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A MATHEMATICAL MODEL OF ENTERPRISE COMPETITIVE ABILITY AND PERFORMANCE THROUGH EMDEN-FOWLER EQUATION (II)*

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Abstract In this paper we work with the ordinary equation $u'' - u^2(u + \bar{u}) = 0$ and obtain some interesting phenomena concerning, blow-up, blow-up rate, life-span of solutions to those equations.

Key words estimate; life-span; blow-up; blow-up rate; performance; competitive ability

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0 Introduction

How to improve the performance and competitiveness of the company is the critical issue of Industrial and Organizational Psychology in Taiwan. We try to design an appropriate mathematical model of the competitiveness and the performance of the 293 benchmark enterprises out of 655 companies. Unexpectedly, we discover the correlation of performance and competitiveness is extremely high. Some benchmark enterprises present the following phenomena:

Competitive ability (Force, $F(P(n))$) is a cubic function of the performance $(P(n))$; that is, there exist positive constant performances $P_i > 0, i \in \{0,1\}$ and a constant k so that

$$F(P(n)) = \begin{cases} k(P(n) - P_0)^3 \quad \text{or} \\ kP(n)(P(n) - P_0)^2 \quad \text{or} \\ kP(n)(P(n) - P_1)(P(n) - P_0), \end{cases}$$

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where n is the surveying rod enterprise's composition department number or the main unit commanders counts, the performance $P(n)$ of the rod enterprise's is lager than P_0 and F is proportional to the second derivative of P with respect to n. For $F(P(n)) = M\frac{\mathrm{d}^2P(n)}{\mathrm{d}n^2}$, let $u(n) := \sqrt{\frac{k}{M}}(P(n) - P_0) \geq 0$, $\bar{u} := \sqrt{\frac{k}{M}}P_0$ then we obtain a stationary one dimensional semilinear wave equation with initial condition

$$\begin{cases} u''(n) - u(n)^2(u(n) + \bar{u}) = 0, & n \geq n_0, \\ u(n_0) = u_0 = \sqrt{\dfrac{k}{M}}(P(n_0) - P_0) \geq 0, & u'(n_0) = u_1. \end{cases} \tag{0.1}$$

It is clear that the function $u^2(u + \bar{u})$ is locally Lipschitz, hence by the standard theory, the local existence of classical solutions is applicable to equation (0.1). We would use our methods used in [1–16] to discuss problem (0.1).

In Section 1, we would deal with the estimates for the existence interval of the solutions of (0.1); in Section 2, with the blow-up rate and blow-up constant; in Section 3, with the global existence, critical point and the asymptotic behavior; in Section 4, with triviality, stability and instability.

Notation and Fundamental Lemmas For a given function u in this work, we use the following abbreviations

$$a_u(n) = u(n)^2, \ E_u(n_0) = u_1^2 - \frac{1}{2}u_0^4 - \frac{2}{3}\bar{u}u_0^3, \ J_u(n) = a_u(n)^{-\frac{1}{4}}.$$

Definition A function $g : \mathbb{R} \to \mathbb{R}$ with a blow-up rate q means that g exists only in finite time; that is, there is a finite number T^* such that

$$\lim_{t \to T^*} g(t)^{-1} = 0 \tag{0.2}$$

and there exists a non-zero $\beta \in \mathbb{R}$ with

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta, \tag{0.3}$$

in this case β is called the blow-up constant of g.

According to the uniqueness of the solutions to equation (0.1), we can rewrite $a_u(n) = a(n)$, $J_u(n) = J(n)$ and $E_u(n) = E(n)$. After some elementary calculations we obtain the following.

Lemma 1 Suppose that u is the solution of (0.1), then we have

$$E(n) = u'(n)^2 - \frac{1}{2}u(n)^4 - \frac{2}{3}\bar{u}u(n)^3 = E(n_0), \tag{0.4}$$

$$a''(n) = 5u'(n)^2 + \frac{1}{2}a(n)^2 - 3E(n_0), \tag{0.5}$$

$$J''(n) = -\frac{1}{4}J(n)^5\left(\frac{1}{2}a(n)^2 - 3E(n_0)\right), \tag{0.6}$$

$$J'(n)^2 - \frac{1}{8}J(n)^{-2} - \frac{E(n_0)}{4}J^6(n) = \frac{1}{6}\bar{u} \tag{0.7}$$

and

$$a'(n) = a'(n_0) + 2E(n_0)(n - n_0) + \int_{n_0}^{n} \left(3u(r)^4 + \frac{10}{3}\bar{u}u(r)^3\right) \mathrm{d}r. \tag{0.8}$$

We prove briefly this Lemma 1 as follows: by (0.1), $E(n) = u'(n)^2 - \frac{1}{2}u(n)^4 - \frac{2}{3}\bar{u}u(n)^3$, we obtain that $E'(n) = 2u'(n)u''(n) - 2u(n)^3 u'(n) - 2\bar{u}u(n)^2 u'(n) = 0$, therefore $E(n) = E(n_0)$.

