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A MATHEMATICAL MODEL OF ENTERPRISE COMPETITIVE ABILITY AND PERFORMANCE THROUGH A PARTICULAR EMDEN-FOWLER EQUATION*

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Abstract In this paper we work with the ordinary diffential equation $u'' - u^3 = 0$ and obtain some interesting phenomena concerning blow-up, blow-up rate, life-spann, zeros and critical points of solutions to this equation.

Key words estimate; life-span; blow-up; blow-up rate; performance; competitive ability

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0 Introduction

How to improve the performance and competitiveness of the company is the critical issue of Industrial and Organizational Psychology in Taiwan. We try to design an appropriate mathematical model of the competitiveness and the performance of the 293 benchmark enterprises out of 655 companies.

Unexpectedly, we discover the corelation of performance and competitiveness is extremely high. Some benchmark enterprises present the following phenomena.

Competitive ability (force, $F(P(n))$) is a cubic function of the performance ($P(n)$); that is, there exist positive constant performance $P_0 > 0$, and a positive constant $k > 0$ such that

$$F(P(n)) = k(P(n) - P_0)^3,$$

where n is the surveying rod enterprise's composition department number or the main unit commanders counts, the performance $P(n)$ of the rod enterprise is larger than P_0 and F is proportional to the second-order derivative of P with respect to n.

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Although the relation between $F(P(n))$ and $P(n)$ is non-linear but not the same as the previous type, there exist positive constant performances $P_0 > 0$, $P_1 > 0$ and a positive constant $k(n) > 0$ such that

$$F(P(n)) = k(P(n) - P_1)(P(n) - P_0)^2.$$

F and the P relations present the following punishment type, there exist positive constant performances $P_0 > 0, P_1 > 0, P_2 > 0$ and a positive constant $k(n) > 0$ such that

$$F(P(n)) = k(P(n) - P_2)(P(n) - P_1)(P(n) - P_0).$$

Now we consider the special case 1. For $F(P(n)) = M\frac{\mathrm{d}^2 P(n)}{\mathrm{d}n^2}$, let $u(n) := \sqrt{\frac{k}{M}}(P(n) - P_0) \geq 0$, then we obtain a stationary one-dimensional semilinear wave equation with initial condition

$$\begin{cases} u'' - u^3 = 0, \quad n \geq n_0, \\ u(n_0) = u_0 = \sqrt{\dfrac{k}{M}}(P(n_0) - P_0), \quad u'(n_0) = u_1. \end{cases} \tag{0.1}$$

It is clear that the function u^3 is locally Lipschitz, hence by the standard theory, the local existence of classical solutions is applicable to problem (0.1).

We would use our methods used in [1–11] to discuss problem (0.1) in two parts: blow-up time and "the singularity and regularity of solution for higher order derivatives".

In Section 1, we would deal with the estimates for the existence interval of the solutions of (0.1); in Section 2, with the blow-up rate and blow-up constant; in Section 3, with the global existence, critical point and the asymptotic behavior; in Section 4, with the null points (zero) and triviality; in Section 5 with the stability and instability. In Sections 6 and 7, we would study the blow-up behavior of $u^{(k)}$ and the regularity of the solution u of (0.1).

Notations and Fundamental Lemmas For a given function u in this work, we use the following abbreviations:

$$a_u(n) = u(n)^2, \ E_u(n_0) = u_1^2 - \frac{1}{2}u_0^4, \ J_u(n) = a_u(n)^{-\frac{1}{2}}.$$

Definition A function $g : \mathbb{R} \to \mathbb{R}$ with a blow-up rate q means that g exists only in a finite time, that is, there is a finite number T^* such that

$$\lim_{t \to T^*} g(t)^{-1} = 0 \tag{0.2}$$

and there exists a non-zero $\beta \in \mathbb{R}$ with

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta, \tag{0.3}$$

in this case β is called the blow-up constant of g.

According to the uniqueness of the solutions to (0.1), we can rewrite $a_u(n) = a(n)$, $J_u(n) = J(n)$ and $E_u(n) = E(n)$. After some elementary calculations we obtain the following:

Lemma 1 Suppose that u is the solution of (0.1), then we have

$$E(n) = u'(n)^2 - \frac{1}{2}u(n)^4 = E(n_0), \tag{0.4}$$

$$6u'(n)^2 = 4E(n_0) + a''(n), \tag{0.5}$$

$$J''(n) = 2E(n_0) J(n)^3, \tag{0.6}$$

$$J'(n)^2 = J'(n_0)^2 - E(n_0) J(n_0)^4 + E(n_0) J(n)^4, \tag{0.7}$$

and

$$a'(n) = a'(n_0) + 2E(n_0)(n - n_0) + 3\int_{n_0}^{n} u(r)^4 \, \mathrm{d}r. \tag{0.8}$$

The following lemmas are easy to prove, so we omit their proofs.

Lemma 2 Suppose that r and s are real constants and $u \in C^2(\mathbb{R})$ satisfies

$$\begin{aligned} &u'' + ru' + su \le 0, \quad u \ge 0, \\ &u(0) = 0, \quad u'(0) = 0, \end{aligned}$$

then u must be null: $u \equiv 0$.

Lemma 3 If $g(t)$ and $h(t, r)$ are continuous with respect to their variables and

$$\lim_{t \to T} \int_0^{g(t)} h(t, r) \, \mathrm{d}r$$

exists, then

$$\lim_{t \to T} \int_0^{g(t)} h(t, r) \, \mathrm{d}r = \int_0^{g(T)} h(T, r) \, \mathrm{d}r.$$

1 Estimates for the Life-Spann

To estimate the existence interval of the solution of (0.1), we separate this section into three parts: $E(n_0) < 0$, $E(n_0) = 0$ and $E(n_0) > 0$. Here the existence interval N of u means that u exists and makes sense only in the interval $[n_0, N)$ such that problem (0.1) possesses the solution $u \in \bar{C}^2(n_0, N)$.

1.1 Estimates for the Existence Intervals Under $E(n_0) \le 0$

We deal with two cases, $E(n_0) < 0$, and $E(n_0) = 0$ and $a'(n_0) > 0$ in this subsection, but the case $E(n_0) = 0$ and $a'(n_0) \le 0$ will be considered in Sections 3 and 4. Here we have the following result.

