Option Pricing Using the Martingale Approach with Polynomial Interpolation

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This study shows that in particular cases, the minimal martingale measure coincides with the Esscher martingale measure. Using the martingale approach can produce an exact solution for the price of a European call option on an asset modeled as an exponential Lévy process when a closed-form expression exists for the Lévy measure under some integrability conditions. If the jump component vanishes, the solution reduces to the Black–Scholes formula. To compute the option price accurately and quickly, this study uses polynomial interpolation with divided differences. A numerical analysis compares the accuracy and CPU time of the latter method with those of three Fourier-based formulas described by Lewis (2001). © 2012 Wiley Periodicals, Inc. Jrl Fut Mark 33:469–491, 2013

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INTRODUCTION

Option pricing usually relies on the risk-neutral approach, which involves choosing a martingale measure for the discounted asset prices. However, the exponential Lévy models contain numerous possible choices for such measures. For example, the Esscher transform can produce an equivalent martingale measure, which in the case of the exponential Lévy process achieves minimum relative entropy (see Chan, 1999), whereas the Lévy property of the process continues to hold. Another equivalent martingale procedure employs the minimal measure described by Föllmer and Schweizer (1991), which minimizes the risk involved in trying to replicate a contingent claim while also preserving the Lévy property of the process. Noting these various options, this article aims to demonstrate that in some cases the minimal martingale measure actually coincides with the Esscher martingale measure.

As our second research goal, we derive an expression for the price of a European call option for an asset modeled as an exponential Lévy process. To evaluate the option, we use the underlying asset and savings account as numeraires; change the probability measure, such that we can write the option price in terms of two probabilities; and offer an exact closed-form formula based on an inversion for obtaining the two probabilities. In this scenario, we need to combine a nontrivial option pricing formula with an efficient algorithm to compute prices quickly and accurately. Black and Scholes (1973) derive a pricing formula that depends on two standard normal distributions; their famous formula is a special case of our proposed formula. In the exponential Lévy model, Carr and Madan (1999) and Lewis (2001) also suggest formulas to evaluate option prices accurately in terms of a characteristic function of the asset return. Although Lewis's (2001) formula appears to take a Black–Scholes form, it suffers from slow numerical computing and cannot reduce to the Black–Scholes formula, even after excluding the jump part.

Finally, we aim to show that the method of polynomial interpolation with divided differences is efficient for estimating option prices. To examine the accuracy and CPU time for our approach, we apply five methods and compute 540 option prices for three sets of parameters of the diffusion-generalized tempered stable process (D-GTSP), 60 strike levels, and three maturities. These results reveal that our pricing formula can be computed efficiently through polynomial interpolation.

To achieve these research goals, we structure the remainder of our article as follows. The next section contains the model for the asset price process, as well as a discussion of the link between the Esscher martingale measure and the minimal measure. After we demonstrate how to calculate the price of an option, we introduce the polynomial interpolation method. Then we consider

D-GTSP as a driver of the underlying asset, compare the pricing errors, and show that the CPU time needed for the method we introduced is much smaller than that required by other pricing formulas. Finally, we conclude with some implications.

MODEL OF ASSET PRICE WITH NO ARBITRAGE CONDITION

We assume a complete stochastic basis, $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T^*}, P)$, where the filtration satisfies the usual conditions, and we consider the range $0 < T < T^*$. A Lévy process is a càdlàg stochastic process X_t with stationary independent increments; the characteristic function, which corresponds to a characteristic triplet (ω, σ, ν), can be expressed as

$$E_P\left[\exp\left(izX_t\right)\right] = \exp\left(t\Psi\left(z\right)\right),$$

where Ψ is the characteristic exponent of X. It satisfies the following Lévy Khinchine representation:

$$\Psi(z) = i\omega z - \frac{1}{2}\sigma^2 z^2 + \int_R (e^{izx} - 1 - izx1_{|x|\le 1})\nu(dx),$$

where $\omega \in R$, $\sigma > 0$, and ν is a Lévy measure in $R\setminus\{0\}$, with $\int_R (x^2 \wedge 1)\nu(dx) < \infty$. For our study purposes, we also require the process X to satisfy the following condition:

$$E_P\left[\exp\left(uX_1\right)\right] < \infty \quad \text{for all } u \in R. \tag{1}$$

This requirement ensures that X_t has finite moments in all orders. The Lévy measure ν should also satisfy the following conditions for all $u \in R$:

$$\int_{|x|\ge 1} e^{ux}\nu(dx) < \infty,$$
$$\int_{|x|\ge 1} x\nu(dx) < \infty, \quad \text{and}$$
$$\int_{|x|\ge 1} x^h e^{ux}\nu(dx) < \infty \quad \text{for } h > 0.$$

Next, we consider a driftless Lévy process $X_t = \sigma W_t + \int_0^t \int_R x(\mu - \nu)(du, dx)$, where W is a P-standard Brownian motion on R, and μ is the random measure of jumps associated to Lévy measure $\nu(dt, dx) = \nu(dx)dt$. From the Lévy Khinchine representation and Equation (1), we recognize that the characteristic function of X_t can be expressed as

$$E_P\left[\exp\left(izX_t\right)\right] = \exp\left(t\left(-\frac{1}{2}\sigma^2 z^2 + \Psi_{\mu\nu}(z)\right)\right),$$

where

$$\Psi_{\mu\nu}(z) = \int_R \left(\exp(izx) - 1 - izx\right)\nu(dx).$$

Next, suppose that the asset price process S can be modeled by the following exponential Lévy process:

$$S_t = S_0 \exp\left(bt + X_t\right) = S_0 \exp\left(bt + \sigma W_t + \int_0^t \int_R x\left(\mu - \nu\right)\left(du, dx\right)\right), S_0 > 0, \tag{2}$$

where S_0 is the current asset price, and $b \in R$. In differential form, Equation (2) is equivalent to

$$\frac{dS_t}{S_{t^-}} = \left(b + \frac{1}{2}\sigma^2\right)dt + \sigma dW_t + \int_R \left(e^x - 1\right)\left(\mu - \nu\right)(dt, dx) + \int_R \left(e^x - 1 - x\right)\nu(dt, dx).$$

Furthermore, the risk-free security can be expressed as $B_t = B_0 e^{rt}, \forall t \in [0, T^*]$, with $B_0 = 1$, where r is a constant interest rate. We can then denote the discounted asset price at time t as

$$S_t^* = \frac{S_t}{B_t} = S_0 \exp\left((b - r)t + \sigma W_t + \int_0^t \int_R x\left(\mu - \nu\right)\left(du, dx\right)\right).$$

Using Ito's formula, the evolution of the discounted asset price is

$$\frac{dS_t^*}{S_{t^-}^*} = \left(b - r + \frac{1}{2}\sigma^2\right)dt + \sigma dW_t + \int_R \left(e^x - 1\right)\left(\mu - \nu\right)(dt, dx) + \int_R \left(e^x - 1 - x\right)\nu(dt, dx). \tag{3}$$

According to the fundamental theorem of asset pricing, we know that the market model defined by $\mathcal{B}$ and asset price $(S_t)_{t\in[0,T]}$ is arbitrage free, if and only if there exists a probability measure P^* equivalent to P, such that the discounted price process $(S_t^*)_{t\in[0,T]}$ is a P^*-martingale. According to the Girsanov theorem for the Lévy processes (Papapantoleon, 2005), we can define the Radon–Nikodym derivative for the change of the measure from P to P^* as follows:

$$\begin{aligned} Z_t = \exp\Bigg[&\int_0^t \beta\sigma dW_u - \frac{1}{2}\int_0^t \beta^2\sigma^2 du + \int_0^t\int_R (\kappa(x) - 1)(\mu - \nu)(du, dx) \\ &- \int_0^t\int_R (\kappa(x) - 1 - \ln(\kappa(x)))\,\mu(du, dx)\Bigg], \end{aligned} \tag{4}$$

where the tuple (β, κ) characterizes the change in the drift of the continuous part, as well as the change in the compensator portion of the jump part of the process.

