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A monopolistic competition model of spatial agglomeration with variable density*

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Abstract. In this paper, we combine the monopolistic competition model of Fujita (1988) with the variable density model by Tabuchi (1986), Liu (1988) and Grimaud (1989), and develop a monopolistic competition model of spatial agglomeration with variable density. We compare the results of the present paper with those of previous work, and show that some previous results cannot be carried over to our generalized model with variable density.

1. Introduction

During the last decade, significant progress has been made in developing a new class of urban land use models, called *general urban land use models.* In these models, the traditional assumption of monocentricity is abandoned, and location of all agents in a city is determined simultaneously; thus, formation of various types of centers is also determined endogenously (for reviews, see e.g., Fujita 1986a, 1990; Stahl 1987). To generate internal forces of spatial agglomeration, most existing such models rely on the concept of *spatial externalities* (or, *nonprice interactions*). Recently, however, several people have attempted to develop an alternative class of general urban land use models, in which agglomeration forces are endogenously generated through *price interactions* alone (e.g., Kanemoto 1985; Papageorgiou and Thisse 1985; Fujita 1988). In particular, Fujita (1988) developed a spatial version of the Chamberlinian monopolistic competition model (with differentiated consumer goods), and demonstrated that pure market processes based on price interactions alone can generate spatial agglomeration of economic activities.

The objectives of this paper are twofold. First, we generalize Fujita's model (1988) by allowing the land-use density to vary over the urban space. Although

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Fujita's assumption of constant land-use density simplifies the analysis, it is very restrictive in the context of urban model. Considering that Fujita's model has a potential for explanation of various types of spatial agglomeration due to product variety (as will be discussed in Sect. 6), it is imperative to generalize the model to the case with variable density. This generalization is achieved by following the recent studies of Tabuchi (1986), Liu (1988) and Grimaud (1989), in which constructors can freely choose the profit-maximizing floor density at each location. Second, in the context of this generalized model, we reexamine the results of Fujita (1988). For example, one of important conclusions of Fujita (1988) is that (socially) optimal land use pattern will be realized at the equilibrium of the competitive land market and monopolistic consumer-goods market (without any government intervention). This result is noteworthy because all previous models (based on spatial externalities) concluded otherwise, i.e., the equilibrium land use pattern is not socially optimal. We show later that Fujita's result holds only when the land-use density is kept constant; hence this result cannot be carried over to our generalized model with variable density. Similarly, we compare and contrast other important results of the two models.

The plan of the paper is as follows. In Sect. 2, we develop the model. In Sect. 3, equilibrium conditions are stated, and several additional specifications are introduced. In Sect. 4, equilibrium urban configurations are obtained. Section 5 compares the equilibrium configurations with optimal configurations. Section 6 concludes the paper.

2. The model

Let us represent the *location space* of the city, generally, by X. In this location space X, we consider spatial interactions among three types of activities: households ($\equiv$ consumers), the local consumer-goods industry (*c-industry*), and floor-space constructors. The c-industry provides a continuum of differentiated consumption goods to households, while constructors supply the floor space to households and c-industry.[1] Given the spatial distribution of households (i.e., customers), each firm in the c-industry chooses its optimal location and (f. o. b.) price of its good. In turn, given the spatial distribution of firms (i.e., suppliers) in the c-industry, each household chooses its optimal location and consumption and consumption-trip pattern (for purchases of these goods). Constructors rent land from absentee landlords, and produce floor space. In turn, floor space is rented to households and firms. An equilibrium is reached when the demand and supply of each good is balanced at each location and the land and floor-space markets are cleared everywhere in X. We explain the behavior of each type of activity-unit in detail below.

[1] As an example of c-industry, we may imagine an industry which consists of a variety of restaurants.

