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PRICING THE AMERICAN OPTIONS FROM THE VIEWPOINTS OF TRADERS

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ABSTRACT. *This paper proposes an arbitrage model, which involves trading the American option (AO) and asset in the frictionless market. The solution of this model provides a trading strategy to maximize the expected arbitrage profit when the market exists an arbitrage opportunity. When there is no arbitrage opportunity in the market, we analyze the arbitrage model by using the duality theory of mathematical programming and show that the initial value of the American option is equal to the expectation of all this option's future possible payoffs. Our results can be used to construct a probability recovering models from the observed market price of the American option.*

Keywords: American Option; Duality Theory; Arbitrage Model

1. Introduction. An American option (AO) is a contract that can be exercised prior to the maturity. However, an AO is always a tradable asset in the market. Instead of exercising the AO, investors holding the ACC will trade this option in the market to maximize the profit of the portfolio when its market price is greater than the early exercise profit.

In this paper, we will propose an arbitrage model, which involves short selling or long buying the American option and assets, from the view point of investors. Namely, the investor in our model can reallocate the position of the American option at each state to maximize the arbitrage profit.

The alive American option always has a market price at each state and can be traded in the practical market. In the practical market, the investors always hold more than one unit of the American option and can reallocate the position of the American option at each state. Therefore, our arbitrage model is constructed by including a variable which is indicated the position of the American options. The objective function in the arbitrage model is defined as the expected arbitrage profit. When the market exists an arbitrage opportunity, the solution of our model provide a trading strategy to maximize the expected arbitrage profit.

On the other hand, the optimal value of the arbitrage model is equal to zero when there is no arbitrage opportunity in the market. We analyze the arbitrage model by using the mathematical technique of duality theory. The dual problem turns out to require the existence of the probability measure. Furthermore, we show that the value of the American option at each state is equal to the conditional expectation of all the possible future payoff under the synthesis probability measures.

2. Notations and Assumption. Following the notation of King (Duality, A. J. King, 2002), We denote $\mathbb{F}=\{\mathscr{F}_t \mid t=0,1,2,\ldots,T\}$ and $\mathbb{F}$ is called a filtration. The true state of the world is revealed to the investors at time t by the atoms of $\mathscr{F}_t$. For any t, let the set $\mathscr{N}_t$ be a partition of Ω which generates the algebra $\mathscr{F}_t$. Thus $\mathscr{N}_{t+1}$ will be a refiner of $\mathscr{N}_t$ for all t. It is convenient to model the relation of $\{\mathscr{N}_t\}_{t=0}^{T}$ by a T-depth scenario tree, in which each atom in $\mathscr{N}_t$ corresponds to an unique node at the depth t in the tree. The leaf nodes $n \in \mathscr{N}_T$ of the scenario tree are one to one correspondence to the elements ω in the sample space Ω. The unique parent node of node $n \in \mathscr{N}_t$ for $1 \le t \le T$ is denoted as $a(n) \in \mathscr{N}_{t-1}$, the set of child nodes of node $n \in \mathscr{N}_t$ for $0 \le t \le T-1$ is denotes as $c(n) \in \mathscr{N}_{t+1}$. Let the probability of each leaf node $n \in \mathscr{N}_T$ is weighted as p_n with $\sum_{n \in N_T} p_n = 1$. Hence, the probability of the internal nodes $n \in N_T, t = 1,2,\ldots,T-1$ is obtained recursively by $p_n = \sum_{m \in c(n)} p_m$ and the conditional probability of the state n occurred under the information of state a(n) is defined by $p_n / p_{a(n)}$:

Denoted $P = \{p_n\}_{n \in \mathscr{N}}$ for all t, the triple $(\Omega, \mathscr{F}, P)$ becomes a probability space. Suppose that $\{X_t\}_{t=0}^{T}$ is stochastic process adapted by the filtration $\mathbb{F}$ then the condition expectation of X_{t+1} under the information of state $n \in \mathscr{N}_t$ and the probability P is defined as $E^P[X_{t+1} \mid N_t] \equiv \sum_{m \in c(m)} X_m \dfrac{p_m}{p_n}$ which is a random variable of states in $\mathscr{N}_t$.

