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Option Trading Strategies with Integer Linear Programming

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ABSTRACT

The problem of how to construct the optimal combination trading strategy for investors when they face a series of options of different exercise prices on the same maturity date can be solved by many standard trading rules. Yet these standard trading rules cannot completely cover the complex and highly changeable combination strategy. This paper proposes an integer linear programming (ILP) model to construct the optimal trading strategy for option portfolio selection. This model focuses on constructing the optimal strategy for an option portfolio of call- and put-options on the same maturity date. Given the investor's belief of the stock price, we also provide an extended ILP model to include this belief. Finally, an empirical study will be presented by using the ILP model applied to the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX, Ticker Symbol: TXO) call and put options.

Keywords: Integer linear programming; Arbitrage opportunity; Option pricing

1. Introduction

An arbitrage opportunity means that you buy and sell something so that you have no probability of negative value at expiration, and yet having a possibility of a positive profit. A market in which prices always full reflect available information is called "efficient". In efficient markets all prices accurately and rapidly adjust to reflect the true intrinsic value of securities. So an opportunity to make excess profits will soon be arbitrage away. Some researches showed that the market which violates market efficiency may overreact to new announcements and it may correct its error slowly. In this situation, a profitable trading rule can devised to take advantage of the slow adjustments of the stock.

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There is a trade-off between risk and return in investment. The risk is a concept which is familiar to everyone but not easy to define in a precise way. Markowitz [5] pioneer measure the risk as the variance of the return and construct a portfolio to control the return and the risk. After Markowitz's research, many risk-measure methods have been developed, such as downside-risk and mean-absolute deviation. A portfolio is called an efficient portfolio if a portfolio may not change the position to increase return at the same risk level or to reduce the risk at the same return level then this. Nowadays, better trade-offs can be achieved by use of financial derivative instruments such as option.

The holder of European call (put) option has the right but not obligation to buy (sell) an underlying asset at a special date for a contractually specified amount, irrespective of the market value of the security on that date. Black and Scholes [1] gave a theoretical pricing formula to compute the "fair" price of such an option based on the assumption that underlying asset follows a geometric Brownian motion. The interest rates, dividend yields and stock volatility remain constants in his pricing formula. In using his pricing formula, one only needs to estimate the volatility of the asset price. There are many ways to estimate the volatility, such as equally-weighted moving average, exponential weighted moving average and implied volatility method. The survey paper of [6] states that the early researches found implied volatility to be better at forecasting future volatility then estimators based on historical data, for option pricing.

Following closing pricing formula of an option, some researches of portfolio also consider the Greeks of the options. Rendleman [8] and [7] use the hedge parameters to construct an optimal option trading strategy which is a linear programming so that the portfolio maintains risk-neutral. But one needs to calculate the Greeks, the correctness of the solution of this model depend on the accuracy of the Greek's computation.

Dert and Oldenkamp [2] construct a linear programming model such that constructed portfolio guaranteed the minimum profit. The underlying assets of this portfolio consist of a stock, options and a risk-free asset. His model not only controls the lower-risk but also considers the casino effect.

Several articles have been devoted to study of the arbitrage opportunity between the communities, futures and the options. Lee and Nayar [4] examined the relationship between the Standard & Poor's 500 Index (SPX) options and the SPX futures contracts and determined the existence of profitable arbitrage opportunities. Draper and Fung [3] used the put-call parity condition to throw light on the relationship between options and futures written against the FTSE Index.

"There is no excess profit in strong or semi-strong efficient market" states in the texts book of Finance. Some of the markets, however, are still weak form efficiency or inefficiency, and some arbitrage opportunities exist in these markets. To construct an efficient portfolio by the return-risk portfolio model may not suit for the lower efficient market. In the following, we will construct an optimal portfolio to master the arbitrage opportunity in the lower efficient market. Some standard trading strategies may have been provided to deal with this problem, but these strategies can not fulfill all the wide variety of markets. To master the arbitrage profit, we first replace the traditional risk measure by our arbitrage probability measure (that is $P\{\text{terminal profit} \geq 0\} = 1$) and maximize the terminal profit. Since the price of the underlying asset at the maturing date is unknown at the current time, our model becomes a stochastic integer linear programming model (SILPM). We also derive our arbitrage constraints to replace our arbitrage probability measure and maximize the expected value of our arbitrage profit at the maturity date. Thus, our SILPM is converted to an integer programming model (ILPM).

