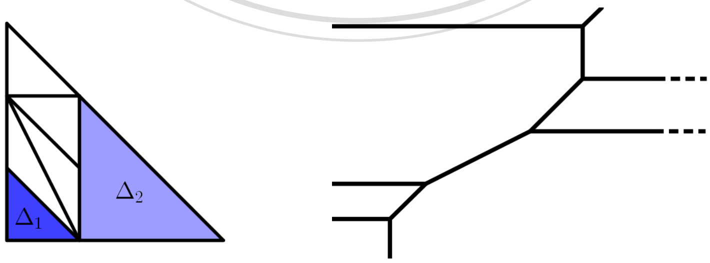


Figure 4.10:

For the left part of this curve, with a similar way in the proof of Theorem 4.2.1, we may first consider the case in Figure 4.11, and find out the polynomial of the curve shown in Figure 4.12. Next, we add the suitable numbers to the coefficients of $y^{\odot 2}$-term and $y^{\odot 3}$-term. Then we will get the polynomial corresponding to the original curve.

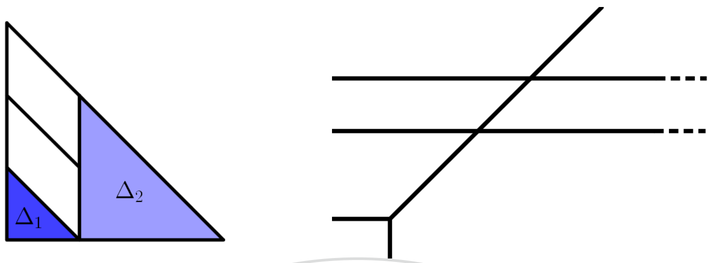


Figure 4.11:

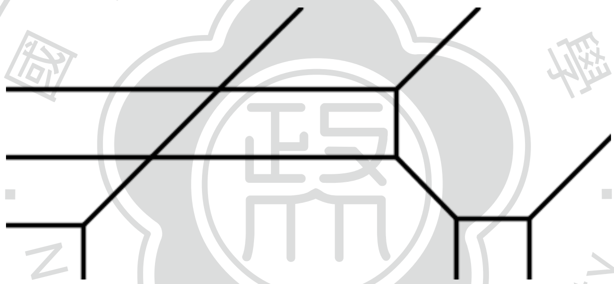


Figure 4.12:

Let $V_i = (v_1^{(i)}, v_2^{(i)})$, $i = 1, \ldots, 9$. Let $g_1(x,y) = (-v_1^{(6)}) \odot x \oplus (-v_2^{(6)}) \odot y \oplus 0$, $g_2(x,y) = (-v_1^{(9)}) \odot x \oplus (-v_2^{(9)}) \odot y \oplus 0$, $g_3(x,y) = (-v_1^{(1)}) \odot x \oplus (-v_2^{(1)}) \odot y \oplus 0$, and $h_1(x,y) = c_0$, where $c_0 = v_1^{(8)} - v_1^{(7)}$. Then we have the polynomial of the right part to be

$$\begin{aligned}
G_1(x,y) &:= g_1(x,y) \odot g_2(x,y) \oplus h_1(x,y) \\
&= ((-v_1^{(6)}) \odot x \oplus (-v_2^{(6)}) \odot y \oplus 0) \odot ((-v_1^{(9)}) \odot x \oplus (-v_2^{(9)}) \odot y \oplus 0) \\
&\quad \oplus c_0 \\
&= ((-v_1^{(6)}) \odot (-v_1^{(9)})) \odot x^{\odot 2} \oplus ((-v_1^{(6)}) \odot (-v_2^{(9)})) \odot (x \odot y) \\
&\quad \oplus ((-v_2^{(6)}) \odot (-v_2^{(9)})) \odot y^{\odot 2} \oplus (-v_1^{(6)}) \odot x \oplus (-v_2^{(9)}) \odot y \oplus c_0.
\end{aligned}$$