From (0.1), (0.4) and the definition of $a(n)$,

$$\begin{aligned} a''(n) &= 2u''(n)u(n) + 2u'(n)^2 \\ &= 2u'(n)^2 + 2a(n)^2 + 3u'(n)^2 - \frac{3}{2}a(n)^2 - 3E(n_0) \\ &= 5u'(n)^2 + \frac{1}{2}a(n)^2 - 3E(n_0). \end{aligned}$$

Using (0.4) and (0.5) we derive $J'(n) = -\frac{1}{4}a(n)^{-\frac{5}{4}}a'(n)$ and

$$\begin{aligned} J''(n) &= -\frac{1}{4}a(n)^{-\frac{5}{4}}a''(n) + \frac{1}{4}\frac{5}{4}a(n)^{-\frac{5}{4}-1}a'(n)^2 \\ &= -\frac{1}{4}a(n)^{-\frac{5}{4}-1}\left(a(n)a''(n) - \frac{5}{4}a'(n)^2\right) \\ &= -\frac{1}{4}a(n)^{-\frac{5}{4}-1}\left[a(n)\left(5u'(n)^2 + \frac{1}{2}a(n)^2 - 3E(n_0)\right) - \frac{5}{4}a'(n)^2\right] \\ &= -\frac{1}{4}J(n)^5\left(\frac{1}{2}a(n)^2 - 3E(n_0)\right). \end{aligned}$$

Also we get $2J'(n)J''(n) = -\frac{1}{4}J(n)^{-3} + \frac{3}{2}E(n_0)J(n)^5 J'(n)$ and

$$\begin{aligned} J'(n)^2 - \frac{1}{8}J(n)^{-2} - \frac{E(n_0)}{4}J^6(n) &= J'(n_0)^2 - \frac{1}{8}J(n_0)^{-2} - \frac{E(n_0)}{4}J^6(n_0) \\ &= \frac{1}{4}u_0^{-3}u_1^2 - \frac{1}{8}u_0 - \frac{u_1^2 - \frac{1}{2}u_0^4 - \frac{2}{3}\bar{u}u_0^3}{4}u_0^{-3} \\ &= \frac{1}{6}\bar{u}. \end{aligned}$$

By using (0.5) and (0.4), $a''(n) = 3a(n)^2 + \frac{10}{3}\bar{u}u(n)^3 + 2E(n_0)$,

$$a'(n) = a'(n_0) + 2E(n_0)(n - n_0) + \int_{n_0}^{n}\left(3a(r)^2 + \frac{10}{3}\bar{u}u(r)^3\right)\mathrm{d}r.$$

The following lemmas are easy to prove, so we omit their proofs.

Lemma 2 Suppose that r and s are real constants and $u \in C^2(\mathbb{R})$ satisfies

$$u'' + ru' + su \le 0, \quad u \ge 0, u(0) = 0, \quad u'(0) = 0,$$

then u must be null, that is, $u \equiv 0$.

Lemma 3 If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit

$$\lim_{t\to T}\int_0^{g(t)} h(t,r)\,\mathrm{d}r$$

exists, then

$$\lim_{t\to T}\int_0^{g(t)} h(t,r)\,\mathrm{d}r = \int_0^{g(T)} h(T,r)\,\mathrm{d}r.$$

1 Estimates for the Life-Span

To estimate the existence interval of the solution of equation (0.1) , we separate this section into three parts: $E(n_0) < 0,\ E(n_0) = 0$ and $E(n_0) > 0$. Here the existence interval N of u means that u exists and makes sense only in the interval $[n_0, N)$ so that problem (0.1) possesses the solution $u \in \bar{C}^2(n_0, N)$.

1.1 Estimates for the Existence Intervals under $E(n_0) \le 0$

We deal with two cases, $E(n_0) < 0$, and $E(n_0) = 0,\ a'(n_0) > 0$ in this subsection, but the case $E(n_0) = 0$ and $a'(n_0) \le 0$ will be considered in Sections 3 and 4 later. Here we have the following result.

Theorem 4 If N is the existence interval of the solution u to (0.1) with $E(n_0) < 0$, then N is finite. Further, for $a'(n_0) \ge 0$ we have the estimate

$$N \le N_1^* = n_0 + \int_0^{J(n_0)^2} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4 + \frac{2}{3}\bar{u}r}} \tag{1.1}$$

for $a'(n_0) < 0$,

$$N \le N_2^* = n_0 + \left(\int_0^{J(n_1)^2} + \int_{J(n_0)^2}^{J(n_1)^2} \right) \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4 + \frac{2}{3}\bar{u}r}}, \tag{1.2}$$

where

$$J(n_1) = \sqrt{\frac{1}{2}} \sqrt{\sqrt{-\Delta + \frac{\frac{1}{3}\bar{u}}{\frac{1}{4}E(n_0)\sqrt{\Delta}}} - \sqrt{\Delta}}, \quad \Delta = \left(\frac{4\bar{u}^2}{9} E(n_0) \right)^{1/3} + \left(\frac{2}{3\bar{u}^2 E(n_0)} \right)^{1/3}.$$

Furthermore, if $E(n_0) = 0$ and $a'(n_0) > 0$, then

$$a(n) = \left(\left(\sqrt{J(n_0)^2 + \frac{3}{4\bar{u}}} - \sqrt{\frac{\bar{u}}{6}} (n - n_0) \right)^2 - \frac{3}{4\bar{u}} \right)^{-2}, \tag{1.3}$$

$$N \le N_3^* := n_0 + \frac{J(n_0)^2}{\sqrt{\frac{\bar{u}}{6}} \left(\sqrt{J(n_0)^2 + \frac{3}{4\bar{u}}} + \sqrt{\frac{3}{4\bar{u}}} \right)}. \tag{1.4}$$

Remark The phenomena of blow-up of u or P means that such benchmark enterprises attain their maximum of performance and competitiveness.