For P^*, the process $dW_t^* = dW_t - \beta\sigma dt$ is a standard Brownian motion, and the measure $\nu^*(dt, dx) = \kappa(x)\nu(dt, dx)$ is the compensator of the random measure μ. Therefore, for W^* and ν^*, the discounted asset price in Equation (3) can be rewritten as

$$\begin{aligned} \frac{dS_t^*}{S_{t-}^*} = &\left(b - r + \left(\frac{1}{2} + \beta\right)\sigma^2\right)dt + \sigma dW_t^* + \int_R (e^x - 1)(\mu - \nu^*)(dt, dx) \\ &+ \int_R ((e^x - 1)\kappa(x) - x)\,\nu(dt, dx). \end{aligned}$$

According to the dynamics of S^*, the discounted asset price S^* follows a martingale under P^*, provided

$$b - r + \left(\frac{1}{2} + \beta\right)\sigma^2 + \int_R ((e^x - 1)\kappa(x) - x)\,\nu(dx) = 0. \tag{5}$$

Accordingly, the evolution of the asset price process S under P^* can be expressed as

$$\begin{aligned} \frac{dS_t}{S_{t-}} = &\left(b + \left(\frac{1}{2} + \beta\right)\sigma^2\right)dt + \sigma dW_t^* + \int_R (e^x - 1)(\mu - \nu^*)(dt, dx) \\ &+ \int_R ((e^x - 1)\kappa(x) - x)\,\nu(dt, dx). \end{aligned}$$

Using the martingale condition in Equation (5), the evolution of the asset price under P^* is

$$\frac{dS_t}{S_{t^-}} = r\,dt + \sigma dW_t^* + \int_R (e^x - 1)\,(\mu - \nu^*)\,(dt, dx).$$

Thus, the asset price at time t equals

$$S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t^*+\int_0^t \int_R x(\mu-\nu^*)(du,dx)-\int_0^t \int_R \left(e^x-1-x\right)\nu^*(du,dx)}. \qquad (6)$$

Also, from the martingale condition in Equation (5), we know that β and κ cannot be specified uniquely. In other words, the market is incomplete, and a perfect hedge does not exist for the arbitrary contingent claim. We use the Esscher transform and minimal measure to find β and κ. Gerber and Shiu (1994) are the first to use the Esscher transform in option pricing. We follow their lead and let $\theta \in R$. Thus, we consider the following Esscher transform:

$$\left.\frac{dP_\theta}{dP}\right|_{\mathcal{F}_t} = Z_t^\theta = \frac{e^{\theta\left(\sigma W_t+\int_0^t \int_R x(\mu-\nu)(du,dx)\right)}}{e^{t\left(\frac{1}{2}\theta^2\sigma^2+\int_R \left(e^{\theta x}-1-\theta x\right)\nu(dx)\right)}}, \qquad (7)$$

where we can choose θ such that the discounted asset price S^* follows a martingale under P_θ. When $Z = Z^\theta$, the comparison of Equations (7) and (4) reveals that the Esscher transform corresponds to the choices $\kappa(x) = e^{\theta x}$ and $\beta = \theta$. The martingale condition in Equation (5) then can be used to solve θ:

$$b - r + \left(\frac{1}{2} + \theta\right)\sigma^2 + \int_R \left((e^x - 1)\,e^{\theta x} - x\right)\nu\,(dx) = 0.$$

Next, we show that the Esscher martingale measure is also the minimal martingale measure introduced by Föllmer and Schweizer (1991) in an exponential Lévy process, assuming certain conditions are satisfied. Föllmer and Schweizer (1991) define the minimal martingale measure as follows.

Definition. An equivalent martingale measure $\bar{P}$ is minimal if any square-integrable P-martingale, orthogonal to the martingale part of S^* under P,[1] remains a martingale under $\bar{P}$.

[1]Two martingales are orthogonal if their product follows a martingale.

We thus write $M_t^Z = \int_0^t (\sigma dW_u + \int_R (e^x - 1)(\mu - \nu)(du, dx))$. From Equation (3), we know that the martingale part of the discounted price process S^* under P is

$$M_t^S = \int_0^t S_{u^-}^* dM_u^Z,$$

We also introduce a minimal measure $\bar{P}$, defined by

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{F}_T} = \bar{Z}_T,$$

where the Radon–Nikodym derivative $\bar{Z}$ satisfies

$$\begin{aligned}\bar{Z}_t &= 1 + \int_0^t \gamma \bar{Z}_{u^-} d M_u^Z \\ &= 1 + \int_0^t \gamma \bar{Z}_{u^-} \left(\sigma d W_u + \int_R (e^x - 1)(\mu - \nu)(du, dx)\right), \end{aligned} \tag{8}$$

and we can choose γ to make S^* a martingale under $\bar{P}$. If a square-integrable P-martingale N is orthogonal to M^S, then

$$E_{\bar{P}}[N_t] = E_P\left[N_t \bar{Z}_t\right] = E_P[N_t].$$

Therefore, N is a $\bar{P}$-local martingale, and the probability measure $\bar{P}$ is the minimal martingale measure.

Using Ito's formula, Z admits the following integral representation:

$$Z_t = 1 + \int_0^t Z_{u^-} \left(\beta\sigma d W_u + \int_R (\kappa(x) - 1)(\mu - \nu)(du, dx)\right). \tag{9}$$

When $Z = \bar{Z}$, we can compare Equation (8) with Equation (9) and obtain

$$\beta = \gamma, \tag{10}$$

and

$$\kappa(x) = (e^x - 1)\gamma + 1. \tag{11}$$

Then we can combine the martingale condition of Equation (5) with Equation (10) and (11) to derive

$$\gamma = \frac{r - b - \frac{1}{2}\sigma^2 - \int_R (e^x - 1 - x)\,\nu(dx)}{\sigma^2 + \int_R (e^x - 1)^2\,\nu(dx)}.$$

According to Papapantoleon's (2005) Theorem 12.1, the equivalent martingale measure exists if and only if $\kappa(x) = (e^x - 1)\gamma + 1 > 0$, and it thus follows that the minimal martingale measure P^* exists if and only if $0 \le \gamma \le 1$.