2.1 Household

In the city, there exists a continuum of homogenous households of size N. All households are assumed to have the same utility function, which depends on the consumption levels of floor space and other consumption goods. For simplicity, it is assumed that each household consumes a fixed amount of floor space S_h. Each household also consumes an imported (numeraire) good z_0 and a continuum of local consumer goods (*c-goods*) provided by the firms in the c-industry. The utility function is assumed to be symmetric with respect to all c-goods. Furthermore, although each firm (in the c-industry) is assumed to provide a distinct c-good, all firms are assumed to have the same production function (which is introduced later). Let $h(y)$ [or $f(y)$] represent the number (more precisely, density) of households (or firms) at each location $y \in X$. Then, since each firm is assumed to supply a distinct c-good, $f(y)$ also represents the number of c-goods supplied at each $y \in X$. Let $t(x, y)$ represent the transportation cost per unit of c-good from location x to location y, which is assumed to be borne by households, and to be the same for all c-goods. Then because of identical production function for all firms, identical transportation cost function for all c-goods, and identical and symmetric utility function (with respect to the c-goods) for all households, in equilibrium all c-goods provided at the same location, y, must have the same (f.o.b.) price $p(y)$. This, in turn, implies that each household at location x purchases the same amount, $z(x,y)$, of a c-good from each one of $f(y)$ firms at $y \in X$. Based on this, we assume that if a household chooses a location $x \in X$ and purchases $z(x,y)$ of a c-good from each of $f(y)$ firms at each $y \in X$, then the utility function of the household is given by

$$u(x) = \int_X B[z(x,y)]f(y)\,\mathrm{d}y + z_0 \ , \tag{2.1}$$

where B is an appropriate numerical function.

The budget constraint of a household at location x is given by

$$z_0 + \int_X [p(y) + t(x,y)]\, z(x,y) f(y)\,\mathrm{d}y + R(x) S_h = Y \ , \tag{2.2}$$

where $R(x)$ represents the rent per unit of floor at x, i.e., *floor-rent* at x, and Y is the income of each household (which is exogenously given). Here, the unit price of z_0 is normalized to unity. Then solving (2.2) for z_0 and substituting it into (2.1), the utility function of each household at location x is now given as[2]

$$u(x) = \int_X \{B[z(x,y)] - [p(y) + t(x,y)]\, z(x,y)\} f(y)\,\mathrm{d}y - R(x) S_h + Y \ . \tag{2.3}$$

Next, for concreteness, we assume that in the utility function (2.1), the function B is specified as

$$B(z) = \begin{cases} (z/\alpha)(1 + \log\beta) - (z/\alpha)\log(z/\alpha) & \text{if} \quad z < \alpha\beta \\ \beta & \text{if} \quad z \geq \alpha\beta \ , \end{cases} \tag{2.4}$$

2 Here, it is assumed that income Y is sufficiently large so that z_0 is always positive in equilibrium.

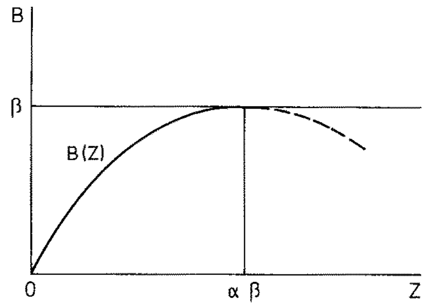

Fig. 1. Function $B(z)$ given by (2.4)

where α and β are positive constants (refer to Fig. 1). This function, which is closely related to the well-known *entropy function*, well expresses households' preference for variety.[3]

Now, the objective of each household is to choose a residential location x and a consumption pattern of c-goods, $z(x,y)$ for all $y\in X$. Notice, however, that because of the additive property of the utility function (2.3), for each pair (x,y) we can determine the optimal value of $z(x,y)$ independently of others. With (2.4), we can obtain the following relation:

$$\max_{z(x,y)} \{B[z(x,y)]-[p(y)+t(x,y)]z(x,y)\}=\beta e^{-\alpha[p(y)+t(x,y)]} , \tag{2.5}$$

where the *optimal demand distribution* function is given by

$$z(x,y)=\alpha\beta e^{-\alpha[p(y)+t(x,y)]} \quad \text{for each} \quad x,y\in X . \tag{2.6}$$

Substituting (2.6) into (2.3), the utility function of a household at x becomes

$$U(x)=\int_X \beta\exp\{-\alpha[p(y)+t(x,y)]\}f(y)\,dy-R(x)S_h , \tag{2.7}$$

where $U(x)\equiv u(x)-Y$.

