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A feasible natural hedging strategy for insurance companies

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HIGHLIGHTS

- We investigate a natural hedging strategy and attempt to find an optimal allocation of insurance products.
- We consider both variance and mispricing effects of longevity risk at the same time.
- This study employs the experienced mortality rates rather than population mortality data.

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ABSTRACT

To offer a means for insurance companies to deal with longevity risk, this article investigates a natural hedging strategy and attempts to find an optimal allocation of insurance products. Unlike prior research, this proposed natural hedging model can account for both the variance and mispricing effects of longevity risk at the same time. In addition, this study employs experience mortality rates, obtained from life insurance companies, rather than population mortality data for life insurance and annuity products.

1. Introduction

The life expectancy of humans worldwide is expected to continue increasing, by 0.2 years annually. As the mortality rate improves, longevity risk, a potential risk that annuitants will live longer than predicted by projected life tables, becomes an increasingly important topic. Thus in the past two decades, a wide range of mortality models have been proposed and discussed (e.g., Lee and Carter, 1992; Brouhns et al., 2002; Renshaw and Haberman, 2003; Koissi et al., 2006; Melnikov and Romaniuk, 2006; Cairns et al., 2006, 2009; Yang et al., 2010). Greater longevity risk implies that life insurers earn profits but annuity insurers suffer losses. More sophisticated mortality models that inform pricing decisions might help companies hedge against longevity risk, for both life insurance and annuity products. However, this solution is often difficult to apply in practice because of the challenges of market competition. That is, insurance companies may have the ability to build accurate mortality models to account for actual future improvements in mortality, but they may not be able to price and sell their annuity products using these derived mortality rates. It would be too expensive to sell them in competitive markets because most insurance companies might assume that current static mortality tables (which ignore future mortality improvements) will remain unchanged or decrease by a constant percentage each year for all ages (e.g., 0.5%) when pricing annuity products. This mispricing problem commonly exists in the countries in which the official static life or annuity tables are issued by governments or actuarial societies and used by insurance companies to price life or annuity products (e.g., Taiwan, Korean, Japan).[1]

Another possible solution uses mortality derivatives, such as survival bonds (Blake and Burrows, 2001) and survival swaps (Cairns et al., 2006; Dowd et al., 2006), which exchange future cash flows on the basis of survivor indices. Although mortality derivatives are convenient, they encounter obstacles in practice. For example, to avoid using appropriate credit-enhancement

[1] The official life or annuity tables can quickly fall out of date. For example, our own professional experience reveals that the annuity table currently used in Taiwan was built 15 years ago, relying on data from before 1987—a 25 year gap between the pricing and real experience bases. Similar situations appear in other countries too.

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mechanisms to manage credit risk, Blake et al. (2006) demonstrate that the maximum maturity of longevity bonds is limited to that of available government debt. However, empirical work by Cairns et al. (2006) shows that a longevity bond horizon of 40–50 years will provide a better hedge for annuity products based on a 65 year old reference population than a longevity bond with a 25 year horizon.

A third solution entails natural hedging. That is, insurance companies might optimize the allocation of their products, annuities, and life insurance offerings in such a way that they hedge against longevity risk. This approach is internal to the insurance company, which makes it more convenient and practical in practice. However, natural hedging remains a relatively new topic in the actuarial field, and few papers have studied this issue. Wang et al. (2003) investigate the influence of changes in mortality factors and propose an immunization model to hedge against mortality risks. Cox and Lin (2007) find that natural hedging employs the interaction between life insurance and annuities with a change in mortality to stabilize aggregate cash outflows. Therefore, natural hedging appears feasible, and mortality swaps can make it widely available. Wang et al. (2010) analyze an immunization model and use effective duration and convexity to find optimal product mixes. Tsai et al. (2010) employ a Conditional Value-at-Risk Minimization (CVaRM) approach to construct an insurer's product mix for insurance companies to hedge against the systematic mortality risk. However, they employ the same mortality rate measure (population mortality rates) for both life insurance and annuity products, because they lack actual mortality data. In practice, life insurance products are typically offered in accordance with the official life table, whereas annuity products are based on the official annuity table, as is the case in Taiwan. Insurance companies will not use the same mortality rate to price both life insurance and annuity products.

For this study, we propose using experience mortality rates from life insurance companies rather than population mortality rates. These incidence data include more than 50,000,000 policies, collected from all Taiwanese life insurance companies. Most life insurance policies with heavy principal repayment contain more than 80% saving premiums, so the pure risk is lower than 20% of the total premiums. These policies are more like saving products than life protection. Without access to real annuity mortality data, we employ the experience mortality rates of life insurance policies with heavy principal repayment (single endowment or serial periodic endowments[2]) as the annuity mortality rates.[3] In this data set, the time effect indices of experience mortality rates with and without principal repayment are correlated, though not perfectly negatively. Therefore, it is not possible to hedge longevity risk perfectly when we consider both life and annuity mortality rates (cf. Wang et al., 2010). We further investigate pricing differences for insurance products that use a period-mortality basis (without consideration of mortality improvement) versus a cohort-mortality basis (with consideration of mortality improvement). Thus, unlike previous literature, we consider a "variance" effect related to uncertainties in the mortality rate and interest rate and a "mispricing" effect induced by mortality improvement. We aim to minimize both variation in the change of the total portfolio value and differences between the pricing bases simultaneously. Using experience mortality rates, our proposed model provides an optimal allocation of insurance products and effectively applies a natural hedging strategy for insurance companies.

The reminder of this article is organized as follows. In Section 2, we review the mortality model and interest rate model, proposing our portfolio model with variance and mispricing effects. In Section 3, we calibrate a Lee–Carter model using the experience mortality rates from the Taiwan Insurance Institute (TII). Section 4 contains the numerical analysis of our model, and then in Section 5, we summarize our findings and offer some conclusions and suggestions for further research.

2. Model setting

Mortality risks and interest rate risks are two main concerns for life insurance companies. As demonstrated by D'Amato et al. (2009), one of the most popular methods for modeling the death rates is the Lee–Carter model (1992), because it is easy to implement and outperforms other models with respect to its prediction errors (e.g., Koissi et al., 2006; Melnikov and Romaniuk, 2006). The Lee–Carter model has also been used widely for mortality trend fitting and projection (Chen and Cox, 2009); even the US Census Bureau relied on it as a benchmark for its long-run forecast of US life expectancy (Hollmann et al., 2000). Therefore, recent Social Security Technical Advisory Panels suggest adopting this method or tactics consistent with it (Lee and Miller, 2001). We accordingly adopt the classical Lee–Carter model to project the mortality process.

Following Eq. (3) of Denuit et al. (2007, p. 92), the classical Lee–Carter (LC) approach is a relational model assuming that, for integer age x and calendar year t,

$$\ln(m_{x,t}) = \alpha_x + \beta_x k_t, \tag{1}$$

where $m_{x,t}$ is the central death rate for a person aged x at time t; α_x denotes the average age-specific mortality factor; β_x is the age-specific improving factor; and k_t is the time-varying mortality index. The parameters β_x and k_t are subject to $\sum_x \beta_x = 1$ and $\sum_t k_t = 0$, respectively, to ensure the model identification. The time effect index k_t can be estimated by using an ARIMA (0, 1, 0) process, that is

$$k_t - k_{t-1} = u_k + e_t, \tag{2}$$

where e_t is a normal distribution with zero mean and variance σ_k^2. We fit the LC model using the close approximation to the singular value decomposition (SVD) proposed by Lee and Carter (1992) because we recognize some missing values in our data.