Let $c_1 = v_2^{(3)} - v_2^{(2)}$ and $c_2 = v_2^{(5)} - v_2^{(4)}$. Let

$$h_2(x,y) = (((-v_2^{(6)}) \odot (-v_2^{(9)}) \oplus (-v_2^{(1)}) \odot (-v_2^{(9)})) \odot c_1) \odot y^{\odot 2}.$$

Then we have

$$G_2(x,y) := G_1(x,y) \odot g_3(x,y) \oplus h_2(x,y),$$

which is of the same type of the original curve.

Next, we compare c_1 and c_2. If $c_1 \geq c_2$, then we add the coefficient of $y^{\odot 3}$-term of G_2 by $c_1 - c_2$; otherwise, we add all terms but $y^{\odot 3}$-term of G_2 by $c_2 - c_1$. We let $h_3(x,y)$ to be this coefficient tuning polynomial.

So we have that the polynomial of the original curve can be represented as

$$\begin{aligned}
&G_2(x,y) \oplus h_3(x,y) \\
&= (G_1(x,y) \odot g_3(x,y) \oplus h_2(x,y)) \oplus h_3(x,y) \\
&= (g_1(x,y) \odot g_2(x,y) \oplus h_1(x,y)) \odot g_3(x,y) \oplus h_2(x,y) \oplus h_3(x,y) \\
&= g_1(x,y) \odot g_2(x,y) \odot g_3(x,y) \\
&\quad \oplus (h_1(x,y) \odot g_3(x,y) \oplus h_2(x,y) \oplus h_3(x,y)).
\end{aligned}$$

By this way, we may have similar results in other cases. We may find out the polynomials of curves corresponding to Δ_1 and Δ_2, and also their product, and then use the algorithm in the proof of Theorem 4.2.1 to construct the polynomial of the original curve.

Now, let us consider the last four cases of which subdivision is not normal.

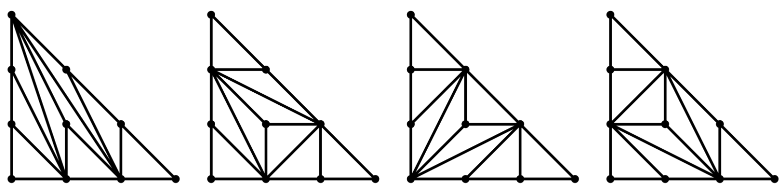


Figure 4.13: The special four types

In Figure 4.13, We see that the first type and the third type are isomorphic, while the second type and the forth type are isomorphic. So we may just consider the first two types.

For the first type, we may observe that it can be obtained from the type shown in Figure 4.15 by adding a suitable number to the coefficient of the $y^{\odot 3}$-term. So this case is done.

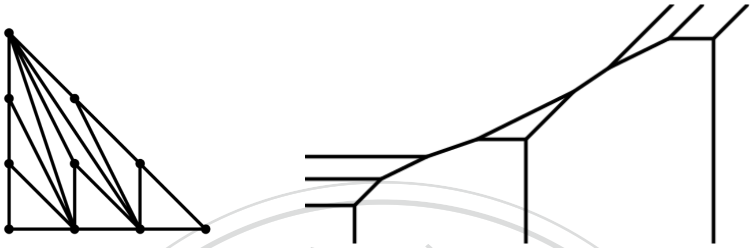


Figure 4.14: A special type of tropical curve

Figure 4.15:

For the second type, we may observe that it can be obtained from the union of three tropical lines shown in Figure 4.17 by adding suitable numbers to the coefficients of x-term, $y^{\odot 2}$-term, and $(x^{\odot 2} \odot y)$-term.