Our portfolio consists of a series of puts and calls expired at the same date. We also show that the arbitrage opportunity always exists if our ILPM has a solution. For calculating the expected value of our arbitrage profit, we need to estimate the volatility of the underlying asset because of that we suppose the dynamic of our underlying asset satisfying the geometric Brownian motion. Since the option price may overreact in the inefficient market, using the implied volatility of the underlying asset to make arbitrage will fall into the theoretical trap. In our empirical study, we used the historical data of the underlying asset to estimate the volatility. Since the derivative is written on some original community, even if the option price is over-reactive, using the historical data of the underlying asset also can obtain the "correct" volatility. Represent the relationship between the terminal profit and the density function of our distribution our underlying asset.

In this paper, we first derived our model in the Section 2 and provided an empirical study in the Section 3. Our empirical study is focus on the index options which was first issued on January 1, 2001. The underlying index is the Taiwan Stock Exchange Capitalization Weighted Stock Index.

2. The Model

2.1 Notations and assumptions

Before we discuss our model and solution procedure more detail, we first define the symbols and notations as follows:

T:	investment horizon;
S_t, $t \in [0, T]$:	value of index on time t;
n:	number of different options;
K_i, $i = 1, 2, \ldots, n$:	the exercise price of the option i, $K_1 < K_2 < \ldots K_n$;
P_i, C_i, $i = 1, 2, \ldots, n$:	prices of European put and call option with exercise price K_i, respectively, expiring at the same time T;
$(d)^+ = \max\{d, 0\}$:	positive part of d;

x_i, y_i: the position of the call and put options, respectively, in the portfolio, $x = (x_1, x_2,..., x_n)$ and $y = (y_1, y_2,..., y_n)$;
r: risk-free rate;
$V(S_t, x, y)$: the value function with position x and y when index at S_t on time t.

The assumptions we made in this model are as follows:

1. There are no transaction cost.
2. We consider a single-period problem, that is, the position of our portfolio do not be changed until the horizon.
3. The marginal costs is not consider in our model.

2.2 Programming model of trading strategy

How do we decide the numbers of position of the put and call to obtain the optimal trading strategy? In this section, we use linear programming to construct this optimal portfolio. First, we suppose holding x_i position of k_i-call and y_i position of k_i-put, $i = 1, 2,..., n$. Here $x_i > 0$ (or $y_i > 0$) means long position, otherwise means short position. The value of our portfolio constructed on time t at the maturity date equal to the difference between the sum of the terminal payoff of each call and put in portfolio and the holding cost of these options, that is,

$$V(S_T, x, y) = \sum_{i=1}^{n}[(S_T - K_i)^+ x_i + (K_i - S_T)^+ y_i] - e^{r(T-t)}\sum_{i=1}^{n}(C_i x_i + P_i y_i).$$

In the situation of a short period to maturity and the low interest rate, we can ignore the time value of the cost and rewrite our profit function as follows:

$$V(S_T, x, y) = \sum_{i=1}^{n}[(S_T - K_i)^+ x_i + (K_i - S_T)^+ y_i] - \sum_{i=1}^{n}(C_i x_i + P_i y_i). \tag{1}$$

An arbitrage opportunity is defined as there exists a portfolio the probability with nonnegative profit of the portfolio equal to one no matter what the change of the price of the underlying asset, that is

$$P\{V(S_T, x, y) \geq 0\} = 1.$$

So our main idea is to construct a chance constrained integer linear programming model constrained by probabilistic constraint.

max $\quad V(S_T, x, y)$
s.t. $\quad P\{V(S_T, x, y) \geq 0\} = 1$
$\quad x_i, y_i \in Z, i = 1, 2,\ldots, n.$

If the terminal index S_T is taking one of a finite scenarios S_j^T, $j = 1, 2,\ldots, s$, each with the probability p_j, we can rewrite the above model as a stochastic integer linear programming as follows:

max $\quad \sum_{j=1}^{s} p_j V(S_j^T, x, y)$
s.t. $\quad V(S_j^T, x, y) \geq 0,\ j = 1, 2,\ldots, s$
$\quad x_i, y_i \in Z, i = 1, 2,\ldots, n.$

We modify the idea of Dert and Oldenkamp [2] to provide a nonnegative profit lemma.

Lemma 1: *If a portfolio* (x, y) *satisfies the following conditions*:

$$V(0, x, y) \geq 0$$
$$V(K_i, x, y) \geq 0, i = 1, 2,\ldots, n$$
$$\frac{\partial}{\partial S_T} V(S_T, x, y) = \sum_{i=1}^{n} x_i \geq 0$$

then there exist an arbitrage opportunity.