In addition, Cox et al. (1985) specify an instantaneous interest rate that follows a square root process, also called the CIR process:

$$dr_t = a(b - r_t)dt + \sigma_r \sqrt{r_t} dZ_r(t), \tag{3}$$

where a denotes the speed of the mean-reverting adjustment; b represents the long-term mean of interest rates; σ_r is the interest rate volatility; and $(Z_r(t))_{t=0}^T$ is a standard Brownian motion that models the random market risk factor. In the CIR model, the interest rate approaches the long-run level b with a speed of adjustment governed by the strictly positive parameter a. We assume that $2ab \geq \sigma_r^2$; therefore, our model precludes an interest rate of zero.

We further assume that the insurance company's portfolio contains zero-coupon bonds, annuities, and life insurance policies for holders of different ages and genders. Specifically, we assume

[2] The heavy principal repayment means that the proportion of the saving premium is greater than 70%. Serial periodic endowments grant policyholders a certain percentage of the face amount each year, such as 5% or 10% annually.

[3] The Taiwan government is currently building a new annuity table, and one of the authors is responsible for building it, using data about experienced mortality rates from life insurance companies with more than 50,000,000 policies. In these data, we find that the experience mortality rates of life insurance policies with heavy principal repayment are lower than those without principal repayment. Therefore, to build the official annuity table in Taiwan, we constructed the annuity mortality rates according to the experience mortality rates of life insurance policies with heavy principal repayment (single endowment or serial periodic endowments).

that at time 0, a group of annuities and life insurance policies gets issued to different cohorts aged $x, x \in \omega$, where ω is a set of all possible ages of annuities and life insurance policyholders at time 0. The factors that affect the total value of this portfolio are the mortality and interest rates. The model is therefore as follows:

$$V(t) = \sum_{s=A,L}\sum_{g=1,2}\sum_{x\in\omega} N_x^{s,g} V(m_x^{s,g}(t), r_t, t) + N_B V^B(r_t, t), \tag{4}$$

where V represents the value of the insurance portfolio; $V^B(r_t, t)$ is the value of one unit of a zero-coupon bond with face value equal to 1; $m_x(t) = m_{x+t,t}$; $m_x^{A,1}(t)$ ($m_x^{A,2}(t)$) denotes the mortality rate of the male (female) annuity aged $x+t$ at time t; $m_x^{L,1}(t)$ ($m_x^{L,2}(t)$) denotes the mortality rate of the male (female) life insurance aged $x+t$ at time t; $V(m_x^{A,1}(t), r_t, t)$ ($V(m_x^{A,2}(t), r_t, t)$) denotes the value of one unit of an annuity policy at time t issued to a cohort of males (females) aged x at time 0; $V(m_x^{L,1}(t), r_t, t)$ ($V(m_x^{L,2}(t), r_t, t)$) denotes the value of one unit of a life insurance policy at time t issued to a cohort of males (females) aged x at time 0; N_B represents the number of units invested in zero-coupon bonds; and $N_x^{A,g}$ ($N_x^{L,g}$) denotes the number of units allocated in annuity (life insurance) policies for male or female policyholders.

In view of Eq. (2), the continuous time limit of the ARIMA (0, 1, 0) process of k_t can be expressed as (see also Biffis et al., 2010, p. 289)

$$dk_t = u_k dt + \sigma_k dZ_k(t), \tag{5}$$

where $(Z_k(t))_{t=0}^T$ is a standard Brownian motion; we provide the proof of Eq. (5) in Appendix A. Equivalently, for integer age y and calendar year t,

$$\begin{aligned} d\ln(m_{y,t}) &= \ln(m_{y,t+dt}) - \ln(m_{y,t}) = \beta_y(k_{t+dt} - k_t) \\ &= \beta_y dk_t = \beta_y(u_k dt + \sigma_k dZ_k(t)). \end{aligned} \tag{6}$$

According to Ito's lemma (Shreve, 2004, p. 148), we can transform the dynamic logarithm of the mortality rate into the dynamic of the mortality rate, as follows:

$$\begin{aligned} d(m_{y,t}) &= d(e^{\ln(m_{y,t})}) = m_{y,t} d(\ln m_{y,t}) + \frac{1}{2} m_{y,t} d(\ln m_{y,t})^2 \\ &= m_{y,t}(\beta_y u_k dt + \beta_y \sigma_k dZ_k(t)) + \frac{1}{2} m_{y,t}(\beta_y^2 \sigma_k^2 dt) \\ &= \left(m_{y,t}\beta_y u_k + \frac{1}{2} m_{y,t}\beta_y^2\sigma_k^2\right) dt + (m_{y,t}\beta_y\sigma_k) dZ_k(t). \end{aligned} \tag{7}$$

If we let $y = x + t$ and $m_x(t) = m_{x+t,t}$, we have

$$\begin{aligned} d(m_x^{s,g}(t)) &= \left(m_x^{s,g}(t)\beta_{x+t}^{s,g} u_k^{s,g} + \frac{1}{2} m_x^{s,g}(t)(\beta_{x+t}^{s,g})^2(\sigma_k^{s,g})^2\right) dt \\ &\quad + (m_x^{s,g}(t)\beta_{x+t}^{s,g}\sigma_k^{s,g}) dZ_k^{s,g}(t) \quad \text{for } s = L \text{ or } A,\ g = 1 \text{ or } 2. \end{aligned} \tag{8}$$

As pointed out by Li and Lee (2005), the populations of the world are becoming more closely linked by communication, transportation, trade, technology, and disease. Wilson (2001) also documents global convergence in mortality levels, such that it appears increasingly improper to prepare mortality forecasts for individual national populations in isolation from one another, and even more so for the regions within a country. Frees et al. (1996) observe a portfolio of annuities for couples and conclude that the times of death of the pair were highly correlated. Carriere (2000) presents alternative models to model the dependence of the time of death of couples and applies them to a data set from a life annuity portfolio. Luciano et al. (2008) use copula methods to capture the dependency between the survival times of members of a couple. Accordingly, to capture the covariance structure of mortality rates of female annuity, male annuity, female life insurance, and male life insurance, we assume that the four mortality risk factors $Z_k^{A,2}, Z_k^{A,1}, Z_k^{L,2}$, and $Z_k^{L,1}$, are dependent standard Brownian motions that satisfy

$$\begin{bmatrix} Z_k^{A,2}(t) \\ Z_k^{A,1}(t) \\ Z_k^{L,2}(t) \\ Z_k^{L,1}(t) \end{bmatrix} \sim N_4\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & 1 \end{bmatrix} t\right), \tag{9}$$

where $N_4(a, b)$ represents a four-dimensional, multi-normal distribution with mean vector a and covariance matrix b.

Following Shreve (2004, p. 171), we can decompose the correlated Brownian motions, $Z_k^{A,2}, Z_k^{A,1}, Z_k^{L,2}$, and $Z_k^{L,1}$, into a linear combination of four independent standard Brownian motions, Z_{k1}, Z_{k2}, Z_{k3}, and Z_{k4}, using the Cholesky decomposition:

$$\begin{bmatrix} dZ_k^{A,2}(t) \\ dZ_k^{A,1}(t) \\ dZ_k^{L,2}(t) \\ dZ_k^{L,1}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} dZ_{k1}(t) \\ dZ_{k2}(t) \\ dZ_{k3}(t) \\ dZ_{k4}(t) \end{bmatrix}, \tag{10}$$

where $\sum_{n=1}^{i} a_{in}a_{jn} = \rho_{ij}$ for $i, j = 1, \ldots, 4$. The proof of Eq. (10) is in Appendix B.