Let the weight of the edges E_1, E_2, and E_3 to be w_1, w_2, and w_3, respectly. Suppose we add c_1, c_2, and c_3 to the coefficients of x, $y^{\odot 2}$, and $(x^{\odot 2} \odot y)$-term of the polynomial of the union shown in Figure 4.17 to obtain the original curve. Then we have

$$\begin{cases} c_1 + c_2 &= w_1, \\ c_2 + c_3 &= w_2, \\ c_3 + c_1 &= w_3. \end{cases}$$

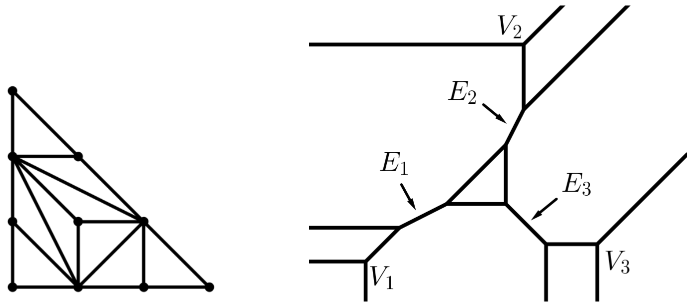


Figure 4.16: A special type of tropical curve

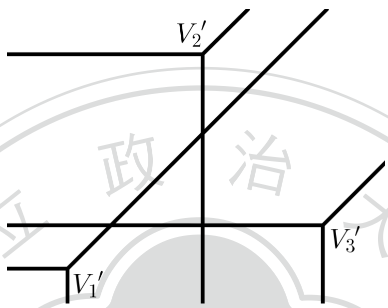


Figure 4.17: The union of three tropical lines

Thus, we have

$$\begin{cases} c_1 &= \dfrac{1}{2}(w_1+w_2+w_3)-w_2, \\ c_2 &= \dfrac{1}{2}(w_1+w_2+w_3)-w_3, \\ c_3 &= \dfrac{1}{2}(w_1+w_2+w_3)-w_1. \end{cases}$$

Let $V_i = (v_1^{(i)}, v_2^{(i)}), i = 1,2,3$. We have

$$\begin{aligned} V_1' &= (v_1^{(1)}+c_1, v_2^{(1)}), \\ V_2' &= (v_1^{(2)}-c_2, v_2^{(2)}-c_2), \\ V_3' &= (v_1^{(3)}, v_2^{(3)}+c_3). \end{aligned}$$

Let g_1, g_2, and g_3 be the tropical polynomials of the tropical lines at V_1', V_2', and V_3', respectly. Suppose

$$G_1(x,y) := g_1(x,y) \odot g_2(x,y) \odot g_3(x,y).$$

For convenience, we let

$$\begin{aligned} G_1(x,y) &= a_1 \odot x^{\odot 3} \oplus a_2 \odot x^{\odot 2} \odot y \oplus a_3 \odot x \odot y^{\odot 2} \oplus a_4 \odot y^{\odot 3} \oplus a_5 \odot x^{\odot 2} \\ &\quad \oplus a_6 \odot x \odot y \oplus a_7 \odot y^{\odot 2} \oplus a_8 \odot x \oplus a_9 \odot y \oplus 0. \end{aligned}$$

Then we have the polynomial of the original curve to be

$$\begin{aligned} G_2(x,y) &= a_1 \odot x^{\odot 3} \oplus (a_2 \odot c_3) \odot x^{\odot 2} \odot y \oplus a_3 \odot x \odot y^{\odot 2} \oplus a_4 \odot y^{\odot 3} \oplus a_5 \odot x^{\odot 2} \\ &\quad \oplus a_6 \odot x \odot y \oplus (a_7 \odot c_2) \odot y^{\odot 2} \oplus (a_8 \odot c_1) \odot x \oplus a_9 \odot y \oplus 0. \end{aligned}$$