Theorem 4 If N is the existence interval of the solution u to (0.1) with $E(n_0) < 0$, then N is finite. Further, for $a'(n_0) \ge 0$ we have the estimate

$$N \le N_1^* = n_0 + \int_0^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}} \tag{1.1}$$

for $a'(n_0) < 0$,

$$N \le N_2^* = n_0 + \left(\int_0^{\left(\frac{1}{2}\frac{-1}{E(n_0)}\right)^{\frac{1}{4}}} + \int_{J(n_0)}^{\left(\frac{1}{2}\frac{-1}{E(n_0)}\right)^{\frac{1}{4}}} \right) \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}}, \tag{1.2}$$

where $\alpha = \left(\frac{1}{2}\frac{-1}{E(n_0)}\right)^{\frac{1}{4}}$. Furthermore, if $E(n_0) = 0$ and $a'(n_0) > 0$, then

$$N \le N_3^* := n_0 + \frac{2a(n_0)}{a'(n_0)}. \tag{1.3}$$

Proof For $E(n_0) < 0$, we know that $a(n_0) > 0$; otherwise we would get $a(n_0) = 0$, that is, $u_0 = 0$, then $E(n_0) = u_1^2 \ge 0$, this contradicts $E(n_0) < 0$. In this situation we separate the proof of this Theorem into two subcases, $a'(n_0) \ge 0$ and $a'(n_0) < 0$.

(i) $a'(n_0) \ge 0$. By (0.5) and (0.7), we find that

$$\begin{cases} a'(n) \ge a'(n_0) - 4E(n_0)(n - n_0) & \forall n \ge n_0, \\ a(n) \ge a(n_0) + a'(n_0)n - 2E(n_0)(n - n_0)^2 & \forall n \ge n_0, \end{cases} \tag{1.4}$$

$$J'(n) = -\sqrt{\frac{1}{2} + E(n_0)J(n)^4} \le J'(n_0) \quad \forall n \ge n_0 \tag{1.5}$$

and $J(n) \le a(n_0)^{-\frac{1}{2}} - \frac{1}{2}a(n_0)^{-\frac{3}{2}}a'(n_0)(n - n_0) \quad \forall n \ge n_0$. Thus, there exists a finite number $N_1^*(u_0, u_1) \le n_0 + \frac{2a(n_0)}{a'(n_0)} = n_0 + \frac{u_0}{u_1}$ such that $J(N_1^*(u_0, u_1)) = 0$ and so $a(n) \to \infty$ as $n \to N_1^*(u_0, u_1)$. This means that $N \le N_1^*(u_0, u_1)$. Now we estimate $N_1^*(u_0, u_1)$. By (1.5) and $J(N_1^*(u_0, u_1)) = 0$ we find that

$$\int_{J(n)}^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0)r^4}} = n - n_0, \quad \forall n \ge n_0 \tag{1.6}$$

and hence we get estimate (1.1).

(ii) $a'(n_0) < 0$. By (1.4), $a'(n_0) < 0$ and the convexity of a, we can find a unique finite number $n_1 = n_1(u_0, u_1)$ such that

$$\begin{cases} a'(n) < 0 = a'(n_1) & \text{for } n \in (n_0, n_1), \\ a'(n) > 0 & \text{for } n > n_1, \end{cases} \tag{1.7}$$

and $a(n_1) > 0$. If not, then $u(n_1) = 0$, thus $E(n) = E(n_1) = u'(n_1)^2 \ge 0$; yet this is a contradiction to $E(n_0) < 0$. Hence, we conclude that $a(n) > 0$, $\forall n \ge n_0$, $u'(n_1) = 0$, $E(n_0) = -\frac{1}{2}u(n_1)^4$and $J(n_1)^4 = \frac{-1}{2E(n_1)}$.

After arguments similar to the step (i), there exists an $N_2^* := N_2^*(u_0, u_1)$ such that the life-spann N of u is bounded by N_2^*, that is, $N \le N_2^*$. By an analogous argument, using (1.7), (0.7) and the fact that $J(n_1)^4 = \frac{-1}{2E(n_0)}$ and $J(N_2^*) = 0$, we conclude that

$$J'(n)^2 = -E(n_0)\left(J(n_1)^4 - J(n)^4\right), \quad \forall n \ge n_1,$$
$$J'(n)^2 = E(n_0)\left(J(n_1)^4 - J(n)^4\right), \quad \forall n \in [n_0, n_1],$$

$$J'(n) = -\sqrt{\frac{1}{2} + E(n_0)J(n)^4}, \quad \forall n \ge n_1, \tag{1.8}$$

$$J'(n) = \sqrt{\frac{1}{2} + E(n_0)J(n)^4}, \quad \forall n \in [n_0, n_1], \tag{1.9}$$

$$\int_{J(n)}^{J(n_1)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}} = n - n_1, \quad \forall n \geq n_1, \tag{1.10}$$

$$\int_{J(n_0)}^{J(n_1)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}} = n_1 - n_0, \tag{1.11}$$

and

$$N_2^* = n_1 + \int_0^{\left(\frac{-1}{2E(n_0)}\right)^{\frac{1}{4}}} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}}. \tag{1.12}$$

Estimate (1.12) is equivalent to (1.2).

(iii) For $E(n_0) = 0$, by (0.6) and $a'(n_0) > 0$, we get that $J'(n_0) < 0, J''(n) = 0$ and $J(n) = a(n_0)^{-\frac{3}{2}} \left(a(n_0) - \frac{1}{2} a'(n_0)(n - n_0)\right)$, $\forall n \geq n_0$. Thus we conclude that

$$a(n) = a(n_0)^3 \left(a(n_0) - \frac{1}{2} a'(n_0)(n - n_0)\right)^{-2}, \quad \forall n \geq n_0 \tag{1.13}$$

and (1.3) is proved.

1.2 Estimates for the Life-Spann under $E(n_0) > 0$

In this subsection we consider the case $E(n_0) > 0$. We have the following blow-up result.