Accordingly, given price distribution $p(.)$, firm distribution $f(.)$ and floor-rent $R(.)$, the locational behavior of each household is now equivalent to choosing a residential location x so as to maximize its utility given by (2.7).

2.2 Firms

As was noted before, the c-industry is assumed to consist of a continuum of firms of size M. The index set of firms is denoted by $\mathbb{M}\equiv\{i\mid 0\leq i\leq M\}$. Each firm produces only one good and is the sole producer of this good, so that $i\in\mathbb{M}$ denotes a specific firm producing a specific good. All firms are assumed to have the same production technology. Moreover, it is assumed that all firms occupy the same

[3] For further discussion of this function, see Anderson, dePalma and Thisse (1988) and Fujita and Smith (1990).

constant amount of floor space S_f, and have the same fixed capital cost K and the same marginal production cost c. Thus, the profit $\pi(x)$ of a firm at x is given as

$$\pi(x) = [p(x)-c] \int_X z(y,x)h(y)\mathrm{d}y - R(x)S_f - K \ , \tag{2.8}$$

where $p(x)$ represents the (f.o.b.) price of each c-good sold at x, and $z(y,x)$ the amount of the c-good purchased from a c-firm at x by a household at y. As before, $h(y)$ represents the household density at y, and $R(x)$ the floor-rent at x. In (2.8), $\int_X z(y,x)h(y)\mathrm{d}y$ represents the total amount of the c-good sold by a firm at x. Substituting (2.6) into (2.8), we have

$$\Pi(x) = \alpha\beta[p(x)-c] \int_X \exp\{-\alpha[p(x)+t(y,x)]\}h(y)\mathrm{d}y - S_f R(x) \ , \tag{2.9}$$

where $\Pi(x) \equiv \pi(x)+K$. Hence, given household distribution $h(.)$ demand distribution $z(.,.)$ and floor space rent $R(.)$, each firm chooses a location x, and an f.o.b. price $p(x)$ so as to maximize its profit given by (2.9).

From the first-order condition for maximization of $\Pi(x)$ with respect to $p(x)$, the equilibrium price of a c-good for each firm at x can be obtained as

$$p(x) = c + (1/\alpha) \equiv p_m \tag{2.10}$$

which reflects the familiar monopolistic pricing and sets the marginal revenue, $p_m - (1/\alpha)$, equal to the marginal cost, c. Note that this price is independent of the location.

Although the *market equilibrium price* of each c-good is a constant, given by (2.10), in the following analysis we treat the price p as a parameter in order to study the equilibrium distributions of households and firms under various price levels.

Define

$$z^0(x,y) = \alpha\beta \exp[-\alpha t(x,y)] \ , \tag{2.11}$$

which is the demand-distribution under zero price [$p(y) = 0$ for each $y \in X$]. We call $z^0(x,.)$ the *potential demand-distribution* of a household at x. Note that since t is fixed, z^0 is also a fixed function. After setting $p(y) = p$ in (2.6), the *actual demand-distribution* of a c-good under each price level, p, is given by

$$z^*(x,y) = \exp(-\alpha p) z^0(x,y) \ . \tag{2.12}$$

Next, without loss of generality we can normalize S_h and S_f (by appropriately changing the units of M and N) so that

$$S_h = 1 = S_f \ . \tag{2.13}$$

Then, by setting $p(y)=p$ or $p(x)=p$, from (2.7) and (2.9) we have

$$U(x)=\gamma(p)\int_X z^0(x,y)f(y)\,\mathrm{d}y-R(x)\ , \tag{2.14}$$

$$\Pi(x)=\delta(p)\int_X z^0(y,x)h(y)\,\mathrm{d}y-R(x)\ , \tag{2.15}$$

where

$$\gamma(p)=(1/\alpha)\exp(-\alpha p)\ ,\quad \delta(p)=(p-c)\exp(-\alpha p)\ . \tag{2.16}$$