Proof For $E(n_0) < 0$, we know that $a(n_0) > 0$; otherwise we would get $a(n_0) = 0$, that is, $u_0 = 0$, then $E(n_0) = u_1^2 \ge 0$, this contradicts $E(n_0) < 0$. In this situation we separate the proof of this Theorem into two subcases, $a'(n_0) \ge 0$ and $a'(n_0) < 0$.

(i) $a'(n_0) \ge 0$. By (0.8) and (0.7) we find that

$$\begin{aligned} a'(n) &= a'(n_0) + 2E(n_0)(n - n_0) + \int_{n_0}^{n} \left(3u(r)^4 + \frac{10}{3}\bar{u}u(r)^3 \right) \mathrm{d}r \\ &\ge a'(n_0) + 2E(n_0)(n - n_0) > 0, \quad \forall n \ge n_0, \end{aligned} \tag{1.5}$$

$$J'(n)^2 = \frac{1}{8}J(n)^{-2} + \frac{E(n_0)}{4}J^6(n) + \frac{1}{6}\bar{u},$$
$$J'(n) = -\sqrt{\frac{1}{8}J(n)^{-2} + \frac{E(n_0)}{4}J^6(n) + \frac{1}{6}\bar{u}} \le J'(n_0), \quad \forall n \ge n_0 \tag{1.6}$$

and

$$J(n) \le a(n_0)^{-\frac{1}{4}} - \frac{1}{4}a(n_0)^{-\frac{5}{4}}a'(n_0)(n-n_0), \quad \forall n \ge n_0.$$

Thus there exists a finite number $N_1^*(u_0,u_1) \le n_0 + \frac{4a(n_0)}{a'(n_0)} = n_0 + 2\frac{u_0}{u_1}$ such that $J(N_1^*(u_0,u_1)) = 0$ and so $a(n) \to \infty$ as $n \to N_1^*(u_0,u_1)$. This means that $N \le N_1^*(u_0,u_1)$.

Now we estimate $N_1^*(u_0,u_1)$. By (1.5) and $J(N_1^*(u_0,u_1)) = 0$ we find that

$$\begin{aligned} n - n_0 &= \int_{J(n)}^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{8}r^{-2} + \frac{E(n_0)}{4}r^6 + \frac{1}{6}\bar{u}}} \\ &= \int_{J(n)}^{J(n_0)} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4}r^8 + \frac{1}{6}\bar{u}r^2}} \\ &= \int_{J(n)^2}^{J(n_0)^2} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4 + \frac{2}{3}\bar{u}r}}, \qquad \forall n \ge n_0 \end{aligned} \tag{1.7}$$

and hence we get estimate (1.1).

(ii) $a'(n_0) < 0$. By (1.4), $a'(n_0) < 0$ and the convexity of a we can find a unique finite number $n_1 = n_1(u_0,u_1)$ such that

$$\begin{cases} a'(n) < 0 = a'(n_1) & \text{for} \quad n \in (n_0, n_1), \\ a'(n) > 0 & \text{for} \quad n > n_1, \end{cases} \tag{1.8}$$

and $a(n_1) > 0$. If not, then $u(n_1) = 0$, thus $E(n) = E(n_1) = u'(n_1)^2 \ge 0$; yet this is a contradiction to $E(n_0) < 0$. Hence, we conclude that $a(n) > 0, \forall n \ge n_0$, $u'(n_1) = 0$, $E(n_0) = -\frac{1}{2}u(n_1)^4 - \frac{2}{3}\bar{u}u(n_1)^3$ and $\frac{1}{8}J(n_1)^{-2} + \frac{E(n_0)}{4}J^6(n_1) + \frac{1}{6}\bar{u} = 0$,

$$J'(n) = \sqrt{\frac{1}{8}J(n)^{-2} + \frac{E(n_0)}{4}J^6(n) + \frac{1}{6}\bar{u}}, \quad n \in [n_0, n_1],$$

$$\begin{aligned} n_1 - n_0 &= \int_{J(n_0)}^{J(n_1)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{8}r^{-2} + \frac{E(n_0)}{4}r^6 + \frac{1}{6}\bar{u}}} \\ &= \int_{J(n_0)^2}^{J(n_1)^2} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4 + \frac{2}{3}\bar{u}r}}. \end{aligned}$$