Again using Ito's lemma, we investigate the change in the total insurance portfolio value with respect to the change of mortality rate and interest rate, as follows:

$$\begin{aligned} dV(t) &= \sum_{s=L,A}\sum_{g=1,2}\sum_{x=i}^{\omega} N_x^{s,g}\frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} dm_x^{s,g}(t) + \frac{\partial V}{\partial r} dr_t \\ &\quad + \sum_{s=L,A}\sum_{g=1,2}\sum_{x=i}^{\omega} \frac{1}{2} N_x^{s,g}\frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g2}}(dm_x^{s,g}(t))^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr_t)^2 \\ &= Q_{0t}\,dt + \sum_{i=1}^{4} Q_{it}\,dZ_{ki}(t) + Q_{5t}\,dZ_r(t), \end{aligned} \tag{11}$$

where $Q_{0t}, Q_{1t}, Q_{2t}, Q_{3t}, Q_{4t}$, and Q_{5t} are given by

$$\begin{aligned} Q_{0t} &= \left\{\sum_{s=L,A}\sum_{g=1,2}\sum_{x\in\omega}\left(N_x^{s,g}\frac{\partial V_x^{s,g}}{\partial m_x^{s,g}}\left(m_x^{s,g}(t)\beta_{x+t}^{s,g}\mu_k^{s,g}\right.\right.\right. \\ &\quad \left. + \frac{1}{2} m_x^{s,g}(t)\beta_{x+t}^{s,g2}\sigma_k^{s,g2}\right) + \frac{1}{2} N_x^{s,g}\frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g2}} \\ &\quad \left. \times (m_x^{s,g}(t)\beta_{x+t}^{s,g}\sigma_k^{s,g})^2\right) + \frac{\partial V}{\partial r} a(b-r) \\ &\quad \left. + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma_r^2 r\right\}, \end{aligned} \tag{12}$$

$$\begin{aligned} Q_{1t} &= \left[\sum_{x\in\omega} N_x^{A,2}\frac{\partial V_x^{A,2}}{\partial m_x^{A,2}} m_x^{A,2}(t)\beta_{x+t}^{A,2}\sigma_k^{A,2}\right. \\ &\quad + a_{21}\sum_{x\in\omega} N_x^{A,1}\frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t)\beta_{x+t}^{A,1}\sigma_k^{A,1} \\ &\quad + a_{31}\sum_{x\in\omega} N_x^{L,2}\frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t)\beta_{x+t}^{L,2}\sigma_k^{L,2} \\ &\quad \left. + a_{41}\sum_{x\in\omega} N_x^{L,1}\frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t)\beta_{x+t}^{L,1}\sigma_k^{L,1}\right], \end{aligned} \tag{13}$$

$$Q_{2t} = \left[a_{22} \sum_{x\in\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t) \beta_{x+t}^{A,1} \sigma_k^{A,1} + a_{32} \sum_{x\in\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} + a_{42} \sum_{x\in\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right], \tag{14}$$

$$Q_{3t} = \left[a_{33} \sum_{x\in\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} + a_{43} \sum_{x\in\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right], \quad \text{and} \tag{15}$$

$$Q_{4t} = \left[a_{43} \sum_{x\in\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right], \tag{16}$$

$$Q_{5t} = \left[\frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} \right]. \tag{17}$$

The proof of Eq. (11) appears in Appendix C. Furthermore, the $\partial V/\partial r$ and $\partial^2 V/\partial r^2$ are of the form

$$\frac{\partial V}{\partial r} = \sum_{s=L,A} \sum_{g=1,2} \sum_{x\in\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial r} + N_B \frac{\partial V^B}{\partial r}, \quad \text{and} \tag{18}$$

$$\frac{\partial^2 V}{\partial r^2} = \sum_{s=L,A} \sum_{g=1,2} \sum_{x\in\omega} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial r^2} + N_B \frac{\partial^2 V^B}{\partial r^2}. \tag{19}$$

With the assumption that mortality risks and financial risks are independent, the terms $dZ_{ki}dZ_r$, $i = 1, 2, 3,$ and 4, equal zero in our model. We therefore apply the concepts of effective durations and convexities to estimate the first- and second-order derivatives:

$$\frac{\partial V}{\partial m} = \frac{V(m^+, r) - V(m^-, r)}{2V(m, r)\Delta m}, \tag{20}$$

$$\frac{\partial V}{\partial r} = \frac{V(m, r^+) - V(m, r^-)}{2V(m, r)\Delta r}, \tag{21}$$

$$\frac{\partial^2 V}{\partial m^2} = \frac{V(m^+, r) + V(m^-, r) - 2V(m, r)}{V(m, r)(\Delta m)^2}, \quad \text{and} \tag{22}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{V(m, r^+) + V(m, r^-) - 2V(m, r)}{V(m, r)(\Delta r)^2}, \tag{23}$$

where $m^+ = m + \Delta m, m^- = m - \Delta m, r^+ = r + \Delta r$, and $r^- = r - \Delta r$; Δm (Δr) represents a positive small change for the mortality rate (interest rate). By virtue of Eq. (11), the variance of the change of total insurance portfolio value is

$$\mathrm{Var}(dV(t)) = \sum_{j=1}^{5} (Q_{jt})^2, \tag{24}$$

which is determined by the parameters of the Lee–Carter and CIR models, as well as the first- and second-order derivatives defined in Eqs. (20)–(23).

Insurers attempt to minimize the variance of their portfolio returns with respect to changes in mortality and interest rates, because their portfolios, containing zero-coupon bonds, annuities, and life insurance policies, are influenced by longevity and interest rate risks. Therefore, we incorporate the variance in Eq. (24) – the variance effect – into the objective function for the optimal allocation of an insurance portfolio across annuity and life insurance policies.

Furthermore, we consider different pricing bases that reflect period mortality and cohort mortality. The mortality rate without an improvement effect is the period mortality rate, which many insurance companies apply to price their insurance policies. The mortality rate with an improvement effect, or cohort mortality rate, contains trends in the future mortality rates. Therefore, the difference of the portfolio values estimated with these two bases represents mispricing error. The relative pricing difference for this portfolio takes the form

$$D = \left[\sum_{s=A,L} \sum_{g=1,2} \sum_{x\in\omega} N_x^{s,g} \times \left(\frac{V_{\text{period}}(m_x^{s,g}(t), r_t, t) - V_{\text{cohort}}(m_x^{s,g}(t), r_t, t)}{V_{\text{cohort}}(m_x^{s,g}(t), r_t, t)} \right) \right], \tag{25}$$

where $V_{\text{period}}(m_x^{s,g}(t), r_t, t)$ denotes the policy value on a period-mortality basis, and $V_{\text{cohort}}(m_x^{s,g}(t), r_t, t)$ denotes the policy value on a cohort-mortality basis. We thus incorporate the pricing difference, or mispricing effect, in our objective function.

There are many setups of objective functions that seek a feasible policy allocation within the risk profile of insurance companies and minimize both the variance effect and the mispricing effect. For analytical tractability, the objective function f is of the form

$$f(N_x^{s,g}, N_B) = \min_{N_x^{s,g}, N_B} (1-\theta) \sum_{j=1}^{5} (Q_{jt})^2 + \theta D^2, \quad s = A, L,\ g = 1, 2, \text{ and } x \in \omega. \tag{26}$$

In view of Eq. (26), if insurance companies put more emphasis on the variance effect, they control for the change in the insurance portfolio due to unexpected shocks in the mortality and interest rates by decreasing the weight θ. If they put more emphasis on the mispricing effect, they aim to minimize mispricing error by increasing the weight θ.