Because $\nu^*(dx) = \kappa(x)\nu(dx) = \gamma e^x \nu(dx) + (1-\gamma)\nu(dx)$, we know that ν^* is a weighted sum of two Lévy measures: the original Lévy measure ν and the transformed Lévy measure $e^x\nu$ from the Esscher transform, with the Esscher parameter $\theta = 1$. From Equation (1), we have determined that $\int_R (x^2 \wedge 1)(\gamma e^x \nu(dx) + (1-\gamma)\nu(dx)) < \infty$. Therefore, the process with ν^* under the minimal measure P^* retains the Lévy property of the process.

When $\theta = \frac{1}{x}\log[(e^x - 1)\gamma + 1]$, for $x \neq 0$, it is easy to show that the minimal measure cannot coincide with the Esscher martingale measure if $0 < \gamma < 1$. Therefore, we consider two cases: $\gamma = 0$ and $\gamma = 1$. If $\gamma = 0$, then $\beta = \theta = 0$; that is, $\frac{dP^*}{dP}\big|_{\mathcal{F}_t} = 1$ for all t. In turn, the measures P and P^* are the same. However, if $\gamma = 1$, then $\beta = 1$ and $\kappa(x) = e^x$, in which case $dW_t^* = dW_t - \sigma dt$ and $\nu^* = e^x\nu$. This condition corresponds to the martingale condition of the Esscher transform with the Esscher parameter $\theta = 1$. The martingale condition from Equation (5) now becomes

$$b = r - \frac{3}{2}\sigma^2 - \int_R \left(e^{2x} - e^x - x\right)\nu(dx) = r - \frac{3}{2}\sigma^2 - \Psi_{\mu\nu}(-2i) + \Psi_{\mu\nu}(-i). \quad (12)$$

As argued, we have thus established the following proposition:

Proposition 1: Assume that the asset price process is modeled as in Equation (2), the Esscher martingale measure is defined as in Equation (7), and the minimal martingale measure is defined as in Equation (8). The minimal martingale measure cannot coincide with the Esscher martingale measure if $0 < \gamma < 1$. If the drift term in Equation (2) is described by Equation (12), then $\gamma = \theta = 1$. If the drift term in Equation (2) is described by $b = r - \frac{1}{2}\sigma^2 - \Psi_{\mu\nu}(-i)$, then $\gamma = \theta = 0$. When $\gamma = 0$ or $\gamma = 1$, the minimal martingale measure coincides with the Esscher martingale measure.

THE MARTINGALE APPROACH FOR OPTION PRICING

Consider a European call option, written on an asset S (modeled as in Equation (2)), and assume that the option matures at date T and has a strike price of K. By virtue of the risk-neutral pricing rule, the value of any contingent claim can be computed as a discounted expectation under the martingale measure, equivalent to the original probability measure P. As we illustrated in the previous section, we can obtain an equivalent martingale measure P^* using the Esscher martingale measure or the minimal martingale measure. Therefore, the price of this call option can be expressed as the risk-neutral conditional expectation of the payoff for any $t \in [0, T]$:

$$c\,(t, S_t) = B_t E_{P^*}\left[(S_T - K)^+ B_T^{-1}\middle|\,\mathcal{F}_t\right]. \tag{13}$$

It is apparent from Equation (6) that the asset price at time T under the equivalent martingale measure P^*, conditional on the filtration $\mathcal{F}_t$, can be described as follows:

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(W_T^*-W_t^*\right)+\int_t^T\int_R x(\mu-\nu^*)(du,dx)+\int_t^T\int_R(-e^x+1+x)\nu^*(du,dx)}. \tag{14}$$

Because we assume the interest rate is constant, we note

$$c\,(t, S_t) = e^{-r(T-t)}E_{P^*}\left[S_T 1_{\{S_T>K\}}\middle|\,\mathcal{F}_t\right] - Ke^{-r(T-t)}E_{P^*}\left[1_{\{S_T>K\}}\middle|\,\mathcal{F}_t\right]. \tag{15}$$

To evaluate the first expectation of the right-hand-side of Equation (15), it is convenient to introduce an auxiliary probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ by setting

$$\begin{aligned}\tilde{Z}_T &= \left.\frac{d\tilde{P}}{dP^*}\right|_{\mathcal{F}_T} \\ &= \frac{S_T/B_T}{S_0/B_0} = e^{-\frac{1}{2}\sigma^2 T+\sigma W_T^*+\int_0^T\int_R x(\mu-\nu^*)(du,dx)-\int_0^T\int_R(e^x-1-x)\nu^*(du,dx)}, \qquad P^*-\text{a.s.}\end{aligned} \tag{16}$$

For every $t \in [0, T]$, we have

$$\tilde{Z}_t = e^{-\frac{1}{2}\sigma^2 t+\sigma W_t^*+\int_0^t\int_R x(\mu-\nu^*)(du,dx)-\int_0^t\int_R(e^x-1-x)\nu^*(du,dx)},$$

and the process $d\tilde{W}_t = dW_t^* - \sigma dt$ follows a standard Brownian motion in the space $(\Omega, \mathcal{F}, \tilde{P})$. In addition, $\tilde{\nu}(dx) = e^x \nu^*(dx)$ is the $\tilde{P}$-compensator of the random measure μ. The asset price at time T, under the probability measure $\tilde{P}$ and conditional on the filtration $\mathcal{F}_t$, equals

$$S_T = S_t e^{\left(r+\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(\tilde{W}_T-\tilde{W}_t\right)+\int_t^T\int_R x(\mu-\tilde{\nu})(du,dx)+\int_t^T\int_R \left(e^{-x}-1+x\right)\tilde{\nu}(du,dx)}. \tag{17}$$

Therefore, we obtain

$$e^{-r(T-t)}E_{P^*}\left[S_T 1_{\{S_T>K\}}\middle|\mathcal{F}_t\right] = S_t E_{P^*}\left[\tilde{Z}_T\tilde{Z}_t^{-1}1_{\{S_T>K\}}\middle|\mathcal{F}_t\right] = S_t E_{\tilde{P}}\left[1_{\{S_T>K\}}\middle|\mathcal{F}_t\right],$$

and Equation (15) can be rewritten as

$$e^{-r(T-t)}E_{P^*}\left[(S_T-K)^+\middle|\mathcal{F}_t\right] = S_t\tilde{P}\left(S_T>K\,|\,\mathcal{F}_t\right) - Ke^{-r(T-t)}P^*\left(S_T>K\,|\,\mathcal{F}_t\right). \tag{18}$$

To compute the conditional probability $P^*(S_T > K\,|\,\mathcal{F}_t)$, we first set $X_t^{*\mu\nu} = \int_0^t\int_R x(\mu-\nu^*)(du,dx)$. Clearly, the compensated jump process $X^{*\mu\nu}$ follows a P^*-martingale, and the characteristic function of $X_t^{*\mu\nu}$ for the probability measure P^* can be given by

$$E_{P^*}\left[\exp\left(izX_t^{*\mu\nu}\right)\right] = \exp\left(t\Psi_{\mu\nu}^*(z)\right),$$

where $\Psi_{\mu\nu}^*(z) = \int_R(e^{izx}-1-izx)\nu^*(dx)$ is the Lévy exponent of $X^{*\mu\nu}$.