After arguments similar to step (i), there exists a $N_2^* := N_2^*(u_0,u_1)$ such that the life-span N of u is bounded by N_2^*, that is, $N \le N_2^*$. By an analogous argument, using (1.7), (0.7) and the fact that $J(N_2^*) = 0$ and

$$J(n_1) = \sqrt{\frac{1}{2}}\sqrt{\sqrt{-\Delta + \frac{\frac{1}{3}\bar{u}}{\frac{1}{4}E(n_0)\sqrt{\Delta}}} - \sqrt{\Delta}},$$

we conclude that

$$J'(n)^2=\frac{1}{8}J(n)^{-2}+\frac{E(n_0)}{4}J^6(n)+\frac{1}{6}\bar{u},\quad \forall n\ge n_1,$$

$$J'(n)=-\sqrt{\frac{1}{8}J(n)^{-2}+\frac{E(n_0)}{4}J^6(n)+\frac{1}{6}\bar{u}},\quad \forall n\ge n_1, \tag{1.9}$$

$$J'(n)=\sqrt{\frac{1}{8}J(n)^{-2}+\frac{E(n_0)}{4}J^6(n)+\frac{1}{6}\bar{u}},\quad \forall n\in[n_0,n_1], \tag{1.10}$$

$$\int_{J(n)}^{J(n_1)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}=n-n_1,\quad \forall n\ge n_1, \tag{1.11}$$

$$\int_{J(n_0)}^{J(n_1)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}=n_1-n_0 \tag{1.12}$$

and

$$N_2^*=n_1+\int_0^{J(n_1)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}. \tag{1.13}$$

This estimate (1.12) is equivalent to (1.2).

(iii) For $E(n_0)=0$, by (0.6) and $a'(n_0)>0$ we get that $J'(n_0)<0$, $J''(n)=-\frac{1}{8}J(n)^{-3}<0$ and $J'(n)=-\sqrt{\frac{1}{6}\bar{u}+\frac{1}{8}J(n)^{-2}}$, $\forall n\ge n_0$. Thus we conclude that

$$\frac{J'(n)}{\sqrt{J^{-2}+\frac{4\bar{u}}{3}}}=-\sqrt{\frac{1}{8}},$$

$$\sqrt{\frac{1}{8}}(n-n_0)=\int_{J(n)}^{J(n_0)}\frac{r\mathrm{d}r}{\sqrt{1+\frac{4\bar{u}}{3}r^2}}=\int_{J(n)}^{J(n_0)}\frac{r\mathrm{d}r}{\sqrt{\frac{4\bar{u}}{3}\left(r^2+\frac{3}{4\bar{u}}\right)}},$$

$$n-n_0=\sqrt{\frac{6}{\bar{u}}}\int_{J(n)}^{J(n_0)}\frac{r\mathrm{d}r}{\sqrt{r^2+\frac{3}{4\bar{u}}}}=\sqrt{\frac{6}{\bar{u}}}\left(\sqrt{J(n_0)^2+\frac{3}{4\bar{u}}}-\sqrt{J(n)^2+\frac{3}{4\bar{u}}}\right),$$

$$\sqrt{\frac{\bar{u}}{6}}(n-n_0)=\sqrt{J(n_0)^2+\frac{3}{4\bar{u}}}-\sqrt{J(n)^2+\frac{3}{4\bar{u}}},$$

$$a(n)^{-\frac{1}{2}}=J(n)^2=\left(\sqrt{J(n_0)^2+\frac{3}{4\bar{u}}}-\sqrt{\frac{\bar{u}}{6}}(n-n_0)\right)^2-\frac{3}{4\bar{u}},$$

$$a(n)=\left(\left(\sqrt{J(n_0)^2+\frac{3}{4\bar{u}}}-\sqrt{\frac{\bar{u}}{6}}(n-n_0)\right)^2-\frac{3}{4\bar{u}}\right)^{-2},\quad \forall n\ge n_0, \tag{1.14}$$

$$\sqrt{\frac{3}{4\bar{u}}}=\sqrt{J(n_0)^2+\frac{3}{4\bar{u}}}-\sqrt{\frac{\bar{u}}{6}}(N_3^*-n_0),$$

$$N_3^* = n_0 + \frac{\sqrt{J(n_0)^2 + \frac{3}{4\bar{u}}} - \sqrt{\frac{3}{4\bar{u}}}}{\sqrt{\frac{\bar{u}}{6}}}$$

and (1.3) is proved. □

1.2 Estimates for the Life-span under $E(n_0) > 0$

In this subsection we consider the case $E(n_0) > 0$, and we have the following blow-up result.

Theorem 5 If N^* is the existence interval of u which solves problem (0.1) with $E(n_0) > 0$, then N^* is finite. Further, in case of $a'(n_0) > 0$, we have

$$N^* \le N_4^*(u_0, u_1) = n_0 + \int_0^{J(n_0)} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4} r^8 + \frac{1}{6}\bar{u} r^2}}. \tag{1.15}$$

In the case of $a'(n_0) = 0$ we have

$$N^* \le N_5^*(u_0, u_1) = n_0 + \int_0^{\infty} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4} r^8 + \frac{1}{6}\bar{u} r^2}}. \tag{1.16}$$

For $a'(n_0) < 0$ and $z(u_0, u_1)$ given by

$$z(u_0, u_1) = n_0 + \int_{J(n_0)}^{\infty} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4} r^8 + \frac{1}{6}\bar{u} r^2}} \tag{1.17}$$

is the zero of a. Further, we have

$$N^* \le N_6^*(u_0, u_1) := z(u_0, u_1) + \int_0^{\infty} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4} r^8 + \frac{1}{6}\bar{u} r^2}}. \tag{1.18}$$