3. Mortality data

The mortality data, collected from the Taiwan Insurance Institute (TII), include more than 50,000,000 policies issued by life insurance companies in Taiwan.[4] The original data are categorized by age, gender, and sorts (i.e., different types of insurance, such as life insurance products with or without repayments). We use these original data to construct four Lee–Carter mortality tables: female annuity (fa), male annuity (ma), female life insurance (fl), and male life insurance (ml). The maximal age of a policyholder in the original data is 85 years. We calibrate the parameters of our model using the approximation method,[5] as we depict in Fig. 1.

To be consistent with the pricing basis with those in practice, we extend the maximal age from 85 to 100 years.[6] To forecast the future mortality rate, in Table 1 we also calculate the standard deviation (σ_k) of k_t^{fa}, k_t^{ma}, k_t^{fl}, and k_t^{ml}.

4 For data characteristics and limitations, please see Yue and Huang (2011).

5 The percentages of mortality data missing are 25.27%, 24.24%, 37.66% and 29.87% for fa, ma, fl, and ml, respectively.

6 We forecast the mortality rates of ages from 1 to 85 using calibrated parameters of the Lee–Carter model and extend the future mortality rates of ages from 86 to 100 according to the Gompertz (1825) mortality model, which has been used overwhelmingly to model elderly mortality rates in previous decades (Dickson et al., 2009). In actuarial notation, the formula for the Gompertz Law can be expressed as $\mu_x = BC^x$, where $B > 0$, $C > 1$, $x > 0$, and μ_x is the force of mortality at age x. With the Gompertz assumption, the (conditional) probability that a person individual now aged x survives to age $x + 1$, denoted by p_x, is $p_x = \exp(-\int_x^{x+1} \mu_t dt) = \exp(-BC^x(C-1)/\log C)$. Because exposures of the elderly – that is, the number of days (or years) they have left to live – vary greatly across different ages, we adapt a *nonlinear maximization weighted least squares* procedure to estimate B and C, $\min_{B,C} \sum_x w_x(p_x - \exp(-BC^x(C-1)/\log C))^2$, where w_x is the population weight of age. We use the function "*fmincon*" in MATLAB to solve for the parameters B and C for each year, using a different set of future predicted mortality rates for ages 1–85.

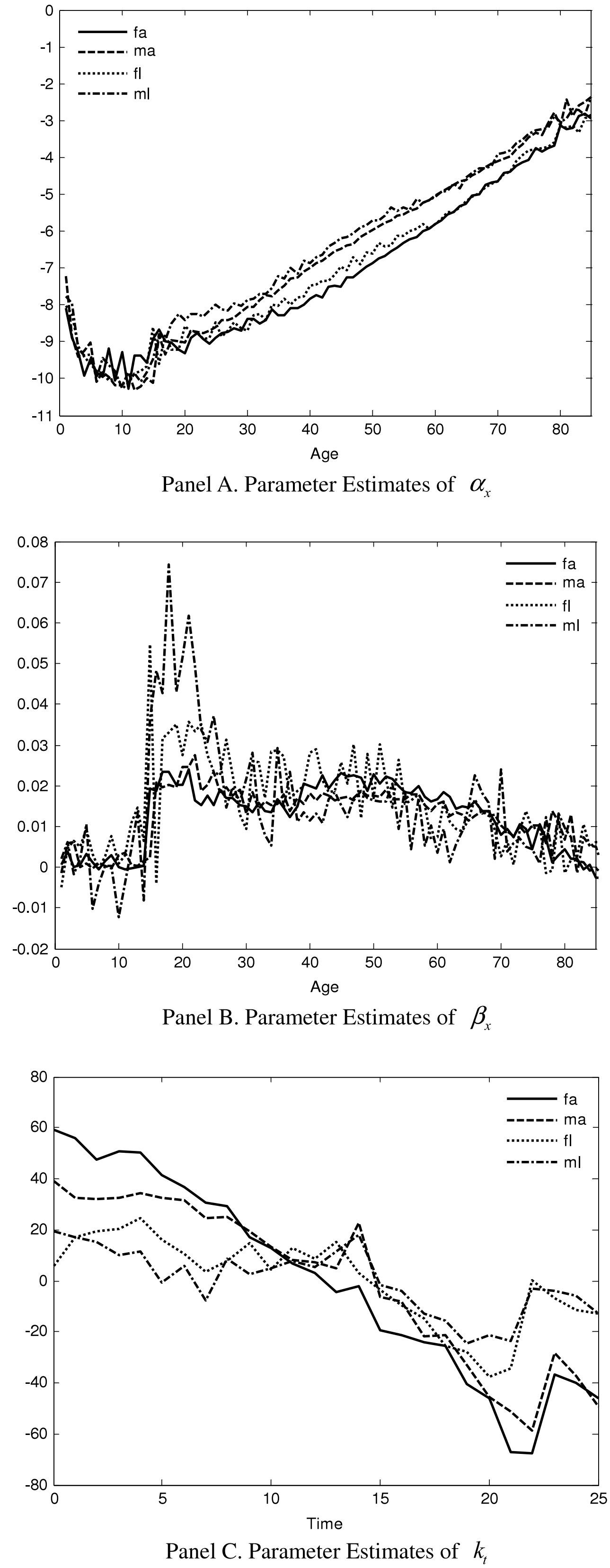

Fig. 1. Parameter estimates of α_x, β_x and k_t in the Lee–Carter model.

Next, we used a Cholesky decomposition to transform the dependent random variables into a linear combination of independent random variables. First, we computed the correlation matrix (M) of k_t^{fa}, k_t^{ma}, k_t^{fl}, and k_t^{ml}:

Table 1
Standard deviation of k_t in four mortality groups.

	κ_t^{fa}	κ_t^{ma}	κ_t^{fl}	κ_t^{ml}
SD	9.34057	10.75928	9.89995	8.66247

$$M = \begin{pmatrix} 1 & 0.7740 & -0.0922 & 0.2438 \\ 0.7740 & 1 & -0.1304 & 0.3583 \\ -0.0922 & -0.1304 & 1 & 0.4558 \\ 0.2438 & 0.3583 & 0.4558 & 1 \end{pmatrix}. \tag{27}$$

Second, the Cholesky decomposition refers to a symmetric positive-definite matrix M with real entries into the product of a lower triangular matrix R and its conjugate transpose R^T, that is, $M = R \times R^T$. In view of Eq. (28), R can be expressed as follows:

$$R = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.7740 & 0.6332 & 0 & 0 \\ -0.0922 & -0.0933 & 0.9914 & 0 \\ 0.2438 & 0.2679 & 0.5076 & 0.7818 \end{pmatrix}. \tag{28}$$

Therefore, using the lower triangular matrix (R), we rewrite Eq. (10) as

$$\begin{bmatrix} dZ_k^{A,2}(t) \\ dZ_k^{A,1}(t) \\ dZ_k^{L,2}(t) \\ dZ_k^{L,1}(t) \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0.7740 & 0.6332 & 0 & 0 \\ -0.0922 & -0.0933 & 0.9914 & 0 \\ 0.2438 & 0.2679 & 0.5076 & 0.7818 \end{bmatrix} \times \begin{bmatrix} dZ_{k1}(t) \\ dZ_{k2}(t) \\ dZ_{k3}(t) \\ dZ_{k4}(t) \end{bmatrix}. \tag{29}$$

4. Numerical analysis

4.1. Scenario 1: $\theta = 0$ (variance effect)

If we assume that the weight θ is zero, we can observe the variance effect alone. We begin with a simple case, in which there are only two policies in the portfolio: an annuity and a life insurance policy. Let $a = 0.1663$, $b = 0.0606$, and $\sigma_r = 4.733\%$ for the parameters of the CIR model. We find the optimal value (unit) of a life insurance hedge against a one-unit value of the annuity due and capture the corresponding hedging relation from these simple cases.