Using Equation (14) and the property of stationary increments of Lévy processes, we know that under the equivalent martingale measure P^*, the process $\sigma(W_T^*-W_t^*)+\int_t^T\int_R x(\mu-\nu^*)(du,dx)$—with a standard deviation derived as $\sqrt{(\sigma^2+\int_R x^2\nu^*(dx))(T-t)}$—is independent of filtration $\mathcal{F}_t$. Moreover, the conditional probability $P^*(S_T>K\,|\,\mathcal{F}_t)$ can be computed as

$$\begin{aligned}
&P^*\left(S_T>K\,|\,\mathcal{F}_t\right)\\
&= P^*\left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(W_T^*-W_t^*\right)+\int_t^T\int_R x(\mu-\nu^*)(du,dx)+\int_t^T\int_R(-e^x+1+x)\nu^*(du,dx)} > K\,\middle|\,\mathcal{F}_t\right)\\
&= P^*\left(Y^*<d_2\right),
\end{aligned}$$

where

$$d_2 = \frac{\ln(S_t/K)+\left(r-\frac{1}{2}\sigma^2-\Psi_{\mu\nu}^*(-i)\right)(T-t)}{\sqrt{\left(\sigma^2+\int_R x^2\nu^*(dx)\right)(T-t)}},$$

and

$$Y^* = -\frac{\sigma\left(W_T^* - W_t^*\right) + \int_t^T \int_R x\,(\mu - \nu^*)\,(du, dx)}{\sqrt{\left(\sigma^2 + \int_R x^2 \nu^*(dx)\right)(T - t)}}. \tag{19}$$

Because W^* and $X^{*\mu\nu}$ are independent, the characteristic function Φ^* of the random variable Y^* under P^* is

$$\Phi^*(z) = E_{P^*}\left[\exp\left(izY^*\right)\right] = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2 + \int_R x^2 \nu^*(dx)} + \Psi_{\mu\nu}^*\left(\frac{-z}{\sqrt{\left(\sigma^2 + \int_R x^2 \nu^*(dx)\right)(T-t)}}\right)(T-t)}.$$

Then, according to Karr's (1993) Theorem 6.5, the probability $P^*(Y^* < d_2)$ can be derived as follows:

$$\Pi_2(d_2) = P^*(Y^* < d_2) = \frac{1}{2\pi} \lim_{L\to-\infty} \lim_{A\to\infty} \int_{-A}^{A} \frac{e^{-izL} - e^{-izd_2}}{iz} \Phi^*(z)\,dz.$$

To compute the conditional probability $\tilde{P}(S_T > K \mid \mathcal{F}_t)$, we let $\tilde{X}_t^{\mu\nu} = \int_0^t \int_R x(\mu - \tilde{\nu})(du, dx)$. The characteristic function of $\tilde{X}_t^{\mu\nu}$ is given by

$$E_{\tilde{P}}\left[\exp\left(iz\tilde{X}_t^{\mu\nu}\right)\right] = \exp\left(t\tilde{\Psi}_{\mu\nu}(z)\right),$$

where $\tilde{\Psi}_{\mu\nu}(z) = \int_R (e^{izx} - 1 - izx)\tilde{\nu}(dx)$ is the Lévy exponent of $\tilde{X}^{\mu\nu}$.

From Equation (17) and the property of stationary increments that marks the Lévy process, we know that with the equivalent martingale measure $\tilde{P}$, the process $\sigma(\tilde{W}_T - \tilde{W}_t) + \int_t^T \int_R x(\mu - \tilde{\nu})(du, dx)$—with a standard deviation derived as $\sqrt{(\sigma^2 + \int_R x^2 \tilde{\nu}(dx))(T - t)}$—is independent of filtration $\mathcal{F}_t$. The conditional probability $\tilde{P}(S_T > K \mid \mathcal{F}_t)$ can be computed as

$$\begin{aligned}
&\tilde{P}\left(S_T > K \mid \mathcal{F}_t\right) \\
&= \tilde{P}\left(S_t e^{\left(r + \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right) + \int_t^T \int_R x(\mu - \tilde{\nu})(du, dx) + \int_t^T \int_R \left(e^{-x} - 1 + x\right)\tilde{\nu}(du, dx)} > K \,\middle|\, \mathcal{F}_t\right) \\
&= \tilde{P}\left(\tilde{Y} < d_1\right),
\end{aligned}$$

where

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2 + \tilde{\Psi}_{\mu\nu}(i)\right)(T - t)}{\sqrt{\left(\sigma^2 + \int_R x^2 \tilde{\nu}(dx)\right)(T - t)}},$$

and

$$\tilde{Y} = -\frac{\sigma\left(\tilde{W}_T - \tilde{W}_t\right) + \int_t^T \int_R x\,(\mu - \tilde{\nu})\,(du, dx)}{\sqrt{\left(\sigma^2 + \int_R x^2 \tilde{\nu}\,(dx)\right)(T-t)}}. \tag{20}$$

Again in this case, $\tilde{W}$ and $\tilde{X}^{\mu\nu}$ are independent, so the characteristic function $\tilde{\Phi}$ of the random variable $\tilde{Y}$ under $\tilde{P}$ is

$$\tilde{\Phi}(z) = E_{\tilde{P}}\left[\exp\left(iz\tilde{Y}\right)\right] = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2 + \int_R x^2 \tilde{\nu}(dx)} + \tilde{\Psi}_{\mu\nu}\left(\frac{-z}{\sqrt{\left(\sigma^2 + \int_R x^2 \tilde{\nu}(dx)\right)(T-t)}}\right)(T-t)}.$$

Then we can derive the probability $\tilde{P}(\tilde{Y} < d_1)$ in the following form:

$$\Pi_1(d_1) = \tilde{P}\left(\tilde{Y} < d_1\right) = \frac{1}{2\pi} \lim_{L \to -\infty} \lim_{A \to \infty} \int_{-A}^{A} \frac{e^{-izL} - e^{-izd_1}}{iz} \tilde{\Phi}(z)\,dz.$$

Through these arguments, we establish the following proposition, which provides a closed-form pricing formula of a European call option in the exponential Lévy model.