For convenience, the following definitions and abbreviations will be used throughout the remainder of this paper:

$$\tilde{\gamma}(p)\equiv\alpha\beta\gamma(p)\ ,\quad \tilde{\delta}(p)\equiv\alpha\beta\delta(p)\ . \tag{2.17}$$

2.3 Constructors

The constructors rent land from the absentee landlords and supply floor space to households and firms. It is assumed that the land not used for floor-construction is used for agriculture, yielding a given constant rent R_a. The market in floor space is assumed to be perfectly competitive. It is also assumed that the construction cost without land rent is a function of floor density. Thus, the (nonland) construction cost, $C(x)$, per unit of land at x is assumed to be given by

$$C(x)=aH(x)^b\ , \tag{2.18}$$

where a and b are constants such that $a>0$ and $b>1$, and $H(x)$ is the floor space density (per unit of land) at x. Then, the profit of construction per unit land (which is used for construction) at x is given by

$$\pi_c=R(x)H(x)-aH(x)^b-L(x)\ , \tag{2.19}$$

where $L(x)$ is the *land rent* at x. Let $g(x)$ be the proportion of land used for construction (of floor space) at each x, where $0\leq g(x)\leq 1$. In equilibrium, competition among constructors drives the profit of each constructor to zero. Hence, provided that $g(x)>0$, it must hold that $\pi_c(x)=0$, which in turn yields that

$$L(x)=R(x)H(x)-aH(x)^b\quad\text{if}\quad g(x)>0\ . \tag{2.20}$$

Furthermore, since constructors take floor rents and land rents as given, setting $\partial\pi_c(x)/\partial H(x)=0$, we can obtain the following condition for choosing the profit-maximizing density at each x:

$$R(x)=abH(x)^{b-1}(\equiv dC(x)/dH(x))\ , \tag{2.21}$$

which represents a familiar equality of the marginal revenue and marginal cost of floor space supply. Upon substation of (2.21) into (2.20), we have

$$L(x)=a(b-1)H(x)^b \quad \text{if} \quad g(x)>0 \ . \tag{2.22}$$

3. Equilibrium conditions and some additional specifications

In this section, first we derive the conditions for the equilibrium of an urban spatial configuration. A *land use equilibrium* describes a state of the urban system that shows no propensity to change. That is, an equilibrium is reached when all households (in the city) achieve the same maximum utility, all firms the same maximum profit, all constructors the same maximum profit (which equals zero by assumption), and land and floor-space markets are cleared everywhere.

To state this equilibrium conditions precisely, recall that in equilibrium, all households (firms) must achieve the same utility level (profit level). Hence, given the equilibrium utility level U^* (or profit level Π^*) (which are yet unknown) we define the *bid-floor-rent function of households*, $\psi(x)$, and the *bid-floor-rent function of firms*, $\Phi(x)$, respectively as follows:

$$\begin{aligned}\psi(x) &\equiv \psi(x; f(.), p, U^*)\\ &= \gamma(p)\int_X z^0(x,y)f(y)\,\mathrm{d}y - U^* \ ,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\Phi(x) &\equiv \Phi(x; h(.), p, \Pi^*)\\ &= \delta(p)\int_X z^0(y,x)h(y)\,\mathrm{d}y - \Pi^* \ .\end{aligned} \tag{3.2}$$

Here, the price level is fixed at a given level $p \geq c$. By definition, $\psi(x)$ [or $\Phi(x)$] represents the maximum rent which can be paid by a household (or firm) per unit of floor space at x while attaining the utility level U^* (for profit level Π^*). Similarly, considering (2.22) and noticing that the equilibrium profit of constructors is always zero, for each x, we define the *bid land rent function of constructors*, $\Gamma(x)$, as follows:

$$\Gamma(x)=a(b-1)H(x)^b \ . \tag{3.3}$$

Now, we say that a spatial $\{h(x), f(x), H(x), g(x), R(x), L(x), U^*, \Pi^*; x \in X\}$ represents a *land use equilibrium* under a price p if and only if the following set of conditions are satisfied: at each $x \in X$,