Proof: We take the derivative of the value function $V(S_T, x, y)$ with respective to S_T obtaining the following equation:

$$\begin{cases} -\sum_{i=1}^{n} y_i & 0 < S_T < K_1 \\ -\sum_{i=l}^{n} y_i + \sum_{i=1}^{l-1} x_i & K_{l-1} < S_T < K_l \\ \sum_{i=1}^{n} x_i & K_n < S_T \end{cases} \tag{2}$$

By (2), the value function $V(S_T, x, y)$ is a piecewise linear function since the derivative function of (1) is a constant between every exercise price. If the value function $V(S_T, x, y)$ is positive at every exercise price and the slope of the value function is nonnegative when the index level greater than K_n, then no matter what the change of terminal index, this value function is always positive. It follows that an arbitrage opportunity will exists in the constructed portfolio.

So, by using the above lemma, our arbitrage model can be rewritten as follows:

Model 1:

max $V(S_T, x, y)$

s.t. $$V(S_T, x, y) = \sum_{i=1}^{n} k_i y_i - \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{n} c_i x_i \geq 0$$

$$V(k_l, x, y) = \sum_{i=l+1}^{n} (k_i - k_l) y_i + \sum_{i=1}^{l-1} (k_i - k_i) x_i - \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{n} c_i x_i \geq 0, \; l = 1, 2, \ldots, n$$

$$V'(S_T, x, y) = \sum_{i=1}^{n} x_i \geq 0$$

$x_i, y_i \in Z, i = 1, 2, \ldots, n$

If the arbitrage opportunity exists, the investors may not be able to earn the infinite wealth in actual market. Because a margin deposit is necessary for short selling an option but the budget of each investor is finite, thus, no one can constructs an option portfolio without budget restriction. So we restrict the short and long position for our holding option and assume that our portfolio does not be forced to liquidate beyond the maturity date. After the above discussion, we set a numbers of contracts, M, being the boundary of each option position in our portfolio. Since the value of the underlying asset at the maturity date is unknown, the objective function in our model one is still a function of random variable S_T. To deal with this stochastic model, we include the option pricing theory to modify our objective function.

2.3 Option pricing formula

We replace our objective function of Model 1 by the expected return of our portfolio. The value of the expected value of $(S_T - k_i)^+$ or $(k_i - S_T)^+$ can be obtained by the theoretical option pricing formula, such as binary tree pricing formula, Black and Schole's [1] famous pricing formula (BS pricing formula). In this paper, we include the BS pricing formula. For using the BS pricing formula, one needs to suppose that the dynamic of the underlying asset follows the geometric Brownian motion, that is,

$$dS_t = S_t(rdt + \sigma dw_t)$$

where dw_t is a Wiener process. This implies that the random variable S_T satisfies the log-normal distribution with mean and variance are

$$S_t \exp\{r(T-t) + \frac{(\sigma(T-t))^2}{2}\}$$

and

$$e^{2r+(\sigma(T-t))^2}[e^{(\sigma(T-t))^2}-1]$$

respectively.

Let $\bar{c}_i$ and $\bar{p}_i$ are the theoretical prices of k_i-call and k_i-put, respectively, which can be obtain by taking the expected final payoff under the risk-neutral probability measure as follows:

$$\bar{c}_i = E[(S_T - k_i)^+] \text{ and } \bar{p}_i = E[(k_i - S_T)^+].$$

Then the objective function of Model 1 can be modified as follows:

$$E[V(S_T, x, y)] = \sum_{i=1}^{n}[(\bar{p}_i - p_i)y_i + (\bar{c}_i - c_i)]x_i.$$

We now need to estimate the volatility of the underlying asset. In the option pricing theory, the implied volatility will adequately represent the trend of the future price of underlying asset in the market. But in the arbitrage theory, the arbitrage opportunity will be cancelled if we use the implied volatility as the parameter of the BS pricing formula. This is because that our arbitrage opportunity constructs on the gap between the theoretical price and the market price but using the implied volatility will force our theoretical price approaching to the actual market price, even if the option market is inefficiency. Thus, the historical volatility will be used as a parameter of the BS pricing formula. Finally, our integer linear programming model is modified as following:

Model 2:

$$\max \quad \sum_{i=1}^{n}[(\bar{p}_i - p_i)y_i + (\bar{c}_i - c_i)]x_i$$

$$\text{s.t.} \quad V(S_T, x, y) = \sum_{i=1}^{n} k_i y_i - \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{n} c_i x_i \geq 0$$

$$V(k_l, x, y) = \sum_{i=l+1}^{n}(k_i - k_l)y_i + \sum_{i=1}^{l-1}(k_l - k_i)x_i - \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{n} c_i x_i \geq 0, \; l = 1, 2, \ldots, n$$

$$V'(S_T, x, y) = \sum_{i=1}^{n} x_i \geq 0$$

$$x_i, \; y_i \in Z \cap [-M, M], \; i = 1, 2, \ldots, n.$$

where $\bar{c}_i$ and $\bar{p}_i$ are the theoretical price of k_i-call and k_i-put, respectively and M be the boundary of numbers of contracts. Note that $x = y = 0$ is the trivial feasible solution, then this model always have an optimal solution. If the optimal solution of this model is a zero vector, this means that the investor may not invest in this situation.

2.4 Application

Some standard option trading strategies have been provided, such as covered calls, protective put, bullish spread, bearish spread and butterfly spread. Our ILPM also can help investors to construct the standard option trading strategies by adjusting the value in the RHS of our constraints. The following example demonstrates this application.

Example: Suppose that the stock price of stock XYZ is NT$38 (New Taiwan Dollar), 30-call option is NT$3 and 35-call is NT$5. Both of the options are delivery at the same month. If investor would like to construct a bullish spread portfolio, how do we decide the position of each option? Imposing all the necessary condition into our ILPM obtained the following model:

$$
\begin{aligned}
\max \quad & (\bar{c}_1 - 3)x_1 + (\bar{c}_2 - 5)x_2 \\
\text{s.t.} \quad & V(0) = -3x_1 - 5x_2 \geq \alpha_1 \\
& V(30) = -3x_1 - 5x_2 \geq \alpha_2 \\
& V(35) = 5x_1 - 3x_1 - 5x_2 \geq \alpha_3 \\
& V'(S_T) = x_1 + x_2 \geq 0 \\
& x_1, x_2 \in Z \cap [-1, 1]
\end{aligned}
$$

where $\bar{c}_i$ are the theoretical option price obtained by the BS pricing formula, α_j are the profit level chosen by the investor's desired. If we like to construct a bullish spread portfolio the selection of α_j should be $\alpha_1 < \alpha_2 < \alpha_3$. If this ILPM has a solution, the terminal profit diagram of our portfolio will higher than the terminal profit diagram of given bullish spread. The other standard strategies follow the same idea to choose the α_j properly.

3. Empirical

3.1 Data and method

The European-style Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) option (TXO) commenced trading on 1/1/2001. In this empirical study, we use model one to testify the arbitrage profit of TXO market during the first half year of 2004. We constructed a short-term portfolio which was delivery at the spot month from the first four day of expiration date to the expiration date. To perform our experiment, the last closing price of each contract of TXO was our market price. Our portfolios only consisted of the contracts with the volume of trade larger than 100 contracts at the constructing day. The volume of selling or buying for each contract in our portfolio was also limited to 10 contracts to ensure our trading strategy practicable. Thus, we totally constructed 24 independent portfolios, and each portfolio contained seven consecutive calls and puts. By the way, the time-stamped and the trading closed price of the TAIEX and the TXO were obtained from the web site of Taiwan Stock Exchange Corporation (TSEC) and Taiwan Futures Exchanges (TFE).

For using BS pricing formula, we also need the following two parameters: risk-free rate and volatility. The average rate of all year was our risk-free rate and the volatility was calculated by the standard derivation of the historical data of the first three months of the delivery month. To compute all the option's price that we need, we used the function *blsprice* in the financial toolbox of MATLAB and solved our integer linear programming model with CPLEX. Before our discussing, we clarified one thing: the final profit diagram of our portfolios are all positive functions of S_T. This may imply that the arbitrage opportunity exists in each day. The only different of each arbitrage portfolio at each day is the level of the profit.

3.2 Results and discussion

In this section, we analyzed the relation between the expected profit and the actual profit of our portfolios, and the relation between the density function of the random variable S_T and the final profit diagram of our portfolios.

To analyze the relation between the expected profit and the actual profit, we first divided our portfolios into two groups according to the difference between the expected profit and actual profit. In particular, the first group consisted of the portfolios with the difference between the expected profit and actual profit less than 100 which is listed in Table 1 and the others is listed in the Table 2 called the second group.