Theorem 5 If N^* is the existence interval of u which solves problem (0.1) with $E(n_0) > 0$, then N^* is finite. Further, in case of $a'(n_0) > 0$, we have

$$N^* \leq N_4^*(u_0, u_1) = n_0 + \int_0^{J(n_0)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}}. \tag{1.14}$$

In the case of $a'(n_0) = 0$ we have

$$N^* \leq N_5^*(u_0, u_1) = n_0 + \int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}}. \tag{1.15}$$

For $a'(n_0) < 0$ and $z(u_0, u_1)$ given by

$$z(u_0, u_1) = n_0 + \int_0^{\sqrt{a(n_0)}} \frac{\mathrm{d}r}{\sqrt{E(n_0) + \frac{1}{2} r^4}} \tag{1.16}$$

is the zero of a. Further we have

$$N^* \leq N_6^*(u_0, u_1) := (z + N_5^*)(u_0, u_1). \tag{1.17}$$

Proof The case of a zero for u is postponed to Section 4.

i) For $a'(0) > 0$, by (0.6) we have

$$\begin{cases} mJ''(n) = (mJ(n))^3, \\ mJ(n_0) = ma(n_0)^{-\frac{1}{2}}, \ mJ'(n_0) = \dfrac{-1}{2} ma(n_0)^{-\frac{3}{2}} a'(n_0), \end{cases}$$

where $m := (2E(n_0))^{\frac{1}{2}}$. Now we set

$$\tilde{E}(n) := 2E(n_0) J'(n)^2 - 2E(n_0)^2 (mJ(n))^4. \tag{1.18}$$

From some calculations, we see that $\tilde{E}(n)$ is a constant and by using (0.8) we obtain that

$$\tilde{E}(n)=\frac{1}{2}m^2=\tilde{E}(n_0)=E(n_0), \tag{1.19}$$

$$\frac{1}{2}=J'(n)^2-E(n_0)J(n)^4,$$

$$a'(n)\geq a'(n_0)+2E(n_0)(n-n_0)>0 \quad \forall n\geq n_0, \tag{1.20}$$
$$J'(n)<0 \quad \forall n\geq n_0,$$

$$J'(n)=-\sqrt{\frac{1}{2}+E(n_0)J(n)^4} \quad \forall n\geq n_0, \tag{1.21}$$

and

$$\int_{J(n)}^{J(n_0)}\frac{\mathrm{d}r}{\sqrt{\frac{1}{2}+E(n_0)r^4}}=n-n_0 \quad \forall n\geq n_0. \tag{1.22}$$

By (1.21), there exists a finite number $N_4^*(u_0,u_1)$ such that $J(N_4^*(u_0,u_1))=0$, and from (1.22), estimate (1.14) follows easily.

ii) From $a'(n_0)=0=u_0$, $E(n_0)=u_1^2$ and (0.8) we obtain

$$a'(n)=2E(n_0)n+3\int_{n_0}^{n}u(r)^4\,\mathrm{d}r \quad \forall n\geq n_0, \tag{1.23}$$
$$a(n)>0 \quad \forall n\geq n_0,$$

thus $J(n)$ can be defined for each $n>n_0$ and $J'(n)<0,\ \forall n>n_0$.

Using (0.6), we conclude that, for each $\check{n}>n_0$,

$$J'(n)=-\sqrt{J'(\check{n})^2-E(n_0)\left(J(\check{n})^4-J(n)^4\right)}, \quad \forall n\geq\check{n}, \tag{1.24}$$

$$\lim_{\check{n}\to 0}J'(\check{n})^2-u_1^2J(\check{n})^4=\frac{1}{2}, \tag{1.25}$$

thus after inducing (1.24) and (1.25), estimate (1.15) follows.

(iii) For $a'(n_0)<0$, by (1.20) we have $a'(n)\geq 0$ for large n.

Suppose z is the first positive number n so that $a'(n)=0$, then $u(z)=0$; otherwise, $u'(z)=0$ and $E(z)=-\frac{1}{2}u(z)^4<0$, this contradicts the assumption $E(n_0)=E(z)>0$. After the time $n=z$, same as the procedures given in the proof of (i), using (1.22) we obtain (1.17).

1.3 Some Properties Concerning the Life-Spann $N_1^*(u_0,u_1)$

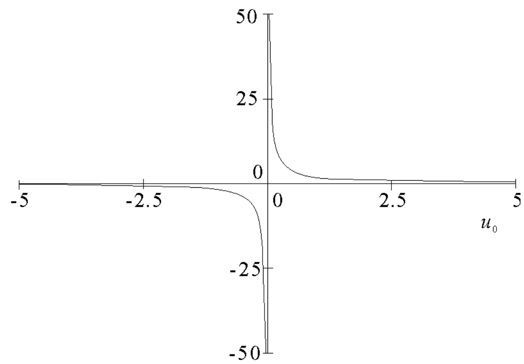
In principle, $N_1^*(u_0,u_1)$ depends on three variables u_0 and u_1. Set $c_k:=\frac{2u_1^2}{u_0^4}$, then

$$N_1^*(u_0,u_1)=\sqrt{2}u_0^{-1}\left(\sqrt[3]{1-c_k}\right)^{-1}\int_0^{\sqrt[4]{1-c_k}}\frac{\mathrm{d}r}{\sqrt{1-r4}}.$$

For convenience, we consider the case $u_1=0$,

$$N_1^*(u_0,0)=\frac{\sqrt{\pi}}{2\sqrt{2}}u_0^{-1}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

Using Maple we obtain the graphs of $N_1^*(u_0,0)$ below:



To grasp the property of the existence interval $N_1^* := N_1^*(u_0, u_1)$ is very difficult, after some computations we get

$$N_1^* = \sqrt{2}\left(u_0^4 - 2u_1^2\right)^{-\frac{1}{4}} \int_0^{\sqrt[4]{1-\frac{2}{u_0^4}u_1^2}} \frac{\mathrm{d}r}{\sqrt{1-r^4}}.$$

By the experience of studying the existence interval N_1^*, we consider its properties with $a'(n_0) \geq 0$ in three cases:

Case 1 $0 < u_0^4 - 2u_1^2 < 1$. In this situation we find that

(i) for fixed $u_1, u_0 \frac{\partial}{\partial u_0} N_1^* < 0$.

(ii) for fixed u_0, the life-spann $N_1^*(u_0, u_1)$ decreases in u_1^2.