Proposition 2: An arbitrage-free price of a European call option, written on an asset modeled as in Equation (2) with an expiration date T and strike price K, can be given, for any $t \in [0, T]$, by

$$c(t, S_t) = S_t \Pi_1(d_1) - K e^{-r(T-t)} \Pi_2(d_2), \tag{21}$$

where

$$\Pi_1(d_1) = \frac{1}{2\pi} \lim_{L_1 \to -\infty} \lim_{A \to \infty} \int_{-A}^{A} \frac{e^{-izL_1} - e^{-izd_1}}{iz} \tilde{\Phi}(z)\,dz,$$

$$\Pi_2(d_2) = \frac{1}{2\pi} \lim_{L_2 \to -\infty} \lim_{A \to \infty} \int_{-A}^{A} \frac{e^{-izL_2} - e^{-izd_2}}{iz} \Phi^*(z)\,dz,$$

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2 + \tilde{\Psi}_{\mu\nu}(i)\right)(T-t)}{\sqrt{\left(\sigma^2 + \int_R x^2 \tilde{\nu}(dx)\right)(T-t)}},$$

$$d_2 = \frac{\ln(S_t/K) + \left(r - \frac{1}{2}\sigma^2 - \Psi^*_{\mu\nu}(-i)\right)(T-t)}{\sqrt{\left(\sigma^2 + \int_R x^2 \nu^*(dx)\right)(T-t)}},$$

$$\tilde{\Phi}(z) = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2+\int_R x^2\tilde{\nu}(dx)} + \tilde{\Psi}_{\mu\nu}\left(\frac{-z}{\sqrt{(\sigma^2+\int_R x^2\tilde{\nu}(dx))(T-t)}}\right)(T-t)},$$

$$\Phi^*(z) = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2+\int_R x^2\nu^*(dx)} + \Psi^*_{\mu\nu}\left(\frac{-z}{\sqrt{(\sigma^2+\int_R x^2\nu^*(dx))(T-t)}}\right)(T-t)},$$

$$\tilde{\Psi}_{\mu\nu}(z) = \int_R \left(e^{izx} - 1 - izx\right)\tilde{\nu}(dx),$$

$$\Psi^*_{\mu\nu}(z) = \int_R \left(e^{izx} - 1 - izx\right)\nu^*(dx), \quad \text{and}$$

$$\tilde{\nu}(dx) = e^x \nu^*(dx).$$

That is, Proposition 2 states that the call option price can be written according to two probabilities. From the risk-neutral pricing rule and Equation (13), we know that the probability measure P^* is the martingale measure that corresponds to the choice of the savings account B as a numeraire asset. According to Equation (16), the probability measure $\tilde{P}$ is the martingale measure that corresponds to the choice of the asset price S as the numeraire. We have also shown that Π_1, represented by the characteristic function $\tilde{\Phi}$, is the distribution function of the random variable $\tilde{Y}$, which follows a martingale under $\tilde{P}$. Furthermore, Π_2, represented by the characteristic function Φ^*, is the distribution function of the random variable Y^*, which follows a martingale under P^*. When the jump component vanishes, $\tilde{Y}$ and Y^* become standardized normal random variables, and Equation (21) yields the Black-Scholes (1973) valuation formula as a special case. From Equations (15) and (21), the expression $S_t e^{r(T-t)}\Pi_1(d_1)$ is the expected value of a variable that equals S_T if $S_T > K$ and is zero otherwise in a risk-neutral world, and $K\Pi_2(d_2)$ is the strike price times the probability that the strike price will be paid in a risk-neutral world.

Bakshi and Madan (2000) similarly show that the call price can be decomposed into a portfolio of Arrow–Debreu securities. However, each Arrow–Debreu security is not explicitly represented as a distribution function of a random variable according to a probability measure. Furthermore, their pricing formula does not reduce to the Black-Scholes (1973) valuation formula when the jump component disappears.

In a condition in which $\sigma_{JD} = \sqrt{(\sigma^2 + \int_R x^2\nu^*(dx))(T-t)}$ and $d_3 = d_2\sigma_{JD}$, Carr and Madan (1999) assume that the characteristic function of the risk-neutral density is known analytically. Therefore, they derive the pricing formula of a European call option, for $\nu_1 > 0$, as

$$c(t, S_t) = \frac{Ke^{-r(T-t)}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{(iz+(\nu_1+1))d_3}\Phi^*\left(-(u-(\nu_1+1)i)\sigma_{JD}\right)}{(iu+\nu_1+1)(iu+\nu_1)}du. \quad (22)$$

Instead, if $x = \ln S_T$, $a < -1$, and $b > 0$, Lewis (2001) applies a generalized Fourier transform and derives a general formula for an option with payoff $f(x)$:

$$c(t, S_t) = \frac{e^{-r(T-t)}}{2\pi} \int_{i\nu_0-\infty}^{i\nu_0+\infty} e^{-izd_3} \Phi^*\left(z\sigma_{JD}\right) \hat{f}(z)\, dz,\ \nu_0 = \operatorname{Im} z,\ z \in \mathcal{S}_c \triangleq \mathcal{S}_f \cap \mathcal{S}^*,$$

where

$$\mathcal{S}^* = \{z : a < -\operatorname{Im} z < b\} \quad \text{and}$$
$$\mathcal{S}_f = \left\{ z : \hat{f}(z) = \int_{-\infty}^{\infty} e^{izx} f(x)\, dx \text{ exists for } z \text{ such that } a < \operatorname{Im} z < b \right\}.$$

This expression provides several variation formulas for a European call option. For example, for $\nu_0 = 1/2$, the call option price $c(t, S_t)$ is given by

$$c(t, S_t) = S_t - \frac{\sqrt{S_t K} e^{-r(T-t)/2}}{\pi} \int_0^\infty \frac{\operatorname{Re}\left[e^{izd_3} \Phi^*\left(-\left(z - \frac{i}{2}\right)\sigma_{JD}\right)\right]}{z^2 + 1/4} dz. \tag{23}$$

If instead $\nu_0 = \nu_1 + 1$, the call option price $c(t, S_t)$ is given by

$$c(t, S_t) = -\frac{K e^{-r(T-t)}}{2\pi} \int_{i\nu_0-\infty}^{i\nu_0+\infty} e^{-izd_3} \Phi^*\left(z\sigma_{JD}\right) \frac{dz}{z^2 - iz}. \tag{24}$$

If we move the contours to exactly $\nu_0 = 1$ and $\nu_0 = 0$, the call option price $c(t, S_t)$ in Black–Scholes terms takes the form:

$$c(t, S_t) = S_t \Pi_3(d_3) - K e^{-r(T-t)} \Pi_4(d_3), \tag{25}$$

where

$$\Pi_3 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\left[\frac{e^{izd_3} \Phi^*\left(-(z - i)\sigma_{JD}\right)}{iz}\right] dz,$$

and

$$\Pi_4 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\left[\frac{e^{izd_3} \Phi^*\left(-z\sigma_{JD}\right)}{iz}\right] dz.$$

If we change the integration variables by letting $u = -z + (\nu_1 + 1)i$, we realize that Equation (24) is equivalent to Equation (22). In addition, Equation (25) represents a special case of Bakshi and Madan's (2000) Expression (12). Although a generalized Fourier transform can evaluate option prices accurately, the martingale method we use provides a more economical explanation for the option valuation. Thus, we combine these findings with our Proposition 2, to derive the following corollary.

Corollary. An arbitrage-free price of a European put option, written on an asset modeled as in Equation (2) with an expiration date T and strike price K, is given, for any $t \in [0, T]$, by

$$p(t, S_t) = Ke^{-r(T-t)}(1 - \Pi_2(d_2)) - S_t(1 - \Pi_1(d_1)),$$

where Π_1, Π_2, d_1, and d_2 are defined as in Proposition 2.