In the market, there are $K+1$ traded securities indexed by $k = 0,1,\ldots,K$ with the vector price process S_n, $n \in N_t$ are adapted by the filtration $\mathbb{F}$, that is, S_n is measurable in the algebra $\mathscr{F}_t$. The vector S_n can be represented as (S_n^o, S_n^1, S_n^K) where security 0 is chosen to be a numeraire which is essentially a bank account process. We introduce the discounted processes denoted by $\beta_n = \dfrac{1}{S_n^0}$. The discounted security prices relative to the numeraire are denoted by $Z_n^k = \beta_n S_n^k$ for $k = 0,1,\ldots,K$. The price Z_n^0 of the numeraire will be exactly one in any state n.

Let θ_n^k denote the investment amount of the asset k at the state n from time t to time $t+1$. A trading strategy $\bar{\theta}_n = (\theta_n^0, \theta_n^1, \ldots, \theta_n^K)$ at state n, $n \in \mathscr{N}_t$ is a vector of decision variables. The total wealth at node n of the investor can be obtained by the inner product of the two vectors as follows:

$$Z_n \bar{\theta}_n = \sum_{k=0}^{K} Z_n^k \theta_n^k \equiv V_n.$$

Note that if θ_n^k is negative, it corresponds to borrow money from the bank or to sell the security *k*. In this representation, we need to note that the investment amount of each asset at the node n always decide after the asset's price revealed. A trading strategy is called self-finance if $Z_n \bar{\theta}_{a(n)} = Z_n \bar{\theta}_n, n \in N_t, t = 0,1,2,\ldots,T,$ which states that time *t* investments are financed only from the proceeds of time $t-1$ holding.

3. An Arbitrage Model. In this section, we will propose an arbitrage model which involves short selling of δ_0 units of the American option and buying $\bar{\theta}_0$ units of the assets at the initial state. We then analyze the optimization dual problem of the arbitrage model in the arbitrage-free market. The definition of arbitrage is followed by Hunt and Kennedy (2000).

Definition 1. The economy ε admits arbitrage if there exists a portfolio ϕ such that one of the following conditions holds:
1. $\phi \cdot A_0 < 0$ and $\phi \cdot A_1(w_j) \geq 0,$ for all j,
2. $\phi \cdot A_0 < 0$ and $\phi \cdot A_1(w_j) \geq 0,$ for all j, which strictly inequality for some j.

If there is no such f then the economy is said to be arbitrage-free.

Let δ_n and C_n denote respectively the holding number of units and the market price of the American option at state *n*. An American option is a contract which gives the buyer a right to receive a payoff F_n when he exercises his right at node $n \in N_t, t = 1,2,\ldots,T$. Hence, the possible payout for $\delta_{a(n)} < 0$ (or income for $\delta_{a(n)} > 0$) at the current state is $\beta_n F_n \delta_{a(n)}, n \in N_t$.

The arbitrage opportunity in this paper is defined as the possibility of finding a self-financing trading strategy which can produce positive reward from nothing. Therefore, the arbitrage model should include the self-finance strategy constraint and the initial wealth constraint.

Hence, the arbitrage model is written as follows:

Arbitrage model

$$\text{Max} \sum_{t=1}^{T} \sum_{n \in N_t} p_n d_n \tag{1}$$

$$\text{s.t. } Z_n \bar{\theta}_{a(n)} - \beta_n F_n \delta_{a(n)} \geq d_n, \qquad n \in N_t, t = 1,2,\ldots,T-1 \tag{2}$$

$$Z_n(\bar{\theta}_n - \bar{\theta}_{a(n)}) - \beta_n C_n(\delta_n - \delta_{a(n)}) = 0 \qquad n \in N_t, t = 1,2,\ldots,T \tag{3}$$

$$Z_0 \bar{\theta}_0 - C_0 \delta_0 = 0 \tag{4}$$

$$d_n \geq 0 \qquad n \in N_t, t = 1,2,\ldots,T \tag{5}$$

$$\delta_n, \theta_n \text{ are unrestricted,} \qquad n \in N_t, t = 1,2,\ldots,T \tag{6}$$

The following theorems show that the objective value of the arbitrage model will equal to zero when there is no arbitrage opportunity in the market.