It may also written as

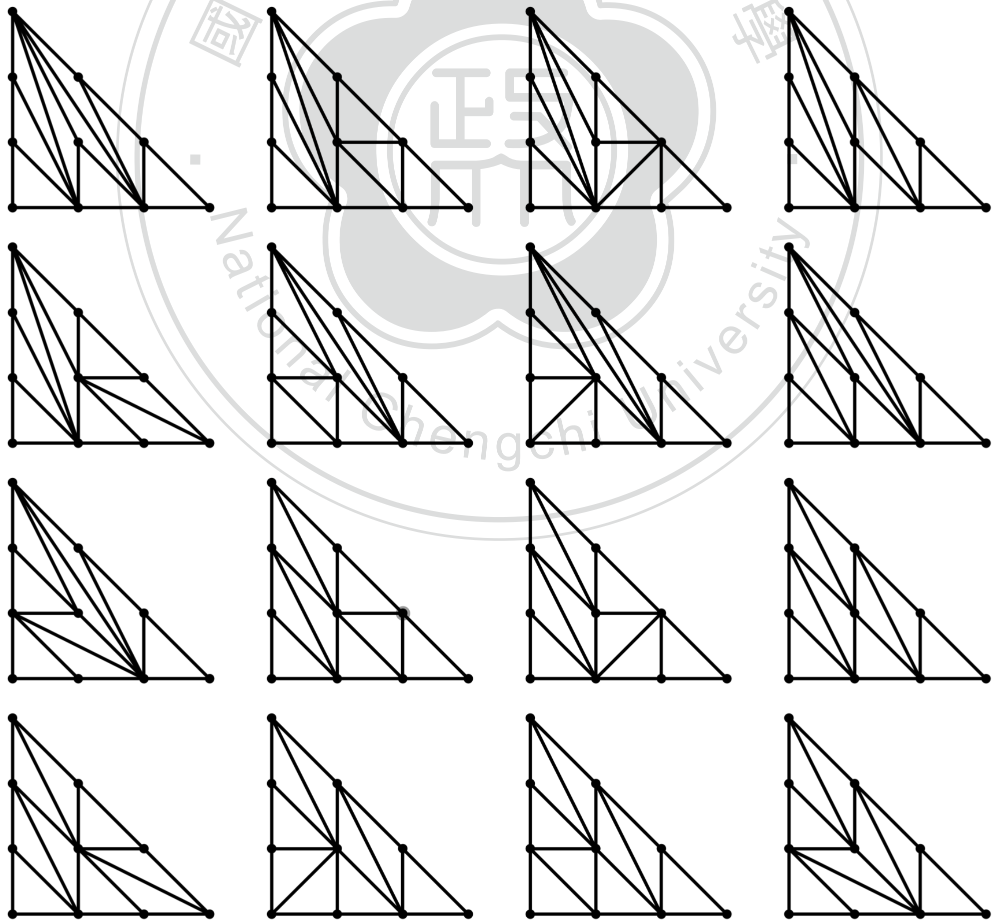
$$G_2(x,y) = G_1(x,y) \oplus (h_1(x,y) \oplus h_2(x,y) \oplus h_3(x,y)),$$