Case 2 $u_0^4 - 2u_1^2 > 1$. The life-spann $N_1^*(u_0, u_1) = N_1^*(s, u_0)$ decreases in $s = \sqrt[4]{1 - \frac{2}{u_0^4}u_1^2}$ for fixed u_0.

Case 3 $u_0^{p+1} - 2u_1^2 = 1$. On the region $\left\{(u_0, u_1) \in \mathbb{R}^2 \,\middle|\, u_0^4 - 2u_1^2 = 1\right\}$, we find that

$$N_1^*(u_0, u_1) = N_1^*(u_0) = \sqrt{2}\int_0^{u_0^{-1}} \frac{1}{\sqrt{1-r^4}}\mathrm{d}r,$$

and that $N_1^*(u_0)$ is monotone decreasing in u_0.

2 Blow-up Rate and Blow-up Constant

In this section we study the blow-up rate and blow-up constant for a, a' and a'' under the conditions in Section 1. We have got the following results.

Theorem 6 If u is the solution of the problem (0.1) with one of the following properties:

(i) $E(n_0) < 0$,

(ii) $E(n_0) = 0,\ a'(n_0) > 0$,

(iii) $E(n_0) > 0$.

Then the blow-up rate of a is 2, and the blow-up constant K_1 of a is 2, that is, for $m = 1, 2, 3, 4, 5, 6$,

$$\lim_{n \to N_m^*} \left(N_m^* - n\right)^2 a(n) = 2. \tag{2.1}$$

The blow-up rate of a' is 3, and the blow-up constant K_2 of a' is 4, that is, for $m = 1, 2, 3, 4, 5, 6$,

$$\lim_{n \to N_m^*} \left(N_m^* - n\right)^3 a'(n) = 4. \tag{2.2}$$

The blow-up rate of a'' is 4, and the blow-up constant K_3 of a'' is 12, that is, for $m = 1,2,3,4,5,6$,

$$\lim_{n\to N_m^*} a''(n)\left(N_m^* - n\right)^4 = 12. \tag{2.3}$$

Proof (i) Under this condition, $E(n_0) < 0,\ a'(n_0) \geq 0$ by (1.1), (1.6) and Lemma 4, we get

$$\int_0^{J(n)} \frac{1}{N_1^* - n} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}} = 1 \quad \forall n \geq n_0, \tag{2.4}$$

$$\lim_{n\to N_1^*} \sqrt{2} \frac{J(n)}{N_1^* - n} = 1. \tag{2.5}$$

Identity (2.5) is equivalent to (2.1) for $m = 1$.

For $E(n_0) < 0,\ a'(n_0) < 0$, by (1.9) we have also

$$\int_0^{J(n)} \frac{\mathrm{d}r}{\sqrt{\frac{1}{2} + E(n_0) r^4}} = N_2^* - n, \quad \forall n \geq n_0. \tag{2.6}$$

Through Lemma 4 and (2.6), therefore we get (2.1) for $m = 2$.

Seeing (1.5) and (1.8), we find

$$\lim_{n\to N_m^*} J'(n) = -\frac{1}{2}\sqrt{2}, \tag{2.7}$$

$$\lim_{n\to N_m^*} a'(n)\left(N_m^* - n\right)^3 = 4, \tag{2.8}$$

$$\lim_{n\to N_m^*} u'(n)^2 \left(N_m^* - n\right)^4 = 2 \tag{2.9}$$

for $m = 1,2$. Using (0.5) and (2.9) we obtain, for $m = 1,2$,

$$\lim_{n\to N_m^*} a''(n)\left(N_m^* - n\right)^4 = 6 \lim_{n\to N_m^*} u'(n)^2 \left(N_m^* - n\right)^4. \tag{2.10}$$

Thus, (2.10) and (2.3) are equivalent.

(ii) For $E(n_0) = 0,\ a'(n_0) > 0$, by (1.1) we get, for $m = 1,2$,

$$a(n) = a(n_0)^3 \left(\frac{1}{2} a'(n_0)\right)^{-2} \cdot \left(N_3^* - n\right) \quad \forall n \geq n_0. \tag{2.11}$$

Therefore, estimates (2.1), (2.2) and (2.3) for $m = 3$ are followed from (2.11).

(iii) For $E(n_0) > 0$, estimates (2.1), (2.2) and (2.3) for $m = 4,5,6$, are similar to the above arguments (i) in the proof of this theorem.

3 Global Existence and Critical Point

In this section we study the following case that $E(n_0) = 0$ and $a'(n_0) < 0$.

Here we take the global existence of the solutions to the problem (0.1) in the following sense:

$$J(n) > 0,\ a'(n)^{-2} > 0,\ a''(n)^{-2} > 0 \quad \forall n \in [n_0, N],$$

where N is the time that u exists, in other words, in any finite time u does not blow up in C^2 sense, even though u blows up in a finite time in some sense, for example, C^k or L^k for some $k \geq 3$.

By [12, p.151] every positive proper solution of problem (0.1) has the asymptotic form

$$u(n) \sim cn^{-1}.$$

This result could happen and will be explained below only in the case that $E(n_0) = 0$ and $a'(n_0) < 0$. Under the condition it is easy to see that $J(n) > 0\ \forall n \in (n_0, N)$, and

$$a(n) = a(n_0)^3 \left(a(n_0) - \frac{1}{2}a'(n_0)(n - n_0)\right)^{-2}, \quad \forall n \in (n_0, N),$$

$$a'(n)^{-2} = a(n_0)^{-6} a'(n_0)^{-2} \left(a(n_0) - \frac{1}{2}a'(n_0)(n - n_0)\right)^6 > 0, \quad \forall n \in (n_0, N),$$

$$a''(n)^{-2} = \frac{4}{9}a(n_0)^{-6} a'(n_0)^{-4} \left(a(n_0) - \frac{1}{2}a'(n_0)(n - n_0)\right)^8 > 0, \quad \forall n \in (n_0, N).$$

Hence we find $\lim\limits_{n\to\infty} a(n) = 0$, $\lim\limits_{n\to\infty} a'(n) = 0$, $\lim\limits_{n\to\infty} a''(n) = 0$ and

$$\lim_{n\to\infty} (n - n_0)^2 a(n) = a(n_0)^3 \left(\frac{1}{-2}a'(n_0)\right)^{-2} = \frac{u_0^4}{u_1^2}, \tag{3.1}$$

$$\lim_{n\to\infty} (n - n_0)^3 a'(n) = a(n_0)^3 a'(n_0) \left(\frac{1}{-2}a'(n_0)\right)^{-3} = -\frac{2u_0^4}{u_1^2}, \tag{3.2}$$

$$\lim_{n\to\infty} (n - n_0)^4 a''(n) = \frac{3}{2}a(n_0)^3 a'(n_0)^2 \left(\frac{1}{-2}a'(n_0)\right)^{-4} = \frac{3}{32}\frac{u_0^4}{u_1^2}. \tag{3.3}$$

Theorem 7 Suppose that u is the solution of problem (0.1) with $E(n_0) = 0$ and $a'(n_0) < 0$, then u can be defined globally and estimates $(3.1), (3.2)$ and (3.3) are valid.