Theorem 1. *A market exists an arbitrage opportunity if and only if the objective value of the arbitrage model is strictly positive. Moreover, the optimal solution of the arbitrage model has at least a* $d_n > 0$ *for some* $n \in N_t, t \leq T$.

Corollary 1. *A market has no arbitrage opportunity if and only if the objective value of the arbitrage model equals to zero, that is* $d_n = 0$ *for all n.*

To obtain the main result of this paper, we begin with analyzing the dual problem of the above model. The first step in calculating the dual is to multiply all the constraints by the dual variables to form the Lagraingian function.

Let $(\theta, d; \eta, w)$ be an optimal solution of the Lagraingain function. We have $\mathscr{L}(\theta, d; \eta, w) = 0$ when there is no arbitrage opportunity in the market, namely the objective value of the arbitrage (primal) model is equal to zero. As a result, the solution (η, w) satisfy the following necessarily conditions:

Feasibility Problem

$$w_n \geq p_n \qquad n \in N_t, t = 1,2,\ldots,T, \quad (7)$$

$$\eta_n Z_n = \sum_{m \in c(n)} w_m Z_m \qquad n \in N_{T-1}, \quad (8)$$

$$\eta_n \beta_n C_n = \sum_{m \in c(n)} \beta_m F_m w_n \qquad n \in N_{T-1}, \quad (9)$$

$$\eta_n Z_n = \sum_{m \in c(n)} (\eta_m + w_m) Z_m \qquad n \in N_t, t = 0,1,2,\ldots,T-2, \quad (10)$$

$$\eta_n \beta_n C_n = \sum_{m \in c(n)} [\beta_m C_m \eta_m + \beta_m F_m w_m] \qquad n \in N_t, t = 0,1,2,\ldots,T-2, \quad (11)$$

Definition 2. [3] A stochastic process $\{X_t\}$ is called a martingale process under the probability measure *Q* if

$$E^Q[X_t \mid N_n] = X_n, \text{ for all } n \in N_s, s < t$$

and *Q* is called a martingale probability measure for the process.

Theorem 2. *Let (η, w) be a feasible solution of the feasibility problem. We define $q_{m,n} = \dfrac{w_m}{\eta_n}$ for $m \in c(n), n \in N_T$ and $q_{m,n} = \dfrac{\eta_m + w_m}{\eta_n}$ for $m \in c(n), n \in N_t$, $t = 1,2,\ldots,T-1$ then, for all $n \in N_t$, the collection $\{q_{m,n}\}_{m \in c(n)}$ is then interpreted as a conditional probability of the state $m \in c(n)$ under the state n. In fact, the discounted stock price process Z_n is a martingale under the probability measure $\{q_{m,n}\}_{m \in c(n)}$*

Now, we produce the American option's price process from *T* backward to 0. At the maturity date, the holder of the American option receive final payoff F_n and the value C_n of the American option equal to the final payoff F_n. At the earlier stage before the maturity, we have

$$C_n = \sum_{m \in c(n)} \frac{\beta_m}{\beta_n} F_m \frac{\eta_m}{\eta_n} = \sum_{m \in c(n)} \frac{\beta_m}{\beta_n} F_m q_{m,n}, n \in N_{T-1}$$

by (9). This implies that American option’s price at state $n \in N_{T-1}$ equal to the conditional expectation of the possible final payoff discounted to the time $T-1$.

At any time $t = 1,2,\ldots,T-1$, the holder of the American option has two possible decisions. One decision is the holder exercising the American option to receive the exercise payoff F_n. The other is the holder holding the American option with the market price C_n. (11) describes this two cases; the first term of in the brackets corresponds to the American option’s market value and the second term corresponds to the American option’s exercise payoff. The current value of the American option equal to the weighted sum of the market value and the exercise payoff in (11). By analyzing (11), we obtain the following equation:

$$C_n = \sum_{m\in c(n)} \frac{\beta_m}{\beta_n}[F_m \frac{w_m}{\eta_n} + C_m \frac{\eta_m}{\eta_n} = \sum_{m\in c(n)} \frac{\beta_m}{\beta_n}[C_m q_{m,n,1} + F_m q_{m,n,2}], n \in N_t \tag{12}$$