Proof i) For $E(n_0) > 0, a'(0) > 0$, by (0.8) and (0.7) we have $J'(n) = -\frac{1}{4} J(n)^5 a'(n) < 0$,

$$J'(n) = -\sqrt{\frac{1}{6}\bar{u} + \frac{1}{8} J(n)^{-2} + \frac{E(n_0)}{4} J^6(n)} \le -\sqrt{\frac{1}{6}\bar{u}}, \tag{1.19}$$

$$J(n) \le J(n_0) - \sqrt{\frac{1}{6}\bar{u}}(n - n_0) \to 0 \quad \text{as } n \to n_0 + \sqrt{\frac{6}{\bar{u}}} J(n_0);$$

therefore $a(n)$ blows up at finite $n = N_4^*$ and $J(N_4^*) = 0$. By (1.18) we obtain (1.14) and

$$n - n_0 = \int_{J(n)}^{J(n_0)} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8} + \frac{E(n_0)}{4} r^8 + \frac{1}{6}\bar{u} r^2}}. \tag{1.20}$$

ii) From $a'(n_0) = 0 = u_0$, $E(n_0) = {u_1}^2 - \frac{1}{2}{u_0}^4 - \frac{2}{3}\bar{u}{u_0}^3 = {u_1}^2$ and (0.8) we obtain $J'(n) = -\frac{1}{4} J(n)^5 a'(n) < 0, n > n_0$ and also estimate (1.15).

iii) For $a'(n_0) < 0$, by (0.8) we have $a'(n) > 0$ for large $n > n_0 + \frac{-a'(n_0)}{2E(n_0)}$.

Suppose z is the first positive number n so that $a'(n) = 0$, then $u(z) = 0$; otherwise, $u'(z) = 0$ and $E(z) = -\frac{1}{2} u(z)^4 - \frac{2}{3}\bar{u} u(z)^3 < 0$, this contradicts the assumption $E(n_0) = E(z) > 0$. By using (0.7), we conclude that

$$J'(n) = \sqrt{\frac{1}{6}\bar{u} + \frac{1}{8} J(n)^{-2} + \frac{E(n_0)}{4} J^6(n)} \quad \text{for} \quad n \in [n_0, z], \tag{1.21}$$

$$n-n_0=\int_{J(n_0)}^{J(n)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}\quad\text{for}\quad n\in[n_0,z],$$

$$z(u_0,u_1)-n_0=\int_{J(n_0)}^{\infty}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}.$$

After the number $n=z$, same as the procedures given in the proof of (i), using (1.18) we obtain (1.17). □

2 Blow-up Rate and Blow-up Constant

In this section we study the blow-up rate and blow-up constant for a, a' and a'' under the conditions in Section 1. We have got the following results.

Theorem 6 If u is the solution of problem (0.1) with one of the following properties that (i) $E(n_0)<0$; (ii) $E(n_0)>0$.

Then the blow-up rate of a is 2, and the blow-up constant K_1 of a is 8, that is, for $m=1,2,4,5,6$,

$$\lim_{n\to N_m^*}(N_m^*-n)^2a(n)=8. \tag{2.1}$$

The blow-up rate of a' is 3, and the blow-up constant K_2 of a' is 32, that is, for $m=1,2,4,5,6$,

$$\lim_{n\to N_m^*}(N_m^*-n)^3a'(n)=32. \tag{2.2}$$

The blow-up rate of a'' is 4, and the blow-up constant K_3 of a'' is $32+\frac{5}{16}$, that is, $m=1,2,4,5,6$,

$$\lim_{n\to N_m^*}a''(n)(N_m^*-n)^4=32+\frac{5}{16}. \tag{2.3}$$

Proof (i) Under this condition, $E(n_0)<0,\ a'(n_0)\geq 0$ by (1.1), (1.6) and Lemma 4 we get

$$n-n_0=\int_{J(n)}^{J(n_0)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}},$$

$$N_1^*-n_0=\int_{0}^{J(n_0)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}},$$

$$\int_0^{J(n)}\frac{1}{N_1^*-n}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}=1,\quad\forall n\geq n_0, \tag{2.4}$$

$$\lim_{n\to N_1^*}\sqrt{8}\frac{J(n)^2}{N_1^*-n}=1. \tag{2.5}$$

This identity (2.5) is equivalent to (2.1) for $m=1$.

For $E(n_0)<0,\ a'(n_0)<0$ by (1.10) we have also

$$\int_0^{J(n_1)}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}}=N_2^*-n_1,$$

$$\int_{J(n)}^{J(n_1)} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}} = n - n_1,\ n \geq n_1$$

$$\int_0^{J(n)} \frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_0)}{4}r^8+\frac{1}{6}\bar{u}r^2}} = N_2^* - n \quad \forall n \geq n_1. \tag{2.6}$$

Through Lemma 4 and (2.6) , therefore we get (2.1) for $m = 2$.

Seeing (1.5) and (1.8) , we find

$$\lim_{n\to N_m^*} a'(n)\, a(n)^{-3/2} = \frac{2}{\sqrt{2}},$$

$$\lim_{n\to N_m^*} \left(a(n)\,(N_m^* - n)^2\right)^{\frac{-3}{2}} a'(n)\,(N_m^* - n)^3 = \sqrt{2},$$

$$\lim_{n\to N_m^*} a'(n)\,(N_m^* - n)^3 = 32,$$

$$\lim_{n\to N_m^*} J'(n)\, J(n) = -\frac{1}{2\sqrt{2}}, \tag{2.7}$$

$$\lim_{n\to N_m^*} a'(n)\,(N_m^* - n)^3 = 32, \tag{2.8}$$

$$\lim_{n\to N_m^*} u'(n)^2\,(N_m^* - n)^4 = \frac{1}{16} \tag{2.9}$$

for $m = 1, 2$. Using (0.5) and (2.9) we obtain for $m = 1, 2$,

$$\lim_{n\to N_m^*} a''(n)\,(N_m^* - n)^4 = 32 + \frac{5}{16}. \tag{2.10}$$

Thus, (2.10) and (2.3) are equivalent.