Table 1 contains 5 portfolios that the difference between the final settle (F. S.) price and the spot (S.) price are all less than 3 variances. Table 2 contains 15 portfolios that the difference between the final settle price and the spot price of these portfolios are all larger than 3 variances. Furthermore, there are 10 portfolios

Table 1. Group one of the 20 portfolios.

Constructing Date	The Price of TAIEX		The Profit of Our Portfolio	
	Spot Price	F. S. Price	Expected Profit	Actual Profit
17th-Feb	6,600	6,605	415	409
16th-Apr	6,818	6,810	339	378
20th-Apr	6,799	6,810	479	457
19th-Apr	6,779	6,810	262	294
14th-Jan	5,574	5,560	391	319

Table 2. Group two of 20 portfolios.

Constructing Date	The Price of TAIEX		The Profit of Our Portfolio	
	S. Price	F. S. Price	Expected Profit	Actual Profit
12th-Feb	6,436	6,605	266	20
13th-Feb	6,549	6,605	356	9
16th-Feb	6,565	6,605	491	627
11th-Mar	6,879	6,577	671	28
12th-Mar	6,800	6,577	775	17
15th-Mar	6,635	6,577	2,506	1,433
16th-Mar	6,589	6,577	606	776
15th-Apr	6,736	6,810	339	197
13th-May	5,918	5,860	1,041	324
14th-May	5,777	5,860	477	249
17th-May	5,482	5,860	1,093	17
18th-May	5,557	5,860	315	27
10th-Jan	5,867	5,560	761	21
11th-Jan	5,735	5,560	392	9
15th-Jan	5,646	5,560	249	121

in the Table 2 with the actual profit do not touch the half level of the expected profit. The portfolios with low (large) difference between spot price and final settle price had low (large) difference between expected profit and actual profit. This is because that the expected value of S_T equal to $S_t \exp\{(r - \frac{1}{2}\sigma^2)(T - t)\}$. The remaining time and the historical volatility were shortly and small, the expected value of the random variable S_T always close to the spot price. If the final settle price of S_T near to the spot price, this implies that the actual outcome also near to the expected value of S_T. Namely, our model forecast the actual outcome of S_T with small error, so the expected profit is close to the actual profit. On the other hand, the expected value of S_T is far away from the actual outcome if the current price is far away from the actual outcome.

The final profit diagrams of 20 portfolios are graphed by months. The figure of February, March, and April's portfolio's are in Figure 1, 2, and 3, respectively. All the figures are on the above of x-axis as expected. The peak of the profit diagrams are roughly in the middle of the serial of the strike prices.

The final profit diagram of the portfolios of May 13, May 17 has two peaks as shown on the Figure 4 which are the solid line and the dotted line.

The mean and the variance of the random variable S_T both are increasing functions of the remaining days. This implies that the longer remaining time is, the more spread-out of the profit diagram.

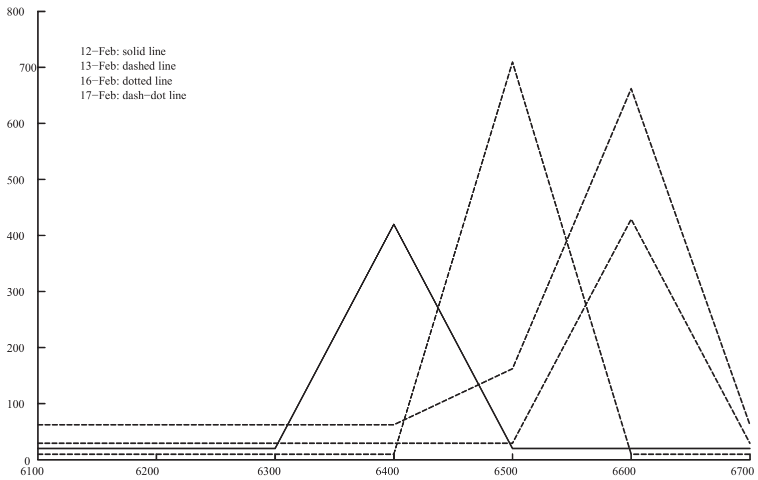

Figure 1. The final profit diagram of the portfolios in February.