Let υ be the percentage shift in the mortality rate (interest rate). Taking a life insurance policy of a woman of age 30 (fl30) as an example, Table 2 exhibits that the effective duration and convexity change only slightly when we vary υ from 5% to 15%. For example, when υ increases from 5% to 15% for the mortality rates, the effective duration and convexity for the mortality rates change from 10.9178 to 10.9863 and from −201.8964 to −203.8521, respectively. Similarly, when υ increases from 5% to 15% for the interest rates, the effective duration and convexity for the interest rates only slightly change from −46.8364 to −48.3241 and −2455.374 to −2498.901, respectively.

Table 3 lists the optimal units of life insurance to hedge against a one-unit value of annuity with 5%, 10%, and 15% shifts. The variance of the change of the total insurance portfolio value defined in Eq. (24) slightly changes with varying levels of υ, which in turn leads to an insignificant change in the optimal units of life insurance needed to hedge against a one-unit value of annuity with

Table 2
Effective duration (dur) and convexity (conv) with different υ (5%, 10%, 15%).

fl30	$\upsilon = 5\%$	$\upsilon = 10\%$	$\upsilon = 15\%$
dur (m)	10.9178	10.9434	10.9863
dur (r)	−46.8364	−47.3903	−48.3241
conv (mm)	−201.8964	−202.6282	−203.8521
conv (rr)	2455.374	2471.613	2498.901

Note: fl30 is female life insurance at age 30. dur (m) and dur (r) denote the effective durations for the mortality rate and spot rate in Eqs. (20) and (21), respectively. Then conv (mm) and conv (rr) denote the effective convexities for the mortality rate and spot rate in Eqs. (22) and (23), respectively.

Table 3
Optimal units of life insurance to hedge against one unit of annuity with different υ $(\theta = 0)$.

		fl30	ml30
1 unit fa60			
$\upsilon = 5\%$	N	7.4418×10^{-9}	0.1223
	$f(N)$	8.4347×10^{-8}	7.7291×10^{-8}
$\upsilon = 10\%$	N	2.6829×10^{-8}	0.1221
	$f(N)$	8.4467×10^{-8}	7.7401×10^{-8}
$\upsilon = 15\%$	N	7.3427×10^{-9}	0.1233
	$f(N)$	8.4669×10^{-8}	7.7587×10^{-8}
1 unit ma60			
$\upsilon = 5\%$	N	2.1111×10^{-9}	0.4218
	$f(N)$	6.2997×10^{-7}	5.4603×10^{-7}
$\upsilon = 10\%$	N	2.1031×10^{-9}	0.4214
	$f(N)$	6.3100×10^{-7}	5.4693×10^{-7}
$\upsilon = 15\%$	N	2.0897×10^{-9}	0.4205
	$f(N)$	6.3274×10^{-7}	5.4843×10^{-7}

Notes: fa60 (ma60) is the female (male) annuity with age 60. fl30 (ml30) is the female (male) life insurance with age 30. N is the optimal units of life insurance policy to hedge against one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

different unexpected change rates.[7] For demonstration purposes, we use 10% shifts for both mortality and interest rates in the scenario analyses.

We calculate corresponding optimal units for life insurance for different genders and ages to hedge against one unit of female annuity with age 60 (fa60). As Table 4 exhibits, to hedge against one unit of fa60, the optimal units of female life insurance policies for different ages approach zero, which indicates that we cannot reduce the total variance by holding female life insurance policies to hedge against the fa60. However, we can reduce the total variance by holding male life insurance policies to hedge against one unit of the fa60. That is, the variances of the portfolio of female annuity and male life insurance policies are smaller than those of the portfolio of female annuity and female life insurance policies.

Similarly, with Table 5 we investigate the optimal units of life insurance policies for different genders and ages to hedge against one unit of male annuity at age 60 (ma60). The results in Table 5 are similar to those in Table 4; comparing the right-hand side of Table 4 with Table 5, we note that, compared with a hedge against one unit of fa60, a higher level of optimal units of male life insurance is needed to hedge against one unit of ma60.

Using Eqs. (13)–(16), we determine the main components of the variance effect according to the effective durations of the annuity and life insurance products, as well as the coefficients of the Cholesky decomposition. We expect to hedge the longevity risk of annuity products by holding some units of life insurance,

7 As Wang et al. (2010, p. 491) indicate, in Tables 8 and 9, the impact of different mortality shocks (10%–25%) on insurers remain relatively small, because their model can help hedge against unexpected mortality shocks.

Table 4
Optimal units of life insurance to hedge against one unit of fa60 $(\theta = 0)$.

1 unit fa60	N	$f(N)$
fl30	7.4×10^{-9}	8.45×10^{-8}
fl40	6.87×10^{-8}	8.45×10^{-8}
fl50	1.55×10^{-9}	8.45×10^{-8}
fl60	1.67×10^{-9}	8.45×10^{-8}
ml30	0.1221	7.74×10^{-8}
ml40	0.0844	7.74×10^{-8}
ml50	0.0348	7.74×10^{-8}
ml60	0.0335	7.74×10^{-8}

Notes: fa60 is the female annuity with age 60. fl$x$ (ml$x$) is the female (male) life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

Table 5
Optimal units of life insurance to hedge against one unit of ma60 $(\theta = 0)$.

1 unit ma60	N	$f(N)$
fl30	2.10×10^{-9}	6.31×10^{-7}
fl40	8.99×10^{-8}	6.31×10^{-7}
fl50	1.45×10^{-9}	6.31×10^{-7}
fl60	1.56×10^{-9}	6.31×10^{-7}
ml30	0.4214	5.46×10^{-7}
ml40	0.2923	5.46×10^{-7}
ml50	0.1203	5.46×10^{-7}
ml60	0.1155	5.46×10^{-7}

Notes: ma60 is the male annuity with age 60. fl$x$ (ml$x$) is the female (male) life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

Table 6
Effective duration and mortality rate of each product (insured).

Age	30	40	50	60
Effective duration				
fa	−2.1416	−2.0101	−1.6590	−1.2928
ma	−2.6402	−2.5003	−2.0483	−1.5060
fl	10.9434	9.3589	7.3291	5.7600
ml	10.6281	8.6618	6.5503	4.8508
Mortality rate				
fa	0.000116	0.000168	0.000424	0.001306
ma	0.000121	0.000436	0.001125	0.003044
fl	0.000174	0.000398	0.001207	0.002715
ml	0.000261	0.001	0.002606	0.006274

Notes: fa (ma) denotes the female (male) annuity. fl(ml) denotes the female (male) life insurance.

because we know that the effective durations of annuity and life insurance products exhibit opposite signs, as Table 6 shows. In view of Eq. (28), we cannot obtain a hedging effect by holding female life insurance, because the coefficients of the Cholesky decomposition between female life insurance and female (male) annuity are −0.0922 (−0.0933). However, the coefficients of the Cholesky decomposition between male life insurance and female (male) annuity are 0.2438 (0.2679), so it is possible to minimize the variance effect by holding some male life insurance policies. In addition, we would need to hold more units of male life insurance to hedge ma60 than fa60, because the magnitude of the male annuity's durations and mortality rates are greater than those of parallel female products, as Table 6 shows.