(ii) For $E(n_0) > 0$, estimates (2.1.1), (2.1.2) and (2.1.3) for $m = 4, 5, 6$, are similar to the above arguments (i) in the proof of this theorem. □

Theorem 7 If u is the solution of problem (0.1) with one of the following properties that $E(n_0) = 0,\ a'(n_0) > 0$, Then the blow-up rate of a is 2, and the blow-up constant K_1 of a is 2, that is,

$$\lim_{n\to {N_3}^*} (N_3^* - n)^2\, a(n) = 2. \tag{2.11}$$

The blow-up rate of a' is 3, and the blow-up constant K_2 of a' is 32, that is,

$$\lim_{n\to N_3^*} (N_3^* - n)^3\, a'(n) = 4. \tag{2.12}$$

The blow-up rate of a'' is 4, and the blow-up constant K_3 of a'' is 12,

$$\lim_{n\to N_3^*} a''(n)\,(N_3^* - n)^4 = 12. \tag{2.13}$$

Proof For $E(n_0) = 0,\ a'(n_0) > 0$, by (1.13) for

$$N_3^* = n_0 + \frac{\sqrt{J(n_0)^2 + \frac{3}{4\bar{u}}} - \sqrt{\frac{3}{4\bar{u}}}}{\sqrt{\frac{\bar{u}}{6}}},$$

we get

$$\begin{aligned} a(n) &= \left(\left(\sqrt{J(n_0)^2+\frac{3}{4\bar u}}-\sqrt{\frac{\bar u}{6}}(n-n_0)\right)^2-\frac{3}{4\bar u}\right)^{-2} \\ &= \left(\left(\sqrt{\frac{3}{4\bar u}}-\sqrt{\frac{\bar u}{6}}(n-N_3^*)\right)^2-\frac{3}{4\bar u}\right)^{-2} \\ &= \left(\left(\sqrt{\frac{3}{4\bar u}}+\sqrt{\frac{\bar u}{6}}(N_3^*-n)\right)^2-\frac{3}{4\bar u}\right)^{-2}, \end{aligned} \tag{2.14}$$

$$a(n)^{\frac{-1}{2}}=\left(\sqrt{\frac{3}{4\bar u}}+\sqrt{\frac{\bar u}{6}}(N_3^*-n)\right)^2-\frac{3}{4\bar u}=\frac{\bar u}{6}(N_3^*-n)^2+\sqrt{\frac{1}{2}}(N_3^*-n),$$

$$\lim_{n\to N_3^*}(N_3^*-n)^{-1}a(n)^{\frac{-1}{2}}=\sqrt{\frac{1}{2}},$$

$$\lim_{n\to N_3^*}(N_3^*-n)^2a(n)=2.$$

By (0.7) we obtain

$$J'(n)^2-\frac{1}{8}J(n)^{-2}=\frac{1}{6}\bar u,$$

$$a'(n)^2-2\left(a(n)(N_3^*-n)^2\right)^3(N_3^*-n)^{-6}=\frac{8}{3}\bar u\left(a(n)(N_3^*-n)^2\right)^{\frac{5}{2}}(N_3^*-n)^{-5},$$

$$(N_3^*-n)^6a'(n)^2-2\left(a(n)(N_3^*-n)^2\right)^3=\frac{8}{3}\bar u\left(a(n)(N_3^*-n)^2\right)^{\frac{5}{2}}(N_3^*-n),$$

$$\lim_{n\to N_3^*}(N_3^*-n)^6a'(n)^2=2\lim_{n\to N_3^*}\left(a(n)(N_3^*-n)^2\right)^3=16,$$

$$\lim_{n\to N_3^*}(N_3^*-n)^3a'(n)=4. \tag{2.15}$$

Using (0.5) and (2.15) we have

$$\lim_{n\to N_3^*}(N_3^*-n)^2a(n)u'(n)^2(N_3^*-n)^4=4,$$

$$\lim_{n\to N_3^*}u'(n)^2(N_3^*-n)^4=2,$$

$$(N_3^*-n)^4a''(n)=5u'(n)^2(N_3^*-n)^4+\frac{1}{2}\left(a(n)(N_3^*-n)^2\right)^2,$$

$$\lim_{n\to N_3^*}(N_3^*-n)^4a''(n)=12.$$

Therefore, estimates (2.11), (2.12) and (2.13) are proved. □

3 Global Existence and Critical Point at Infinite

In this section we study the following case that $E(n_0) = 0$ and $a'(n_0) < 0$.

Here we take the global existence of the solutions to problem (0.1) in the following sense:

$$J(n) > 0,\ a'(n)^{-2} > 0,\ a''(n)^{-2} > 0, \quad \forall n \in [n_0, N],$$

where N is the time that u exists, in other words, in any finite time u does not blow up in C^2 sense, even though u blows up in a finite time in some sense, for example, C^k or L^k for some $k \geq 3$.