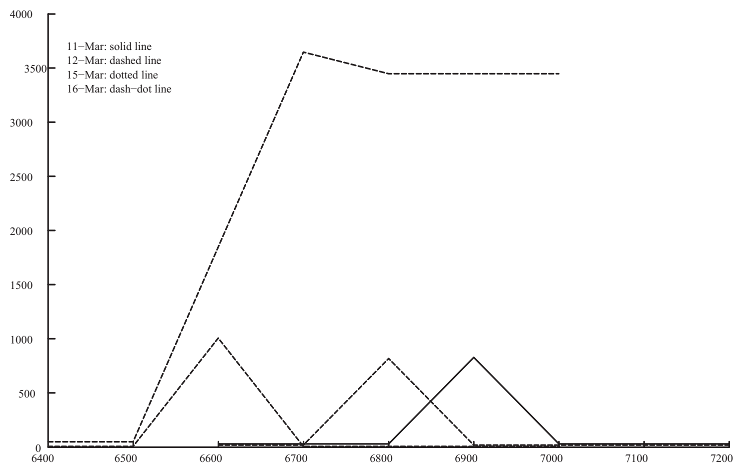

Figure 2. The final profit diagram of the portfolios in March.

We had constructed a portfolio with nonnegative profit, but the objective function does not forecast the actual profit very precisely. The miss forecasting may come from that the behavior of final settle price may not satisfy the log-normal distribution. This is because of that there is three-fourths of the final settle price falling out of 3 variances of the expected value of the random variable S_T in our empirical study.

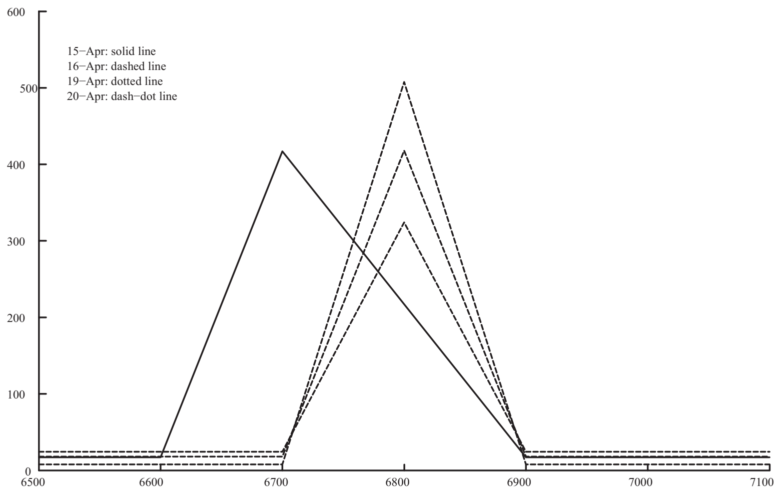

Figure 3. The final profit diagram of the portfolios in April.

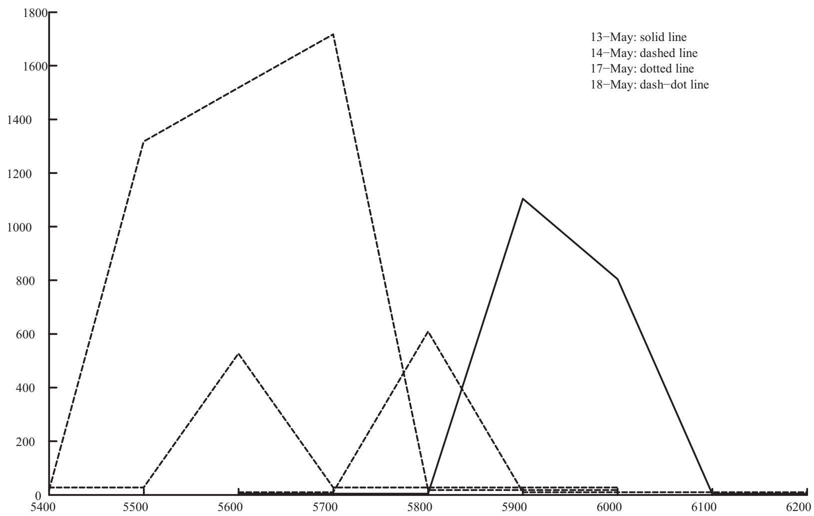

Figure 4. The final profit diagram of the portfolios in April.

4. Conclusions

In this paper, we address an integer linear programming model which also provides a strategy to construct an optimal portfolio to capture the arbitrage opportunity in the weak form efficiency market. We first provide a lemma to master the arbitrage and then establish our integer linear programming model. In our empirical study, we found that our portfolio always has a nonnegative profit no matter what the change of the terminal price of the underlying asset. But the relation between the expected profit and actual profit is depended on the distribution of the terminal price S_T that we supposed. There are many future research at this topic can be done, such as adding the transaction cost in the model or including the underlying asset in the constructed portfolio.

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