Proposition 3: The price of the call option is given as in Equation (21), and the delta hedge ratio equals

$$\frac{\partial c(t, S_t)}{\partial S_t} = \Pi_1(d_1) + \frac{\frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} e^{-izd_1}\tilde{\Phi}(z)\,dz}{\sqrt{\left(\sigma^2 + \int_R x^2\tilde{\nu}(dx)\right)(T-t)}} - \frac{Ke^{-r(T-t)}\frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} e^{-izd_2}\Phi^*(z)\,dz}{S_t\sqrt{\left(\sigma^2 + \int_R x^2\nu^*(dx)\right)(T-t)}}.$$

Proof: The delta hedge ratio can be deduced from

$$\frac{\partial c(t, S_t)}{\partial S_t} = \Pi_1(d_1) + S_t\frac{\partial \Pi_1(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_t} - Ke^{-r(T-t)}\frac{\partial \Pi_2(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_t}.$$

Because

$$\frac{\partial d_1}{\partial S_t} = \frac{1}{S_t\sqrt{\left(\sigma^2 + \int_R x^2\tilde{\nu}(dx)\right)(T-t)}},$$

$$\frac{\partial d_2}{\partial S_t} = \frac{1}{S_t\sqrt{\left(\sigma^2 + \int_R x^2\nu^*(dx)\right)(T-t)}},$$

$$\pi_1(d_1) = \frac{\partial \Pi_1(d_1)}{\partial d_1} = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} e^{-izd_1}\tilde{\Phi}(z)\,dz, \text{ and}$$

$$\pi_2(d_2) = \frac{\partial \Pi_2(d_2)}{\partial d_2} = \frac{1}{2\pi}\lim_{A\to\infty}\int_{-A}^{A} e^{-izd_2}\Phi^*(z)\,dz,$$

the delta hedge ratio takes the following form:

$$\frac{\partial c\,(t, S_t)}{\partial S_t} = \Pi_1\,(d_1) + \frac{\pi_1\,(d_1)}{\sqrt{\left(\sigma^2 + \int_R x^2 \tilde{\nu}\,(dx)\right)(T-t)}} - \frac{K e^{-r(T-t)} \pi_2\,(d_2)}{S_t \sqrt{\left(\sigma^2 + \int_R x^2 \nu^*\,(dx)\right)(T-t)}}.$$

Then it becomes obvious that π_i for $i = 1, 2$ refers to the probability density functions of Π_i.

In the Black–Scholes model, the delta hedge ratio is identical to the first probability element. Without jumps, we attain the Black–Scholes delta hedge ratio $\Pi_1(d_1) = N(d_1)$.

OPTION PRICING USING POLYNOMIAL INTERPOLATION

The numerical computation of one option price using Equation (21) takes 0.1288 seconds when we set the error tolerance to 10^{-6}, according to an adaptive Lobatto quadrature rule, whereas the Black–Scholes formula takes 0.0002 seconds. This difference suggests that we need to find a method that is able to compute option prices both accurately and quickly. Interpolation is a critical tool for obtaining accurate estimates from tabulated values, and it is easy to tabulate numerical approximations for the values of a cumulative distribution function. Therefore, we propose using polynomial interpolation to estimate $\Pi_1(d_1)$ and $\Pi_2(d_2)$. Maron (1982) provides a discussion of the polynomial interpolation problem; we also detail the technical aspects of our polynomial interpolation with divided differences in the Appendix.

First, let $\Phi_1 = \tilde{\Phi}$, and $\Phi_2 = \Phi^*$. For $i = 1, 2$, $\bar{\Pi}_i$ takes the following form:

$$\bar{\Pi}_i\,(d_i) = \frac{1}{2\pi} \lim_{A_i \to \infty} \int_{-A_i}^{A_i} \frac{e^{-izL_i} - e^{-izd_i}}{iz} \Phi_i\,(z)\,dz.$$

Second, we can identify an A_i, such that

$$A_i = \inf\left\{ z > 0 \,\middle|\, 0 \le \frac{|\mathrm{Re}\,(\Phi_i\,(z))| + |\mathrm{Im}\,(\Phi_i\,(z))|}{z} \le \varepsilon, \varepsilon \to 0 \right\}.$$

Third, when $L_i < 0$ and $\pi_i(L_i)$ approaches 0, $\bar{\Pi}_i(d_i)$ also moves toward $\Pi_i(d_i)$. We can find an L_i, such that $L_i = \sup\left\{ L < 0 \,\middle|\, 0 \le \pi_i(L) \le \varepsilon, \varepsilon \to 0 \right\}$. If $U_i > 0$

and $\pi_i(U_i)$ approaches 0, then $\bar{\Pi}_i(U_i)$ moves toward 1. We find that $U_i = \inf\left\{U > 0 \,\middle|\, 0 \le \pi_i(U) \le \varepsilon, \varepsilon \to 0\right\}$. Therefore, if $d_i \le L_i$, then $\Pi_i(d_i)$ moves toward 0, whereas if $d_i \ge U_i$, then it moves toward 1. In addition, M_i satisfies $\pi_i(M_i) = \max_{L_i < d_i < U_i} \pi_i(d_i)$. We then observe that $\Pi_i(d_i)$ is a convex function when d_i is in $[L_i, M_i]$, and $\Pi_i(d_i)$ is a concave function when d_i is in $[M_i, U_i]$. It is difficult to obtain the desired accuracy using interpolation methods for a single interval $[L_i, U_i]$.

To obtain accurate values of the probabilities $\Pi_i(d_i)$ and $L_i \le d_i \le U_i$, $i = 1, 2$, we split $[L_i, U_i]$ into five interpolation intervals: $[L_i, M_i - 1.5]$, $[M_i - 1.5, M_i]$, $[M_i, M_i + 1.5]$, $[M_i + 1.5, M_i + 3]$, and $[M_i + 3, U_i]$. In turn, we can form divided difference tables with $n = 40$ Chebyshev knots for each subinterval, which support our computation of the value of $\Pi_i(d_i)$, $L_i \le d_i \le U_i$.[2]

INTERPOLATION FOR D-GTSP OPTION PRICING

The generalized tempered stable process, introduced by Koponen (1995), is a Lévy process with Lévy density of the form

$$\nu(x) = \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} 1_{\{x>0\}} + \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_- |x|} 1_{\{x<0\}}, \tag{26}$$

where the parameters satisfy $c_\pm > 0$, $\lambda_- > 0$, $\lambda_+ > 2$, and $\alpha_\pm < 2$. For $\alpha_\pm < 2$, we have

$$\int_R |x|^2 \nu(dx) = c_+ \lambda_+^{-(2-\alpha_+)} \Gamma(2-\alpha_+) + c_- \lambda_-^{-(2-\alpha_-)} \Gamma(2-\alpha_-) < \infty,$$

where $\Gamma(x)$ is the gamma function, and then

$$\int_{|x|\le 1} |x|^2 \nu(dx) < \int_R |x|^2 \nu(dx) < \infty, \text{ and}$$

$$\int_{|x|\ge 1} \nu(dx) < \int_{|x|\ge 1} x^2 \nu(dx) \le \int_R |x|^2 \nu(dx) < \infty.$$

Therefore, $\int_R (x^2 \wedge 1)\nu(dx) < \infty$, as required.