This result concerning global existence of the solutions to problem (0.1) could happen and will be explained below only in the case that $E(n_0) = 0$ and $a'(n_0) < 0$. Under the condition it is easy to see that $J(n) > 0\ \forall n \in (n_0, N)$ and for $I(n) = a(n)^{-1/2}$, by using (0.4) and (0.5) we obtain

$$\begin{aligned} I''(n) &= \frac{-1}{2} a(n)^{-\frac{3}{2}} a''(n) + \frac{3}{4} a(n)^{-\frac{5}{2}} a'(n)^2 \\ &= \frac{-1}{2} a(n)^{-\frac{3}{2}} \left(5u'(n)^2 + \frac{1}{2} a(n)^2\right) + \frac{3}{4} a(n)^{-\frac{5}{2}} a'(n)^2 \\ &= \frac{1}{2} u(n)^{-3} u'(n)^2 - \frac{1}{4} u(n) = \frac{1}{2} u(n)^{-3} \left(u'(n)^2 - \frac{1}{2} u(n)^4\right) \\ &= \frac{1}{3} \bar{u} > 0, \\ I'(n) &= I'(n_0) + \frac{1}{3} \bar{u}(n - n_0), \end{aligned}$$

$$\begin{aligned} a(n) &= \left(u_0^{-1} - u_0^{-2} u_1 (n - n_0) + \frac{\bar{u}}{6} (n - n_0)^2\right)^{-2}, \\ a'(n) &= 2\left(u_0^{-2} u_1 - \frac{\bar{u}}{3}(n - n_0)\right)\left(u_0^{-1} - u_0^{-2} u_1 (n - n_0) + \frac{\bar{u}}{6}(n - n_0)^2\right)^{-3}, \\ a''(n) &= -\left(u_0^{-1} - u_0^{-2} u_1 (n - n_0) + \frac{\bar{u}}{6}(n - n_0)^2\right)^{-4} \\ &\quad \cdot \left(\frac{7\bar{u}^2}{9}(n - n_0)^2 - \frac{14\bar{u}}{3} u_0^{-2} u_1 (n - n_0) + 6u_0^{-4} u_1^2 + \frac{2\bar{u}}{3} u_0^{-1}\right). \end{aligned}$$

Hence we find the limit $\lim\limits_{n\to\infty} a(n) = 0,\ \lim\limits_{n\to\infty} a'(n) = 0,\ \lim\limits_{n\to\infty} a''(n) = 0$ and

$$\lim_{n\to\infty} (n - n_0)^4 a(n) = \left(\frac{\bar{u}}{6}\right)^{-2}, \tag{3.1}$$

$$\lim_{n\to\infty} (n - n_0)^5 a'(n) = -\frac{144}{\bar{u}^2}, \tag{3.2}$$

$$\lim_{n\to\infty} (n - n_0)^6 a''(n) = -\frac{1008}{\bar{u}^2}. \tag{3.3}$$

Theorem 8 Suppose that u is the solution of problem (0.1) with $E(n_0) = 0$ and $a'(n_0) < 0$, then u can be defined globally and estimates (3.1), (3.2) and (3.3) are valid.

4 Triviality, Stability and Instability

In this section we discuss the triviality of solution for the problem (0.1) under the case that $E(n_0)=0,\ a'(n_0)=0$.

Proposition If u is the solution of problem (0.1) with $E(n_0)=0$and $a'(n_0)=0$, then u must be null.

Proof Under the conditions $E(n_0)=0$, $a'(n_0)=0$ by using (0.4), it is easy to see that $u_0=0=u_1$, herein the supremum below exists

$$n_1:=\sup\left\{\alpha : a(n)\le 1\ \forall n\in\Big[n_0\,,\alpha\Big]\right\},$$

and then

$$2u'(n)^2=u(n)^4+\frac{4}{3}\bar{u}u(n)^3\ge 0,$$

$$a''(n)=5u'(n)^2+\frac{1}{2}a(n)^2=3u(n)^4+\frac{10}{3}\bar{u}u(n)^3=3a(n)^2+\frac{10}{3}\bar{u}u(n)\,a(n).$$

By Lemma 2 we conclude that

$$a''(n)\le\left(3+\frac{10}{3}\bar{u}\right)a(n),\quad a(n)\equiv 0\equiv u(n)\ \text{ in }\ [n_0,n_1].$$

Continue these steps we get the assertion of this theorem. □

We now consider the applications of the theorems above to the stability theory for the problem

$$\begin{cases} u''(n)=u(n)^2\,(u(n)+\bar{u}), \\ u(n_0)=\varepsilon_1, u'(n_0)=\varepsilon_2. \end{cases} \tag{4.1}$$

We say problem (4.1) is stable under condition F, if any nontrivial global solution $u\in C^2(\mathbb{R}^+)$ of (4.1) under condition F satisfies

$$\|u\|_{C^2}\to 0\quad\text{for}\quad |\varepsilon_1|+|\varepsilon_2|\to 0.$$

According to Theorems 4–8 we have the following result.