We also compare the differences of optimal hedging strategies between holding one unit of annuity due and one unit of deferred annuity (see Tables 7 and 8). For deferred annuities, we need to hold less life insurance to hedge the mortality uncertainty. In addition, with increasing insured ages, we need fewer life insurance units to hedge against the corresponding annuities.

Table 7
Optimal units of male life insurance to hedge one unit of fa60 and one unit of fa30 with 30 years deferred ($\theta = 0$).

	N	$f(N)$
(a) Female annuity due		
1 unit fa60		
ml30	0.1221	7.7401×10^{-8}
ml40	0.0844	7.7402×10^{-8}
ml50	0.0348	7.7401×10^{-8}
ml60	0.0335	7.7401×10^{-8}
(b) Female deferred annuity		
1 unit fa30 with 30 years deferred		
ml30	0.0155	8.9079×10^{-10}
ml40	0.0085	8.8841×10^{-10}
ml50	0.0060	9.1811×10^{-10}
ml60	0.0041	8.8982×10^{-10}

Notes: fa$x$ is the female annuity with age x. ml$x$ is the male life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

Table 8
Optimal units of male life insurance to hedge one unit of ma60 and one unit of ma30 with 30 years deferred ($\theta = 0$).

	N	$f(N)$
(a) Male annuity due		
1 unit ma60		
ml30	0.4848	5.4884×10^{-7}
ml40	0.2924	5.4693×10^{-7}
ml50	0.1204	5.4693×10^{-7}
ml60	0.1156	5.4693×10^{-7}
(b) Male deferred annuity		
1 unit ma30 with 30 years deferred		
ml30	0.0336	3.5081×10^{-9}
ml40	0.0224	3.5093×10^{-9}
ml50	0.0097	3.5081×10^{-9}
ml60	0.0092	3.5081×10^{-9}

Notes: ma$x$ is the male annuity with age x. ml$x$ is the male life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

Although the duration of younger policyholders' life insurance is longer than that of older holders, the latter's mortality rate is much higher. Therefore, the insurance company is more likely to suffer an instant claim by an older policyholder, and it needs fewer units of older life insurance policies to offset the longevity risk of the corresponding annuities.

4.2. Scenario 2: $\theta = 1$ (mispricing effect)

We now ignore the variance effect and discuss the mispricing effect alone. The value of the objective function then is given by

$$f(N_x^{s,g}, N_B) = \min_{N_x^{s,g}, N_B} D^2, \quad s = A, L, g = 1, 2 \text{ and } x \in \omega. \tag{26$'$}$$

In Fig. 2, according to the experience mortality data from Taiwanese life insurance companies (50,000,000 policies), we reveal the levels of underpricing for annuity policies (life insurance policies with heavy principal repayment) and overpricing for life insurance policies for different ages if we ignore the impacts of mortality improvement (cohort-mortality basis) on these products. By excluding the variance effect in the objective function, we can obtain an optimal strategy, such that the mispricing effect in Eq. (26′) is zero. That is, the value of the objective function is always zero in the optimal situation. In the following analysis, we show just the optimal units of the life insurance policies,

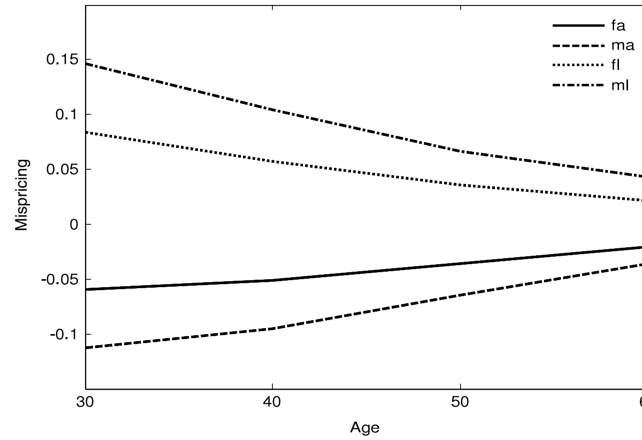

Fig. 2. Pricing differences of each product between period and cohort bases. Notes: fa (ma) is the female (male) annuity. fl (ml) is the female (male) life insurance.

Table 9
Optimal units of female life insurance to hedge one unit of fa60 and ma60 ($\theta = 1$).

fl	N	ml	N
1 unit fa60			
fl30	0.2489	ml30	0.1423
fl40	0.3629	ml40	0.2000
fl50	0.5811	ml50	0.3113
fl60	0.9896	ml60	0.4869
1 unit ma60			
fl30	0.4360	ml30	0.2493
fl40	0.6358	ml40	0.3504
fl50	1.0180	ml50	0.5454
fl60	1.7338	ml60	0.8531

Notes: fa60 (ma60) is the female (male) annuity with age 60. fl$x$ (ml$x$) is the female (male) life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product. $f(N)$ is the objective function defined in Eq. (26).

which correspond to holding one unit of an annuity policy. In addition, according to our experience mortality data, Fig. 2 reveals that for annuity products (life insurance policies with heavy principal repayment), because the ma curve is below the fa curve, underpricing is a more serious problem for male policyholders than for female holders. In parallel, for life insurance products in our experience mortality data, because the ml curve is above the fl curve, overpricing is more serious among male than among female policyholders. Finally, we observe that the magnitude of the mispricing problem decreases as the issuance age increases.

According to Tables 9 and 10 but in contrast with our previous results, we can hedge the mispricing effect of the longevity risk of annuity products with female life insurance products. However, as the issuance age increases, the optimal units of the hedging mispricing effect increase as well, an outcome that differs completely from the results provided in Scenario 1. Therefore, according to the experience mortality data from Taiwanese life insurance companies, we demonstrate that insurance companies probably need different hedging strategies to reduce the variance effect versus the mispricing effect.

4.3. Scenario 3: $0 < \theta < 1$

In practice, insurance companies must take both variance and mispricing effects into account simultaneously to hedge longevity risk. Therefore, the value of θ should be between 0 and 1 but not equal to 0 or 1. In this section, we take weight $\theta = 0.001$ as an example and find that there exists an interaction of optimal hedging strategies between the two effects on average. First, when we consider the variance effect, the optimal units of life insurance

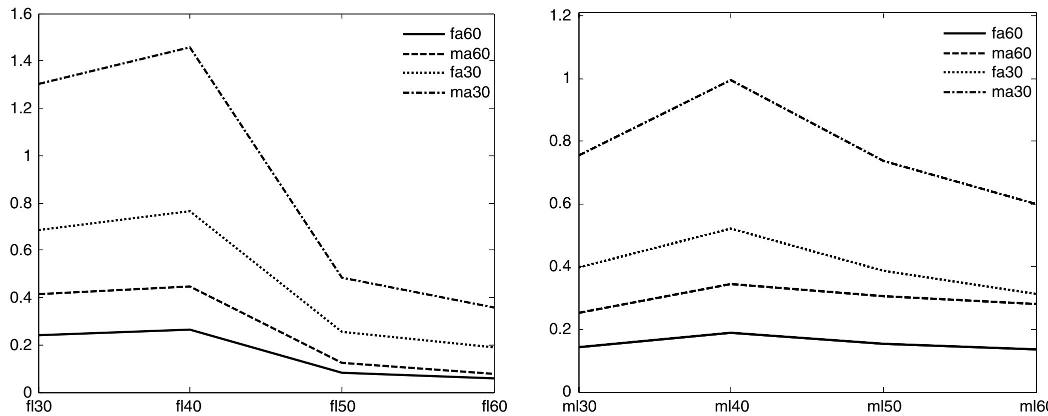

Fig. 3. Optimal units of male and female life insurance to hedge against one unit of annuity for different ages and genders ($\theta = 0.001$). Note: fa$x$ (ma$x$) is the female (male) annuity with age x. fl$x$ (ml$x$) is the female (male) life insurance with age x.