4 Existence of Zero and Triviality

In this section we discuss the triviality of the solution for problem (0.1) under the case that $E(n_0) = 0,\ a'(n_0) = 0$.

Proposition If u is the solution of problem (0.1) with $E(n_0) = 0$and $a'(n_0) = 0$, then u must be null.

Proof Under the conditions $E(n_0) = 0$, $a'(n_0) = 0$, by using (0.5), it is easy to see that $u_0 = 0 = u_1$, herein the supremum below exists:

$$n_1 := \sup\{\alpha : a(n) \leq 1,\ \forall n \in [n_0, \alpha]\},$$

and then $2u'(n)^2 = u(n)^4 \geq 0$, $a''(n) = 6u'(n)^2 = 3u(n)^4 = \frac{3}{2}a(n)^2$. By Lemma 2 we conclude that

$$a''(n) \leq 6a(n), \quad a(n) \equiv 0 \equiv u(n) \quad \text{in } [n_0, n_1].$$

Continue these steps we get the assertion of this theorem.

For the case that $E(n_0) > 0 > a'(n_0)$, we have the following result.

Theorem 8 Suppose that u is the solution to problem (0.1) with $E(n_0) > 0 > a'(n_0)$ and $z(u_0, u_1)$ given by

$$z(u_0, u_1) = n_0 + \int_0^{\sqrt{a(n_0)}} \frac{\mathrm{d}r}{\sqrt{E(n_0) + \frac{1}{2}r^4}}, \tag{4.1}$$

then $z(u_0, u_1)$ is the zero of a. Furthermore, we have

$$\lim_{n\to z^-(u_0,u_1)} a(n)\Big(z(u_0, u_1) - n\Big)^{-2} = E(n_0)^2, \tag{4.2}$$

$$\lim_{n\to z^-(u_0,u_1)} \Big(z(u_0, u_1) - n\Big)^{-1} a'(n) = -2E(n_0)^{\frac{3}{2}}, \tag{4.3}$$

$$\lim_{n\to z^-(u_0,u_1)} a''(n) = 2E(n_0). \tag{4.4}$$

Proof 1) For $E(n_0) > 0 > a'(n_0)$, by (0.4) we obtain that

$$a'(n) = -2\sqrt{E(n_0)\,a(n) + \frac{1}{2}a(n)^3}, \tag{4.5}$$

$$z(u_0, u_1) = n_0 + \int_0^{a(n_0)} \frac{\mathrm{d}r}{2\sqrt{E(n_0)\,r + \frac{1}{2}r^3}}, \tag{4.6}$$

$$n = n_0 + \int_{a(n)}^{a(n_0)} \frac{\mathrm{d}r}{2\sqrt{E(n_0)\,r + \frac{1}{2}r^3}}, \tag{4.7}$$

and

$$\begin{aligned} z(u_0, u_1) &= n_0 + \int_0^{a(n_0)} \frac{\mathrm{d}r}{2\sqrt{r}\sqrt{E(n_0) + \frac{1}{2}r^2}} = n_0 + \int_0^{\sqrt{a(n_0)}} \frac{\mathrm{d}r}{\sqrt{E(n_0) + \frac{1}{2}r^4}} \\ &= n_0 + 2^{\frac{1}{4}} E(n_0)^{\frac{-1}{2}} \int_0^{(2E(n_0))^{\frac{-1}{4}}\sqrt{a(n_0)}} \frac{\mathrm{d}r}{\sqrt{1 + r^4}}. \end{aligned} \tag{4.8}$$

Thus, (4.1) is proved.

2) To claim (4.2), by (4.6), (4.7) and Lemma 3, it induces that

$$z(u_0, u_1) - n = \int_0^{a(n)} \frac{\mathrm{d}r}{2\sqrt{E(n_0)\,r + \frac{1}{2}r^3}} = 2^{\frac{1}{4}} E(n_0)^{\frac{-1}{4}} \int_0^{(2E(n_0))^{\frac{-1}{4}}\sqrt{a(n)}} \frac{\mathrm{d}r}{\sqrt{1 + r^4}},$$

$$(z(u_0, u_1) - n)^{-1} \int_0^{(2E(n_0))^{\frac{-1}{4}}\sqrt{a(n)}} \frac{\mathrm{d}r}{\sqrt{1 + r^4}} = \sqrt[4]{\frac{E(n_0)}{2}},$$

$$\begin{aligned} \sqrt[4]{\frac{E(n_0)}{2}} &= \lim_{n\to z^-(u_0,u_1)} (z(u_0, u_1) - n)^{-1} \int_0^{(2E(n_0))^{\frac{-1}{4}}\sqrt{a(n)}} \frac{\mathrm{d}r}{\sqrt{1 + r^4}} \\ &= \lim_{n\to z^-(u_0,u_1)} (z(u_0, u_1) - n)^{-1} (2E(n_0))^{\frac{-1}{4}} \sqrt{a(n)} \\ &\quad \cdot \lim_{n\to z^-(u_0,u_1)} \int_0^1 \frac{\mathrm{d}s}{\sqrt{1 + \Big((2E(n_0))^{\frac{-1}{4}} \sqrt{a(n)}s\Big)^4}} \\ &= \lim_{n\to z^-(u_0,u_1)} (z(u_0, u_1) - n)^{-1} (2E(n_0))^{\frac{-1}{4}} \sqrt{a(n)}. \end{aligned} \tag{4.9}$$

Thus we get conclusion (4.2).