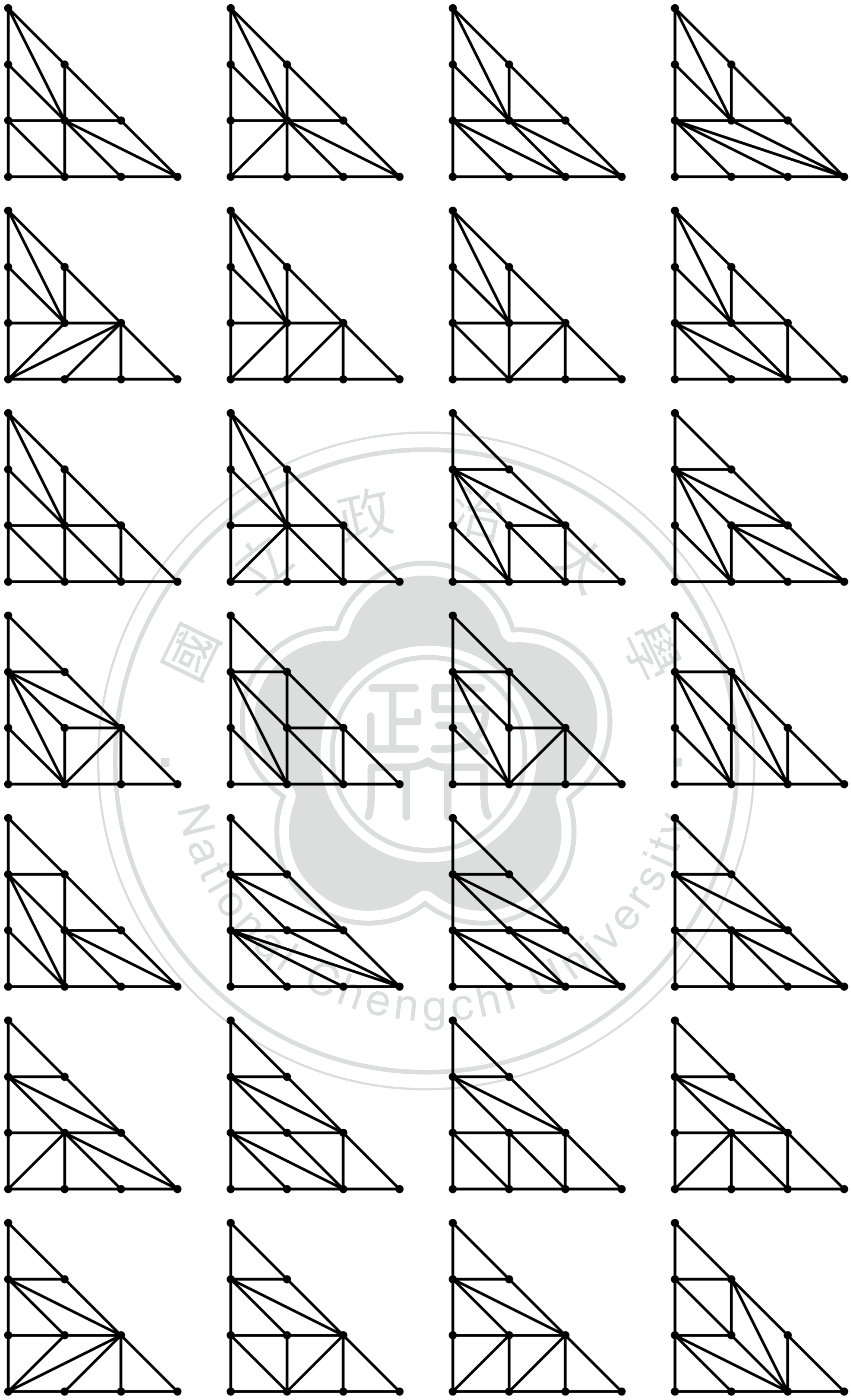
where $h_1(x,y)$ is the tropical polynomial obtained by adding c_1 to the coefficient of x-term of G_1; $h_2(x,y)$ is the tropical polynomial obtained by adding c_2 to the coefficient of $y^{\odot 2}$-term of G_1; $h_3(x,y)$ is the tropical polynomial obtained by adding c_3 to the coefficient of $(x^{\odot 2} \odot y)$-term of G_1.