Corollary 9.1 Problem (4.1) is stable under $E_u(n_0)=0,\ \varepsilon_1\varepsilon_2<0$ and unstable under the following one of the followings

(i) $E_u(n_0)<0$,

(ii) $E_u(n_0)=0<\varepsilon_1\varepsilon_2$,

(iii) $E_u(n_0)>0$.

According to Theorems 4 and 5, we can obtain the following conclusions when we reconsider the problem

$$\begin{cases} u''(n)=u(n)^2\,(u(n)+\bar{u}), \\ u(n_0)=\varepsilon u_0, u'(n_0)=\varepsilon^2 u_1 \end{cases} \tag{4.2}$$

for

$$E_\varepsilon(n_0)=\varepsilon^4\left(u_1^2-\frac{1}{2}u_0^4-\frac{2}{3}\varepsilon^{-1}\bar{u}u_0^3\right),\ J_\varepsilon(n_0)=(\varepsilon u_0)^{-\frac{1}{2}},$$

$$J_\varepsilon(n_1)=\sqrt{\frac{1}{2}}\sqrt{\sqrt{-\Delta_\varepsilon+\frac{\frac{1}{3}\bar{u}}{\frac{1}{4}E_\varepsilon(n_0)\sqrt{\Delta_\varepsilon}}}-\sqrt{\Delta_\varepsilon}},$$

$$\Delta_{\varepsilon}=\left(\frac{4\bar{u}^{2}}{9}E_{\varepsilon}\left(n_{0}\right)\right)^{1/3}+\left(\frac{2}{3\bar{u}^{2}E_{\varepsilon}\left(n_{0}\right)}\right)^{1/3}.$$

Theorem 9.2 If N_{ε}^{*} is the existence interval of the solution u_{ε} to (4.2) with $E_{\varepsilon}(n_0)<0$, then N_{ε} is finite. Further, for $\varepsilon a'(n_0)\geq 0$, we have the estimate

$$N_{\varepsilon}^{*}\leq N_{1,\varepsilon}^{*}\left(u_{0},u_{1}\right)=n_{0}+\int_{0}^{J_{\varepsilon}(n_{0})^{2}}\frac{\mathrm{d}r}{\sqrt{\frac{1}{2}+E_{\varepsilon}\left(n_{0}\right)r^{4}+\frac{2}{3}\bar{u}r}}$$

for $\varepsilon a'(n_0)<0$,

$$N_{\varepsilon}^{*}\leq N_{2,\varepsilon}^{*}\left(u_{0},u_{1}\right)=n_{0}+\left(\int_{0}^{J_{\varepsilon}(n_{1})^{2}}+\int_{J_{\varepsilon}(n_{0})^{2}}^{J_{\varepsilon}(n_{1})^{2}}\right)\frac{\mathrm{d}r}{\sqrt{\frac{1}{2}+E_{\varepsilon}\left(n_{0}\right)r^{4}+\frac{2}{3}\bar{u}r}}.$$

Furthermore, if $E_{\varepsilon}(n_0)=0$ and $\varepsilon a'(n_0)>0$, then

$$N_{\varepsilon}^{*}\leq N_{3,\varepsilon}^{*}\left(u_{0},u_{1}\right)=n_{0}+\frac{J_{\varepsilon}\left(n_{0}\right)^{2}}{\sqrt{\frac{\bar{u}}{6}}\left(\sqrt{J_{\varepsilon}\left(n_{0}\right)^{2}+\frac{3}{4\bar{u}}}+\sqrt{\frac{3}{4\bar{u}}}\right)}. \tag{4.3}$$

Theorem 9.3 If N_{ε}^{*} is the existence interval of the solution u_{ε} to (4.2) with $E_{\varepsilon}(n_0)>0$, then N_{ε}^{*} is finite. Further, fin case of $\varepsilon a'(n_0)\geq 0$, we have

$$N_{\varepsilon}^{*}\leq N_{4,\varepsilon}^{*}\left(u_{0},u_{1}\right)=n_{0}+\int_{0}^{J_{\varepsilon}(n_{0})}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E_{\varepsilon}(n_{0})}{4}r^{8}+\frac{1}{6}\bar{u}r^{2}}}.$$

In the case of $\varepsilon a'(n_0)=0$ we have

$$N_{\varepsilon}^{*}\leq N_{5,\varepsilon}^{*}\left(u_{0},u_{1}\right)=n_{0}+\int_{0}^{\infty}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E(n_{0})}{4}r^{8}+\frac{1}{6}\bar{u}r^{2}}}.$$

For $\varepsilon a'(n_0)<0, z_{\varepsilon}\left(u_0,u_1\right)$ given by

$$z_{\varepsilon}\left(u_{0},u_{1}\right)=n_{0}+\int_{J_{\varepsilon}(n_{0})}^{\infty}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E_{\varepsilon}(n_{0})}{4}r^{8}+\frac{1}{6}\bar{u}r^{2}}}$$

is the zero of u_{ε}. Further we have

$$N_{\varepsilon}^{*}\leq N_{6,\varepsilon}^{*}\left(u_{0},u_{1}\right)=\left(z_{\varepsilon}+\int_{0}^{\infty}\frac{r\mathrm{d}r}{\sqrt{\frac{1}{8}+\frac{E_{\varepsilon}(n_{0})}{4}r^{8}+\frac{1}{6}\bar{u}r^{2}}}\right)\left(u_{0},u_{1}\right).$$

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