3) Using (4.8) and (4.5) we obtain that

$$\begin{aligned}&\lim_{n\to z^-(u_0,u_1,p)} (z(u_0,u_1,p)-n)^{-1} a'(n)\\&=-2\lim_{n\to z^-(u_0,u_1,p)} \sqrt{a(n)(z(u_0,u_1,p)-n)^{-2}\left(E(0)+\frac{2}{p+1}a(n)^{\frac{p+1}{2}}\right)}\\&=-2E(0)^{\frac{3}{2}}.\end{aligned}$$

4) Applying (0.5), (4.2) and (4.3), we find

$$\begin{aligned}&\lim_{n\to z^-(u_0,u_1)} a(n)(z(u_0,u_1)-n)^{-2} a''(n)\\&=\frac{3}{2}\lim_{n\to z^-(u_0,u_1)} \left(a'(n)(z(u_0,u_1)-n)^{-1}\right)^2-4E(n_0)\lim_{n\to z^-(u_0,u_1)} a(n)(z(u_0,u_1)-n)^{-2}\\&=2E(n_0)^3.\end{aligned}$$

Hence (4.4) is proved.

5 Stability, Instability and Asymptotic Analysis

We now consider the applications of the theorems above to the stability theory for the problem

$$\begin{cases} u''(n)=u(n)^3,\\ u(n_0)=\varepsilon_1, u'(n_0)=\varepsilon_2.\end{cases} \tag{5.1}$$

We say problem (5.1) is stable under condition F, if any nontrivial global solution $u\in C^2(\mathbb{R}^+)$ of (5.1) under condition F satisfies $\|u\|_{C^2}\to 0$ for $|\varepsilon_1|+|\varepsilon_2|\to 0$. According to Theorems 4–8 we have the following result.

Corollery 9.1 Problem (5.1) is stable under $E_u(n_0)=0$, $\varepsilon_1\varepsilon_2<0$, and unstable under one of the followings:

(i) $E_u(n_0)<0$,

(ii) $E_u(n_0)=0<\varepsilon_1\varepsilon_2$,

(iii) $E_u(n_0)>0$.

According to Theorems 4 and 5, we can obtain the following conclusions, when we reconsider the problem

$$\begin{cases} u''(n)=u(n)^3,\\ u(n_0)=\varepsilon u_0, u'(n_0)=\varepsilon^2 u_1, \varepsilon>0.\end{cases} \tag{5.2}$$

Theorem 9.2 If N_ε is the existence interval of the solution u_ε to (5.2) with $E(n_0)<0$, then N_ε is finite. Further, for $a'(n_0)\ge 0$, we have the estimate $N_\varepsilon\le N^*_{1,\varepsilon}=n_0+\varepsilon^{-1}(N^*_1-n_0)$ for $a'(n_0)<0$,

$$N_\varepsilon\le N^*_{2,\varepsilon}=n_0+\varepsilon^{-1}(N^*_2-n_0).$$

Furthermore, if $E(n_0)=0$ and $a'(n_0)>0$, then

$$N_\varepsilon\le N^*_3:=n_0+\varepsilon^{-1}(N^*_2-n_0). \tag{5.3}$$

Theorem 9.3 If N_ε^* is the existence interval of u which solves problem (5.2) with $E(n_0) > 0$, then N_ε^* is finite. Further, in case of $a'(n_0) > 0$ we have

$$N_\varepsilon^* \le N_{4,\varepsilon}^*(u_0, u_1) = n_0 + \varepsilon^{-1}(N_4^* - n_0).$$

In the case of $a'(n_0) = 0$ we have

$$N_\varepsilon^* \le N_{5,\varepsilon}^*(u_0, u_1) = n_0 + \varepsilon^{-1}(N_5^* - n_0).$$

For $a'(n_0) < 0$, $z_\varepsilon(u_0, u_1)$ given by

$$z_\varepsilon(u_0, u_1) = n_0 + \varepsilon^{-1}(z(u_0, u_1) - n_0)$$

is the zero of u_ε. Further we have

$$N_\varepsilon^* \le N_{6,\varepsilon}^*(u_0, u_1) := \left(z_\varepsilon + N_{5,\varepsilon}^*\right)(u_0, u_1).$$

6 Regularity of Solution to (0.1)

In this section we study the regularity of the positive solution u of the nonlinear problem (0.1). Using (0.4) we have

$$u'(n)^2 = E(n_0) + \frac{1}{2}u(n)^4, \tag{6.1}$$

where $E(n_0) = u_1^2 - \frac{1}{2}u_0^4$.

6.1 Regularity of Solution to (0.1)

Now consider the regularity of the positive solution u of problem (0.1). We have the following results.

Theorem 10 If u is the positive solution of problem (0.1) with the life-spann N^*, then $u \in C^q(n_0, N^*)$ for any $q \in \mathbb{N}$ and

$$u^{(2k)} = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i} u^{C_{k,i}}, \tag{6.2}$$

$$u^{(2k+1)} = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i} C_{k,i} u^{C_{k,i}-1} u' = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} O_{k,i} u^{C_{k,i}-1} u' \tag{6.3}$$

for positive integer k, where $\left[\frac{C_{k,0}}{4}\right]$ denotes the Gaussian integer number of $\frac{C_{k,0}}{4}$,

$$C_{k,i} = 4(k-i) - 2n + 1, O_{k,i} = E_{k,i} C_{k,i},\ E_{0,0} = 1,$$

$$E_{k,0} = O_{k-1,0}\left[\frac{1}{2}(C_{k-1,0} + 1)\right] = E_{k-1,0} C_{k-1,0}\left[\frac{1}{2}(C_{k-1,0} + 1)\right],$$

$$\begin{aligned} E_{k,k-1} &= O_{k-1,k-2}(C_{k-1,k-2} - 1) E(0) \\ &= E_{k-1,k-2} C_{k-1,k-2}(C_{k-1,k-2} - 1) E(0), \end{aligned}$$

and

$$E_{k,l} = O_{k-1,l-1}\left(C_{k-1,l-1}-1\right)E\left(0\right) + O_{k-1,l}\left[\frac{1}{2}\left(C_{k-1,l}+1\right)\right]$$
$$= E_{k-1,l-1}C_{k-1,l-1}\left(C_{k-1,l-1}-1\right)E\left(0\right) + E_{k-1,l}C_{k-1,l}\left[\frac{1}{2}\left(C_{k-1,l}+1\right)\right]$$

for positive integer $l, 0 < l < k$.