Let $X_t^{\mu\nu} = \int_0^t \int_R x(\mu - \nu)(dx, du)$ be a generalized tempered stable process, where $\mu(dx, dt)$ is a jump measure with the Lévy measure $\nu(dx, dt) = \nu(dx)dt$,

[2]For example, if $M_i - 1.5 \le d_i \le M_i$, then $[a, b] = [M_i - 1.5, M_i]$ and $\Pi_i(d_i) = q_{k_m, k_m+m}(d_i)$.

where ν is defined by Equation (26). The characteristic function of the process $X_t^{\mu\nu}$ under the probability measure P in turn is given by

$$E_P\left[\exp\left(izX_t^{\mu\nu}\right)\right] = \exp\left(t\Psi_{\mu\nu}(z)\right),$$

where

$$\Psi_{\mu\nu}(z) = \phi_{\mu\nu}(z, 0, c_+, \alpha_+, \lambda_+) + \phi_{\mu\nu}(-z, 0, c_-, \alpha_-, \lambda_-),$$

and if ($\alpha_\pm \neq 1$ and $\alpha_\pm \neq 0$), then

$$\phi_{\mu\nu}(z, h, c, \alpha, \lambda) = c\Gamma(-\alpha)\left[(\lambda - h - iz)^\alpha - (\lambda - h)^\alpha + iz\alpha(\lambda - h)^{\alpha-1}\right],$$

whereas if $\alpha_+ = 1$ or $\alpha_- = 1$, then

$$\phi_{\mu\nu}(z, h, c, \alpha, \lambda) = c\left((\lambda - h - iz)\ln\left(1 - \frac{iz}{\lambda - h}\right) + iz\right).$$

Finally, if $\alpha_+ = 0$ or $\alpha_- = 0$, then

$$\phi_{\mu\nu}(z, h, c, \alpha, \lambda) = -c\left(\frac{iz}{\lambda - h} + \ln\left(1 - \frac{iz}{\lambda - h}\right)\right).$$

Using the characteristic exponent, the cumulant, $c_n(X_t^{\mu\nu})$, of the generalized tempered stable process $X_t^{\mu\nu}$ is given by

$$c_n\left(X_t^{\mu\nu}\right) = t\varphi_n(0),$$

where $\varphi_1(0) = 0$. In addition, for $n \in N$, $n \geq 2$,

$$\varphi_n(h) = c_+(\lambda_+ - h)^{\alpha_+ - n}\,\Gamma(n - \alpha_+) + (-1)^n c_-(\lambda_- + h)^{\alpha_- - n}\,\Gamma(n - \alpha_-).$$

When the asset price process S is defined by Equations (2) and (26), S is a D-GTSP model. To evaluate the price of a contingent claim, we use the martingale measure given by the methods mentioned in Section 2. First, a Lévy process under P is also a Lévy process under the equivalent martingale measure P^*, as derived by the Esscher transform or by the minimal measure. Second, the measures P and P^* on the path space corresponding to two generalized tempered stable processes are mutually equivalent if and only if their coefficients $\alpha_\pm$ and $c_\pm$ coincide (Cont and Tankov, 2004). Therefore, the property of the GTSP can be preserved under the equivalent martingale measure P^θ, obtained from P through the Esscher transform, because the coefficients $\alpha_{\pm}$ and $c_{\pm}$ under P^{θ} remain the same as those under P.

We consider the drift term in Equation (2) as in Equation (12). From Proposition 1, we know that the equivalent martingale measure P^* possesses simultaneously the properties of the Esscher martingale measure and the minimal measure. In turn, under the equivalent martingale measure P^*, we know that $\nu^*(dx) = e^x \nu(dx)$, and the characteristic exponent of $X_t^{*\mu\nu}$ can be derived as

$$\Psi^*_{\mu\nu}(z) = \phi_{\mu\nu}(z, 1, c_+, \alpha_+, \lambda_+) + \phi_{\mu\nu}(-z, -1, c_-, \alpha_-, \lambda_-).$$

Furthermore, the characteristic function of the random variable Y^* defined by Equation (19) under P^* is equal to

$$\Phi^*(z) = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2+\varphi_2(1)} + \Psi^*_{\mu\nu}\left(\frac{-z}{\sqrt{(\sigma^2+\varphi_2(1))(T-t)}}\right)(T-t)}.$$

We next focus on the properties of the D-GTSP under the equivalent measure $\tilde{P}$ with respect to P^*, which is given by Equation (16). The characteristic exponent of $\tilde{X}^{\mu\nu}$ is

$$\tilde{\Psi}_{\mu\nu}(z) = \phi_{\mu\nu}(z, 2, c_+, \alpha_+, \lambda_+) + \phi_{\mu\nu}(-z, -2, c_-, \alpha_-, \lambda_-),$$

and the characteristic function of the random variable $\tilde{Y}$ defined by Equation (20) under $\tilde{P}$ equals

$$\tilde{\Phi}(z) = e^{-\frac{1}{2}\frac{\sigma^2 z^2}{\sigma^2+\varphi_2(2)} + \tilde{\Psi}_{\mu\nu}\left(\frac{-z}{\sqrt{(\sigma^2+\varphi_2(2))(T-t)}}\right)(T-t)}.$$

By Proposition 2, we can evaluate the price of a European call option under the D-GTSP model.

Next, we compute option prices using five methods and compare their mean errors and CPU times:

1. M1 from Equation (21).
2. M2, or a polynomial interpolation with divided differences.
3. M3 from Equation (22), with $\nu_1 = 0.12$.
4. M4 from Equation (23).
5. M5 from Equation (25).

All five methods must approximate the integrals, and we thus need to define an error tolerance to confirm if the termination test for the integral is satisfied.