Where $q_{m,n,1} = \frac{\eta_m}{\eta_n}, q_{m,n,2} = \frac{w_m}{\eta_n}$ and $t = 0,1,2,\ldots,T-2$. Here, $q_{m,n,1}$ and $q_{m,n,2}$ separate the conditional probability $q_{m,n}$ into two parts by using (10). The numbers $q_{m,n,1}$ and $q_{m,n,2}$ are interpreted as conditional probability of holding the American option and exercising the American option at the next stage under the stage *n*, respectively. Thus, we regard the collection $\{q_{m,n,1}\}_{m\in c(n)} \cup \{q_{m,n,2}\}_{m\in c(n)}$ as a conditional probability under the state n. By using (12), the current value of the American option is regarded as the conditional expectation under the state n in $\mathcal{N}_t$ of all possible outcomes, holding and exercising, at the next stage.

At any state *n* of time t_0, we obtain $\eta_n = \sum_{t=t_0+1}^{T} \sum_{m\in\mathcal{N}_t} w_n$ by deriving (8) and (10) from *T* backward to t_0. This implies $1 = \sum_{t=t_0+1}^{T} \sum_{m\in\mathcal{N}_t} \frac{w_n}{\eta_n}$. In order to describe the new probability measure, we denote a new symbol $\mathcal{T}_n, n \in \mathcal{N}_{t_0}$ defined as $\mathcal{T}_n \equiv \{m \in \mathcal{N}_t \mid \text{there exists a path P connecting n and m, but } a(n) \notin P, t \geq t_0\}$ that is, $\mathcal{T}_n$ is a set of nodes in the sub tree of the scenario tree with the root n. For example, the set $\mathcal{T}_0$ contains all nodes on the scenario tree and

$$\eta_0 = \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} w_n = \sum_{n\in\mathcal{T}_0} w_n \tag{13}$$

We set $y_n = w_n \eta_0^{-1}$ and claim that the collection $Q = \{y_n\}_{n\in\mathcal{T}_0}$ is a probability measure.

Theorem 3. *Let* (w, η) *be an optimal solution of the feasibility problem, then the collection* $Q = \{y_n\}_{n\in\mathcal{T}_0}$ *is a probability measure.*

Proof. By (13), we have $1 = \sum_{n\in T_0} \frac{w_n}{\eta_0} = \sum_{n\in T_0} y_n$ and $\eta_0 \geq w_n \geq p_n$ for all $n \in \mathcal{T}_0$ So the collection *Q* is a probability measure.

Imposing (8) into (10) and deriving (10) from *T* backward to 0, we have the following equation

$$\beta_0 C_0 = \sum_{m\in\mathcal{T}_0} \beta_m F_m \frac{w_n}{\eta_0} = \sum_{m\in\mathcal{T}_0} \beta_m F_m y_m \,. \tag{14}$$

Thus, we conclude our discussion by the following theorem:

Theorem 4. *If the market exists no arbitrage opportunity then there exists a probability measure such that the initial value of the American option equals to the expectation of all the possible future payoff. In fact, the discounted value of the American option at each state* $n \in \mathcal{N}_t$ *satisfies the following formula*

$$\beta_n C_n = \sum_{m\in\mathcal{T}_n} \beta_m F_m \frac{w_n}{\eta_n} = \frac{\sum_{m\in\mathcal{T}_0} \beta_m F_m y_m}{\sum_{m\in\mathcal{T}_0} y_m}$$

4. Conclusion. We have provided an arbitrage model which involves trading the American option and the assets. In our arbitrage model, the investor can reallocate the position of the American option at each state. Therefore, the arbitrage model is then constructed by including a crucial variable denoting the position of the American options. By inspecting the dual properties of the arbitrage model, we obtain a new theorem in the arbitrage-free market. Namely, the initial price of American option is equal to the expectation of all the possible payoff at each state under the synthesis probability measure. The synthesis probability is endowed by the solution of the dual variables.

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