Proof Let v_k be the k-th derivative of u; that is $v_k := u^{(k)}$, and denote $v_0^k = u^k$, $v_0 = u$, $v_1 = u'$, $v_2 = u''$, $v_1^2 = (u')^2$, etc. Now let us use the mathematical induction to prove (6.2). When $k = 1$, we have

$$v_2 = \sum_{i=0}^{\left[\frac{C_{1,0}}{4}\right]} E_{1,i}u^{C_{1,i}} = E_{10}u^{C_{1,0}} = v_0^3$$

and

$$C_{0,0} = 1, \quad C_{1,0} = 3, \quad E_{1,0} = E_{0,0}C_{0,0}\left[\frac{1}{2}\left(C_{0,0}-1\right)+1\right] = 1.$$

Suppose that $k \in \mathbb{N}$ and $v_{2k} = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i} \cdot v_0^{C_{k,i}}$. Then by (6.1) we obtain

$$v_{2k+1} = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i}C_{k,i}v_0^{C_{k,i}-1}v_1,$$

$$v_{2k+2} = \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i}C_{k,i}v_0^{C_{k,i}-1}v_2 + \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} E_{k,i}C_{k,i}\left(C_{k,i}-1\right)v_0^{C_{k,i}-2}v_1^2,$$

$$\begin{aligned} v_{2k+2} &= \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} O_{k,i}\left[\frac{1}{2}\left(C_{k,i}+1\right)\right]v_0^{C_{k,i}+2} + \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} O_{k,i}\left(C_{k,i}-1\right)E\left(0\right)v_0^{C_{k,i}-2} \\ &= \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} O_{k,i}\left[\frac{1}{2}\left(C_{k,i}+1\right)\right]v_0^{C_{k+1,i}} + \sum_{i=0}^{\left[\frac{C_{k,0}}{4}\right]} O_{k,i}\left(C_{k,i}-1\right)E\left(0\right)v_0^{C_{k+1,i+1}} \\ &= O_{k,0}\left[\left(\frac{1}{2}\left(C_{k,0}+1\right)\right)\right]v_0^{C_{k+1,0}} + O_{k,0}\left(C_{k,0}-1\right)E\left(0\right)v_0^{C_{k+1,1}} \\ &\quad + O_{k,1}\left[\left(\frac{1}{2}\left(C_{k,1}+1\right)\right)\right]v_0^{C_{k+1,1}} + O_{k,1}\left(C_{k,1}-1\right)E\left(0\right)v_0^{C_{k+1,2}} \\ &\quad + O_{k,2}\left[\left(\frac{1}{2}\left(C_{k,2}+1\right)\right)\right]v_0^{C_{k+1,2}} + \cdots \\ &\quad + O_{k,\left[\frac{C_{k,0}}{4}\right]}\left(C_{k,\left[\frac{C_{k,0}}{4}\right]}-1\right)E\left(0\right)v_0^{C_{k+1,\left(\left[\frac{C_{k,0}}{4}\right]+1\right)}}. \end{aligned}$$

Hence

$$v_{2k+2} = \sum_{i=0}^{\left[\frac{C_{k+1,0}}{4}\right]} E_{k+1,i} \cdot v_0^{C_{k+1,i}},$$

which completes the induction, and we obtain (6.2). Using (6.2), we get (6.3).

7 The Blow-up Rate and Blow-up Constant for $u^{(k)}$

Finding out the blow-up rate and blow-up constant of $u^{(k)}$ of (0.1) is our main result.

Theorem 11 If u is the solution of problem (0.1) with one of the following properties that:

(i) $E(n_0) < 0$,

(ii) $E(n_0) = 0,\ a'(n_0) > 0$,

(iii) $E(n_0) > 0$.

Then the blow-up rate of $u^{(2k)}$ is $1+2k$, and the blow-up constant of $u^{(2k)}$ is $\left|E_{k,0}\left(\sqrt{2}\right)^{1+2k}\right|$; that is, for $k \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\lim_{n\to N_m^*} u^{(2k)}(n)\left(N_m^*-n\right)^{1+2k} = (\pm 1)^{C_{k,0}} E_{k,0}\left(\sqrt{2}\right)^{1+2k} := K_{2k}. \tag{7.1}$$

The blow-up rate of $u^{(2k+1)}$ is $2k+2$, and the blow-up constant of $u^{(2k+1)}$ is $\left|E_{k,0}C_{k,0}\left(\sqrt{2}\right)^{2k+1}\right|$; that is, for $k \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\lim_{n\to N_m^*} u^{(2k+1)}(n)\left(N_m^*-n\right)^{2+2k} = (\pm)^{C_{k,0}} E_{k,0}C_{k,0}\left(\sqrt{2}\right)^{2k+1} := K_{2k+1}, \tag{7.2}$$

where $C_{k,0} = 2k+1,\ E_{k,0} = \prod\limits_{i=0}^{k-1}\left[\frac{4i^2+5i+2}{2}\right]$.

Proof Under condition (i), $E(n_0) < 0,\ a'(n_0) \geq 0$ by (1.6) and (1.1), we get

$$\int_0^{J(n)} \frac{1}{N_1^*-n}\frac{\mathrm{d}r}{\sqrt{\frac{1}{2}+E(n_0)\,r^4}} = 1, \quad \forall n \geq n_0. \tag{7.3}$$

Using Lemma 3 and (1.6), we obtain $\lim\limits_{n\to N_1^*}\sqrt{2}\frac{J(n)}{N_1^*-n} = 1$, in other words,

$$\lim_{n\to N_1^*} a(n)\left(N_1^*-n\right)^2 = 2, \tag{7.4}$$

and then

$$\lim_{n\to N_1^*} u(n)\left(N_1^*-n\right) = \pm\sqrt{2}. \tag{7.5}$$

Here $C_{k,i} = 2k+1-4i$, hence we have $C_{k,i} > C_{k,j}$ as $i < j$. From (6.1) and (7.5), it follows

$$\lim_{n\to N_1^*} u^{(2k)}(n)\left(N_1^*-n\right)^{C_{k,0}} = (\pm 1)^{C_{k,0}} E_{k,0}\left(\sqrt{2}\right)^{C_{k,0}}.$$

Since $C_{k,0} = 1+2k$, we get (7.1) for $m = 1$.