Table 10
Optimal units of female life insurance to hedge one unit of fa30 and ma30 with 30 years deferred ($\theta = 1$).

1 unit fa30 with 30 years deferred			
fl	N	ml	N
fl30	0.7082	ml30	0.4049
fl40	1.0327	ml40	0.5691
fl50	1.6535	ml50	0.8859
fl60	2.8160	ml60	1.3856
1 unit ma30 with 30 years deferred			
fl30	1.3481	ml30	0.7708
fl40	1.9659	ml40	1.0834
fl50	3.1476	ml50	1.6864
fl60	5.3606	ml60	2.6377

Notes: fa30 (ma30) is the female (male) annuity with age 30. fl x (ml x) is the female (male) life insurance with age x. N is the optimal units of life insurance policy to hedge one unit of annuity product.

decrease as the age of the life insurer increases. However, when we consider the mispricing effect, the optimal units increase as the age of the life insurer increases. Therefore, as shown in Fig. 3, the trade-off contributes to a humped curve for the relationship between the optimal units of male or female life insurance needed to hedge against one unit of annuity for different ages and genders.

Second, the results for the optimal units of male and female life insurance to hedge against one unit of annuity for different ages and genders, as expressed in Fig. 3, show similar patterns across different ages and genders. The main differences entail the magnitude of the optimal units. In general, the optimal units of male life insurance to hedge against one unit of annuity are slightly greater than those for female life insurance. Optimal units of male or female life insurance to hedge against an annuity for a younger age also are greater than those to hedge against an older age. In addition, the optimal units of male or female life insurance to hedge against one unit of male annuity are greater than those for a female annuity.

5. Conclusions and suggestions

The natural hedging model we propose can account for two important effects of longevity risk at the same time. The first is variance in the change of the total portfolio value, and the second is the mispricing effect. We can hedge against variations in the future mortality rate and interest rate with the first effect and against present mispricing with the second effect.

Previous research on natural hedging has only addressed the variance effect, but in practice, insurance companies also have a mispricing problem due to mortality improvements over time. We contribute to the existing literature by confirming that it is not reasonable to ignore the mispricing effect when hedging longevity risk. Unlike the previous literature, instead of using the same population mortality rates for both life and annuity policies, we employ the experience mortality rates of life insurance policies with heavy principal repayment as the proxy for annuity mortality rates, because most life insurance policies with heavy principal repayment contain more than 80% saving premiums. Using experience mortality rates, we separate the mortality rate by gender and use correlations across these four types of mortality rates to hedge against variations in the future mortality rate.

In addition, Wang et al. (2010) investigate the natural hedging strategy to deal with longevity risks for life insurance companies under a constant interest rate environment. We integrate the interest rate dynamic into the natural hedging strategy by assuming that the interest rate follows the CIR model, which avoids the problem of a negative nominal interest rate. These differences make our model more general and easier to implement.

We employ a Lee–Carter model to forecast future mortality rates and calculate the level of mispricing in practice. Unlike Wang et al. (2010), we consider a "variance" effect related to uncertainties in the mortality rate and interest rate and a "mispricing" effect induced by mortality improvement. Therefore, this approach can help insurance companies determine the relative significance of variance and mispricing effects according to a weight θ. In this sense, we provide an alternative natural hedging strategy for life insurance companies. When we consider the variance effect only, the optimal units of life insurance depend mainly on the effective duration and mortality rate. Thus, according to the experience mortality rates, the numerical examples show that as the age of a man increases, the optimal units of male life insurance to hedge one unit of the annuity policy decrease if we consider the variance effect only. However, when we consider the mispricing effect only, the optimal units are determined totally by the period–cohort pricing difference of each product. The optimal allocation strategy, obtained by considering the variance effect, arrives at an opposite conclusion for the mispricing effect, based on our experience mortality data. We obtain optimal allocation solutions for both effects to hedge against longevity risk.

Cairns et al. (2009) compare eight stochastic mortality models and find that those with cohort effects, such as the cohort extension of Cairns et al.'s (2006) model and Renshaw and Haberman's (2006) model, provide the best fits to data from England and Wales and US men, respectively. Consequently, in further research, the natural hedging strategy should be examined by various age–period–cohort mortality models that seek to handle longevity risks for insurance companies.

Appendix A. Proof of Eq. (5)

We can rewrite equation (2) as follows:

$$k_{t+n} - k_{t+n-1} = u_k + \sigma_k \varepsilon_{t+n}, \quad n = 1, \dots, T-t, \tag{A.1}$$

where $\varepsilon_{t+n}, n = 1, \dots, T-t$, are independent standard normal random variables. If we sum Eq. (A.1) with $n = 1, \dots, T-t$

$$\begin{aligned} k_T - k_t &= (T-t)u_k + \sigma_k \sum_{n=1}^{T-t} \varepsilon_{t+n} \\ &= (T-t)u_k + \sigma_k \sqrt{T-t}\tilde{\varepsilon}, \end{aligned} \tag{A.2}$$

where $\tilde{\varepsilon}$ is a standard normal random variable. In view of Eq. (A.2), if we let $T = t + dt$, we obtain

$$dk_t = k_{t+dt} - k_t = u_k dt + \sigma_k \sqrt{dt}\tilde{\varepsilon}. \tag{A.3}$$

As Tsay (2002, p. 223) shows, a standard Brownian motion $\{Z_k\}$ satisfies $dZ_k(t) = \sqrt{dt}\tilde{\varepsilon}$. Therefore, we obtain

$$dk_t = k_{t+dt} - k_t = u_k dt + \sigma_k dZ_k(t). \tag{A.4}$$

This completes the proof of Appendix A.

Appendix B. Proof of Eq. (10)

Let $Z_k^{A,2}(t) = \tilde{Z}_1(t), Z_k^{A,1}(t) = \tilde{Z}_2(t), Z_k^{L,2}(t) = \tilde{Z}_3(t)$, and $Z_k^{L,1}(t) = \tilde{Z}_4(t)$. In view of Eq. (10), we know $\tilde{Z}_i(t) = \sum_{h=1}^{i} a_{ih} Z_{kh}(t)$. Because $Z_{kj}(t), j = 1, \dots, 4$, are independent, the covariance of $\tilde{Z}_i(t)$ and $\tilde{Z}_j(t)$ is

$$\begin{aligned} \rho_{ij} t &= \mathrm{Cov}(\tilde{Z}_i(t), \tilde{Z}_j(t)) = E(\tilde{Z}_i(t)\tilde{Z}_j(t)) \\ &= \sum_{n=1}^{i} a_{in} a_{jn} E(Z_{kn}^2(t)) = \sum_{n=1}^{i} a_{in} a_{jn} t. \end{aligned} \tag{B.1}$$

Thus, $\sum_{n=1}^{i} a_{in} a_{jn} = \rho_{ij}$ for $i, j = 1, \dots, 4$. For example, when $i = j = 1$, we know $a_{11} = \rho_{11} = 1$. Similarly, we obtain $a_{21} = \rho_{12}$ when $i = 1$ and $j = 2$ and $a_{22} = \sqrt{1-\rho_{12}^2}$ when $i = 2$ and $j = 2$. Equivalently, $\tilde{Z}_1(t) = Z_{k1}(t)$ and $\tilde{Z}_2(t) = \rho_{12} Z_{k1}(t) + \sqrt{1-\rho_{12}^2} Z_{k2}(t)$, consistent with Shreve's (2004, p. 171) results.