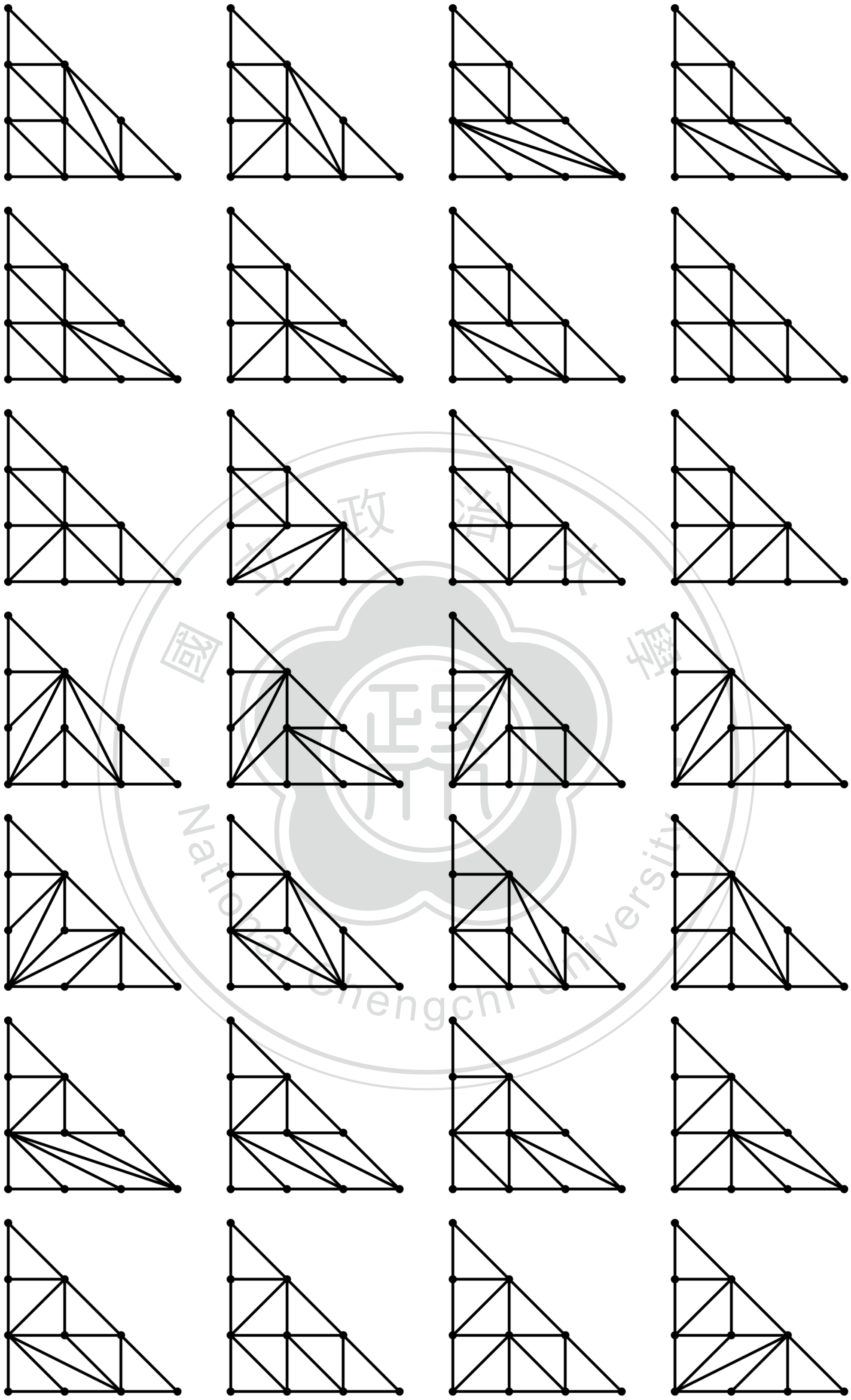
Therefore, we complete the proof of Theorem 3.2.3. $\square$

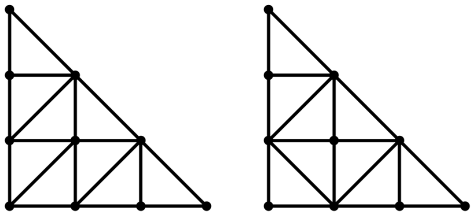
Appendix A

All types of maximal Newton subdivisions of degree three









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