TABLE I
The CPU Times (Error Tolerances) for Various Pricing Errors

Pricing Errors	*M1*	*M2*	*M3*	*M4*	*M5*
$10^{-3} \sim 10^{-4}$	38.61 (1.0E-04)	0.19 (1.0E-04)	149.06 (1.0E-04)	51.22 (1.0E-04)	320.06 (1.0E-04)
$10^{-4} \sim 10^{-5}$	38.61 (1.0E-04)	0.19 (1.0E-04)	149.06 (1.0E-04)	55.89 (1.0E-05)	320.06 (1.0E-04)
$10^{-5} \sim 10^{-6}$	48.52 (1.0E-05)	0.19 (1.0E-05)	149.06 (1.0E-04)	65.30 (1.0E-06)	380.17 (1.0E-05)
$10^{-6} \sim 10^{-7}$	69.56 (1.0E-06)	0.19 (1.0E-06)	160.64 (1.0E-05)	73.20 (1.0E-07)	453.86 (1.0E-06)
$10^{-7} \sim 10^{-8}$	97.64 (1.0E-07)	0.19 (1.0E-07)	181.64 (1.0E-06)	84.66 (1.0E-08)	584.97 (1.0E-07)
$10^{-8} \sim 10^{-9}$	127.58 (1.0E-08)	0.20 (1.0E-09)	212.83 (1.0E-07)	108.50 (1.0E-09)	584.97 (1.0E-07)

Note. Let the asset price be at 100 and the interest rate at 0.02. Three sets of D-GTSP parameters are given by the following:
1. $\sigma^2 = 0.01$, $\alpha_+ = 0.10$, $c_+ = 526.37$, $\lambda_+ = 310.55$, $\alpha_- = -0.17$, $c_- = 526.69$, $\lambda_- = 94.58$;
2. $\sigma^2 = 0.0016$, $\alpha_+ = -0.52$, $c_+ = 1455.80$, $\lambda_+ = 3122.20$, $\alpha_- = -0.50$, $c_- = 1470.60$, $\lambda_- = 93.32$;
3. $\sigma^2 = 0.005$, $\alpha_+ = 0.42$, $c_+ = 60.12$, $\lambda_+ = 265.78$, $\alpha_- = 0.295$, c_-=60.19, $\lambda_- = 79.34$.

We use eight different error tolerances to approximate the integrals, from 10^{-4}, 10^{-5}, ..., to 10^{-11}. For each error tolerance, we compute 540 option prices with the five methods, using three sets of D-GTSP parameters, 60 strike levels (70–129), and three maturity dates (5, 30, and 270 days).

To evaluate pricing errors, we must estimate exact values of the 540 option prices. Because all 540 option prices reported by any two methods are in agreement up to eight decimal places when we set the error tolerance to 10^{-11}, we denote the exact option price as the average value of the option prices computed by these five methods, based on this error tolerance. Pricing errors are therefore deviations from these exact option prices. Thus, we can obtain the error tolerances and CPU times of all five methods for the six ranges of the mean error; we provide the results in Table I.

Table I shows that when the desired mean error level is set to range between 10^{-6} and 10^{-7}, the lowest required error tolerances for methods M1–M5 are 10^{-6}, 10^{-6}, 10^{-5}, 10^{-7}, and 10^{-6}, and their corresponding CPU times are 69.56, 0.19, 160.64, 73.20, and 453.86, respectively. Thus, M2, which computes 540 option prices in 0.19 seconds, is considerably faster than the other methods for every level of mean error. The performance of M1 is next best for mean error levels between 10^{-3} and 10^{-7}; however, M4 performs second best for mean error levels between 10^{-7} and 10^{-9}.

Because the values of L_1 and L_2—which are independent of the initial asset prices and the strike prices—can be obtained before we evaluate the probabilities $\Pi_1(d_1)$ and $\Pi_2(d_2)$ in M1, we omit the CPU times for finding these values when we compute the values for $\Pi_1(d_1)$ and $\Pi_2(d_2)$ for M1. For the same reason, we exclude CPU times to form the divided difference tables when we assess $\Pi_1(d_1)$ and $\Pi_2(d_2)$ for M2.

CONCLUSION

This article shows that the Esscher martingale measure coincides with the minimal martingale measure in some cases. Therefore, we can consider the equivalent martingale measure that simultaneously possesses the properties of the minimal measure and the Esscher measure. Using the same martingale approach as that used by Musiela and Rutkowski (1997) for the proof of Black and Scholes's (1973) theorem, we develop a Black–Scholes form of the European option in an exponential Lévy model. Our formula yields the Black–Scholes expression as a special case when the jump component vanishes. This martingale approach can be applied to revisit contingent claims valuation in the context of Musiela and Rutkowski (1997) under the exponential Lévy model. It is easy to find a smooth function to interpolate the Lévy cumulative distribution function, so we use polynomial interpolation with divided differences to estimate the probabilities in our formula and attain the option price. We illustrate this method for the D-GTSP option-pricing model; the use of polynomial interpolation is much faster than any other methods and permits real-time pricing and hedging.

APPENDIX

Interpolation with Divided Differences

Suppose $n+1$ points $Q_j(x_j, y_j)$ for $j = 0, \ldots, n$ appear on the *xy*-plane, where $y_j = \Pi(x_j)$, and Π is either Π_1 or Π_2. Polynomial interpolation finds polynomials that interpolate one or more of these points—that is, whose graph goes through some or all of $Q_0, Q_1, \ldots, Q_n$. The Chebyshev nodes $x_0, x_1, \ldots, x_n$ in $[a, b]$ can be chosen to ensure that the interpolation error is as uniformly small as possible. To this end, we first obtain the Chebyshev nodes $\xi_0, \xi_1, \ldots, \xi_n$ on $[-1, 1]$ by calculating

$$\xi_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \quad j = 0, 1, \ldots, n.$$

To obtain the desired nodes $x_0, x_1, \ldots, x_n$ in $[a, b]$, we reflect the Chebyshev nodes $\xi_0, \xi_1, \ldots, \xi_n$ into $[a, b]$ using the form

$$x_j = a + \frac{b-a}{2}\left(\xi_j + 1\right), \quad j = 0, 1, \ldots, n.$$

Then we define the mth divided difference at Q_k to be

$$\Delta^m y_k = \frac{\Delta^{m-1} y_{k+1} - \Delta^{m-1} y_k}{x_{k+m} - x_k}, \quad k = 0, \ldots, n - m.$$

We can form the divided difference table for $n + 1$, given Chebyshev knots $Q_0, Q_1, \ldots, Q_n$. Using Deuflhard and Hohmann's (2003) Theorem 7.10, we determine that a polynomial $q_{k,k+m}$ that interpolates the $m + 1$ consecutive knots $Q_k, Q_{k+1}, \ldots, Q_{k+m}$ takes the expression

$$q_{k,k+m}(x) = q_{\text{prev}}(x) + \delta_m(x),$$

where $\delta_m(x) = \Delta^m y_k \Pi_{\text{prev}}(x - x_j)$; $q_{\text{prev}}(x)$ is either $q_{k+1,k+m}(x)$ or $q_{k,k+m-1}(x)$; and Π_{prev} denotes the product as x_j varies over previously used nodes. For an accurate estimate of the cumulative probability for a given x, this formula makes it easy to find $q_{k_0,k_0}, q_{k_1,k_1+1}, q_{k_2,k_2+2}, \ldots, q_{k_m,k_m+m}$ successively, adding new nodes to ensure the best centering of x, until $\delta_m(x)$ ceases at the desired accuracy. Assume that k_j for each $j = 1, \ldots, m$ is chosen so that the interval $[x_{k_j}, x_{k_j+j}]$ provides the best centering of x. For example, let $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$, and $x_5 = 0.5$. When $x = 0.13$, we add the closest nodes in the following order: $x_1, x_2, x_0, x_3, x_4, x_5$. When $x = 0.36$, we add the closest nodes in the following order: $x_4, x_3, x_5, x_2, x_1, x_0$.

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