By (1.5), (7.4) and (6.2) we find that

$$\lim_{n\to N_1^*} J'(n) = -\frac{1}{\sqrt{2}}, \tag{7.6}$$

$$\sqrt{2} = \lim_{n\to N_1^*}\left(a(n)\left(N_1^*-n\right)^2\right)^{-\frac{3}{2}} \cdot \lim_{n\to N_1^*} a'(n)\left(N_1^*-n\right)^3,$$

$$\lim_{n\to N_1^*} u'(n)\left(N_1^*-n\right)^2 = \pm\sqrt{2}, \tag{7.7}$$

and

$$\begin{aligned}
&\lim_{n\to N_1^*} u^{(2k+1)}(n)\left(N_1^*-n\right)^{C_{k,0}+1}\\
&=\lim_{n\to N_1^*}\sum_{i=0}^{k-1} E_{k,i}C_{k,i}u^{C_{k,i}-1}(n)\cdot u'(n)\cdot\left(N_1^*-n\right)^{C_{k0}+1}\\
&=\lim_{n\to N_1^*} E_{k,0}C_{k,0}u^{C_{k,0}-1}(n)\cdot u'(n)\cdot\left(N_1^*-n\right)^{C_{k,0}+1}\\
&=\lim_{n\to N_1^*} E_{k,0}C_{k,0}u^{C_{k,0}-1}(n)\cdot\left(N_1^*-n\right)^{C_{k,0}-1}\cdot u'\cdot\left(N_1^*-n\right)^{2}\\
&=(\pm)^{C_{k,0}}E_{k,0}C_{k,0}\left(\sqrt{2}\right)^{C_{k,0}},
\end{aligned}$$

thus (7.2) is proved for $m=1$.

For $E(0)<0,\ a'(0)<0$, by (1.9) we have

$$\int_0^{J(n)}\frac{\mathrm{d}r}{\left(N_2^*-n\right)\sqrt{\frac{1}{2}+E\left(n_0\right)r^4}}=1\quad\forall n\geq n_0. \tag{7.8}$$

Using Lemma 3, (7.8) and (6.1), therefore we gain the estimate (7.1) for $m=2$, and by (1.8) we get estimate (7.2) for $m=2$.

Under (ii), $E(n_0)=0,\ a'(n_0)>0$, we have

$$a(n)=a\left(n_0\right)^3\left(\frac{1}{2}a'\left(n_0\right)\left(N_3^*-n\right)\right)^{-2}\qquad\forall n\geq n_0. \tag{7.9}$$

In view of (7.9) and (6.1), we get estimate (7.1) for $m=3$. Also, we have $J'(n)=J'(n_0),\ \forall n\geq n_0$ and $\lim\limits_{n\to N_1^*} a(n)^{-\frac{3}{2}}a'(n)=-\frac{1}{2}a\left(n_0\right)^{-\frac{3}{2}}a'\left(n_0\right)$.

By (7.9) and (6.2), estimate (7.2) for $m=3$ is completely proved.

Under (iii), the proofs of estimates (7.1) and (7.2) for $m=4,5,6$ are similar to the above, we omit the argumentation.

Theorem 12 If u is the solution of problem (0.1) with $E(n_0)>0$ and $a'(n_0)<0$, then we have

$$\lim_{n\to z^-(u_0,u_1)} u^{(2k)}(n)\left(z\left(u_0,u_1\right)-n\right)^{-C_{k,k-1}}=(\pm)^{C_{k,k-1}}E_{k,k-1}E\left(n_0\right)^{\frac{C_{k,k-1}}{2}} \tag{7.10}$$

and

$$\lim_{n\to z^-(u_0,u_1)} u^{(2k+1)}(n)\left(z\left(u_0,u_1\right)-n\right)^{-C_{k,k-1}+1}=E_{k,k-1}C_{k,k-1}E\left(n_0\right)^{C_{k,k-1}-1} \tag{7.11}$$

for $k\in\mathbb{N}$, where z is the null point (zero) of u and

$$C_{k,k-1}=5-2n,\quad E_{k,k-1}=\prod_{i=0}^{k-1}(5-2i)(4-2i)E\left(n_0\right)^{k-1}.$$

Proof For $E(n_0)>0$ and $a'(n_0)<0$, we have

$$\lim_{n\to z^-(u_0,u_1)} u^{(2k)}(n)\left(z\left(u_0,u_1\right)-n\right)^{-C_{k,k-1}}$$

$$
\begin{aligned}
&= \lim_{n\to z^-(u_0,u_1)} \sum_{i=0}^{k-1} E_{k,i} u^{C_{k,i}}(n)\,(z(u_0,u_1)-n)^{-C_{k,k-1}} \\
&= \lim_{n\to z^-(u_0,u_1)} E_{k,k-1} u^{C_{k,k-1}}(n)\,(z(u_0,u_1)-n)^{-C_{k,k-1}} \\
&= (\pm 1)^{C_{k,k-1}} E_{k,k-1} E(n_0)^{\frac{C_{k,k-1}}{2}}\,.
\end{aligned}
$$

Therefore, (7.10) is proved.

From (6.2) , we obtain that

$$
\begin{aligned}
&\lim_{n\to z^-(u_0,u_1)} u^{(2k+1)}(n)\,(z(u_0,u_1)-n)^{-C_{k,k-1}+1} \\
&= \lim_{n\to z^-(u_0,u_1)} \sum_{i=0}^{k-1} E_{k,i} C_{k,i} u^{C_{k,i}-1}(n)\,u'(n)\,(z(u_0,u_1)-n)^{-C_{k,k-1}+1} \\
&= \lim_{n\to z^-(u_0,u_1)} E_{k,k-1} C_{k,k-1} u^{C_{k,k-1}-1}(n)\,u'(n)\,(z(u_0,u_1)-n)^{-C_{k,k-1}+1} \\
&= E_{k,k-1} C_{k,k-1} E(n_0)^{C_{k,k-1}}\,.
\end{aligned}
$$

Thus, (7.11) is obtained.

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