Appendix C. Proof of Eq. (11)

According to Ito's lemma, because mortality risks and financial risks are independent, the instantaneous change in the total insurance portfolio value is of the form

$$\begin{aligned} dV(t) &= \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} dm_x^{s,g}(t) + \frac{\partial V}{\partial r} dr_t \\ &\quad + \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} \frac{1}{2} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g2}} (dm_x^{s,g}(t))^2 \\ &\quad + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} (dr_t)^2. \end{aligned} \tag{C.1}$$

Substituting Eqs. (3) and (8) into Eq. (C.1) yields

$$\begin{aligned} dV(t) &= \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} \Bigg[\Bigg(m_x^{s,g}(t) \beta_{x+t}^{s,g} u_k^{s,g} \\ &\quad + \frac{1}{2} m_x^{s,g}(t) (\beta_{x+t}^{s,g})^2 (\sigma_k^{s,g})^2 \Bigg) dt \\ &\quad + (m_x^{s,g}(t) \beta_{x+t}^{s,g} \sigma_k^{s,g}) dZ_k^{s,g}(t) \Bigg] \\ &\quad + \frac{\partial V}{\partial r} [a(b - r_t)dt + \sigma_r \sqrt{r_t} dZ_r(t)] \\ &\quad + \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} \frac{1}{2} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g2}} \\ &\quad \times (m_x^{s,g}(t) \beta_{x+t}^{s,g} \sigma_k^{s,g})^2 dt + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma_r^2 r_t dt \\ &= \Bigg\{ \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} \Bigg(N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} \Bigg(m_x^{s,g}(t) \beta_{x+t}^{s,g} \mu_k^{s,g} \\ &\quad + \frac{1}{2} m_x^{s,g}(t) \beta_{x+t}^{s,g2} \sigma_k^{s,g2} \Bigg) + \frac{1}{2} N_x^{s,g} \frac{\partial^2 V_x^{s,g}}{\partial m_x^{s,g2}} \\ &\quad \times (m_x^{s,g}(t) \beta_{x+t}^{s,g} \sigma_k^{s,g})^2 \Bigg) + \frac{\partial V}{\partial r} a(b - r_t) + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} \sigma_r^2 r_t \Bigg\} dt \\ &\quad + \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} \\ &\quad \times (m_x^{s,g}(t) \beta_{x+t}^{s,g} \sigma_k^{s,g}) dZ_k^{s,g}(t) + \frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} dZ_r(t) \\ &= Q_{0t} dt + \sum_{s=L,A} \sum_{g=1,2} \sum_{x=i}^{\omega} N_x^{s,g} \frac{\partial V_x^{s,g}}{\partial m_x^{s,g}} \\ &\quad \times (m_x^{s,g}(t) \beta_{x+t}^{s,g} \sigma_k^{s,g}) dZ_k^{s,g}(t) + \frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} dZ_r(t). \end{aligned} \tag{C.2}$$

According to Eq. (10), we have $Z_k^{A,2}(t) = Z_{k1}(t), Z_k^{A,1}(t) = \sum_{j=1}^{2} a_{2j} Z_{kj}(t), Z_k^{L,2}(t) = \sum_{j=1}^{3} a_{3j} Z_{kj}(t)$ and $Z_k^{L,1}(t) = \sum_{j=1}^{4} a_{4j} Z_{kj}(t)$. Thus, Eq. (C.2) can be rewritten as

$$\begin{aligned} dV(t) &= Q_{0t} dt + \sum_{x=i}^{\omega} N_x^{A,2} \frac{\partial V_x^{A,2}}{\partial m_x^{A,2}} m_x^{A,2}(t) \beta_{x+t}^{A,2} \sigma_k^{A,2} dZ_k^{A,2}(t) \\ &\quad + \sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t) \beta_{x+t}^{A,1} \sigma_k^{A,1} dZ_k^{A,1}(t) \\ &\quad + \sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} dZ_k^{L,2}(t) \\ &\quad + \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} dZ_k^{L,1}(t) + \frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} dZ_r(t) \\ &= Q_{0t} dt + \sum_{x=i}^{\omega} N_x^{A,2} \frac{\partial V_x^{A,2}}{\partial m_x^{A,2}} m_x^{A,2}(t) \beta_{x+t}^{A,2} \sigma_k^{A,2} dZ_{k1}(t) \\ &\quad + \sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t) \beta_{x+t}^{A,1} \sigma_k^{A,1} (a_{21} dZ_{k1}(t) + a_{22} dZ_{k2}(t)) \\ &\quad + \sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} \\ &\quad \times (a_{31} dZ_{k1}(t) + a_{32} dZ_{k2}(t) + a_{33} dZ_{k3}(t)) \\ &\quad + \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} (a_{41} dZ_{k1}(t) + a_{42} dZ_{k2}(t) \\ &\quad + a_{43} dZ_{k3}(t) + a_{44} dZ_{k4}(t)) + \frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} dZ_r(t) \\ &= Q_{0t} dt + \Bigg[\sum_{x=i}^{\omega} N_x^{A,2} \frac{\partial V_x^{A,2}}{\partial m_x^{A,2}} m_x^{A,2}(t) \beta_{x+t}^{A,2} \sigma_k^{A,2} + a_{21} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t) \beta_{x+t}^{A,1} \sigma_k^{A,1} \\
&+ a_{31} \sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} \\
&+ \left. a_{41} \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right] dZ_{k1}(t) \\
&+ \left[a_{22} \sum_{x=i}^{\omega} N_x^{A,1} \frac{\partial V_x^{A,1}}{\partial m_x^{A,1}} m_x^{A,1}(t) \beta_{x+t}^{A,1} \sigma_k^{A,1} \right. \\
&+ a_{32} \sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} \\
&+ \left. a_{42} \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right] dZ_{k2}(t) \\
&+ \left[a_{33} \sum_{x=i}^{\omega} N_x^{L,2} \frac{\partial V_x^{L,2}}{\partial m_x^{L,2}} m_x^{L,2}(t) \beta_{x+t}^{L,2} \sigma_k^{L,2} \right. \\
&+ \left. a_{43} \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right] dZ_{k3}(t) \\
&+ \left[a_{44} \sum_{x=i}^{\omega} N_x^{L,1} \frac{\partial V_x^{L,1}}{\partial m_x^{L,1}} m_x^{L,1}(t) \beta_{x+t}^{L,1} \sigma_k^{L,1} \right] dZ_{k4}(t) \\
&+ \frac{\partial V}{\partial r} \sigma_r \sqrt{r_t} dZ_r(t) \\
&= Q_{0t}\, dt + \sum_{i=1}^{4} Q_{it}\, dZ_{ki}(t) + Q_{5t}\, dZ_r(t). \qquad \text{(C.3)}
\end{aligned}
$$

This completes the proof of Appendix C.

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