(a) Floor space market equilibrium conditions:

$$R(x)=\mathrm{Max}\{\psi(x), \Phi(x), 0\} \ , \tag{3.4}$$

$$R(x)=\psi(x) \quad \text{if} \quad h(x)>0 \ , \tag{3.5}$$

$$R(x) = \Phi(x) \quad \text{if} \quad f(x) > 0 \ , \tag{3.6}$$

$$R(x) = ab[\mathrm{H}(x)]^{b-1} g(x) \ , \tag{3.7}$$

$$h(x) + f(x) = H(x) g(x) \quad \text{if} \quad R(x) \geq 0 \ , \tag{3.8}$$

(b) Land market equilibrium conditions:

$$L(x) = \mathrm{Max}\{\Gamma(x), R_a\} \ , \tag{3.9}$$

$$0 \leq g(x) \leq 1 \ , \tag{3.10}$$

$$L(x) = \Gamma(x) \quad \text{if} \quad g(x) > 0 \ , \tag{3.11}$$

$$L(x) = R_a \quad \text{if} \quad g(x) < 1 \ , \tag{3.12}$$

(c) Total activity-unit number constraints:

$$\int_X h(x) \mathrm{d}x = N \ , \tag{3.13}$$

$$\int_X f(x) \mathrm{d}x = M \ , \tag{3.14}$$

(d) Nonnegativity constraints:

$$h(x) \geq 0 \ , \quad f(x) \geq 0 \ , \quad R(x) \geq 0 \ , \quad L(x) \geq 0 \ , \quad H(x) \geq 0 \ . \tag{3.15}$$

Conditions (3.4) to (3.6) state that each unit of floor space must be occupied by either a household or a firm which bids a higher (positive) floor rent at that location. Conditions (3.5) and (3.6) state that households or firms may locate at x only if they have succeeded in bidding for the floor space at that location. Equation (3.7) represents the equilibrium floor rent as equal to the marginal cost of providing floor space. Equation (3.8) represents the equality of demand and supply of floor space at each x. Conditions (3.9), (3.11) and (3.12) together imply that each unit of land must be used by either a constructor or a farmer who bids a higher rent. Condition (3.10) states an obvious physical constraint. Conditions (3.11) says that constructors may use the land at x only if they are the highest bidder at that location. Condition (3.12) says that if some land at x is not used for floor space construction, then the land rent there equals the agriculture rent. Conditions (3.13) and (3.14) ensure that all households and firms must locate somewhere in the city. It can be readily seen that if all conditions above are satisfied, then all households (firms) achieve the same maximum utility level U^* (profit level Π^*), all constructors earn zero profit, and land and floorspace markets are cleared everywhere.

Next, in order to obtain explicit solutions to the form of equilibrium configurations, we introduce several simplifying assumptions. First, we assume that the location space of the city is one-dimensional, i.e., $X = \mathbb{R} \equiv (-\infty, \infty)$, and the land density at each $x \in \mathbb{R}$ equals unity. That is, the city locates on a long narrow strip of land having width 1. Second, observe from (3.1) and (3.2) that the characters of the equilibrium configurations are governed by the nature of the potential demand-distribution function, $z^0(x,y)$. In particular, given $z^0(x,y)$ by (2.11), if we specify the transport cost function $t(x,y)$ in various forms, we can obtain different function forms of the term $e^{-\alpha t(x,y)}$. For simplicity, in this paper we consider the case with a linear trip distribution given by

$$e^{-\alpha t(x,y)} = 1 - \tau |x-y| \ , \tag{3.16}$$

where t is a positive constant.[4] From (2.12), this implies that the demand distribution function $z(x,y)$ is also linearly decreasing with respect to the distance between x and y. Third, we consider only symmetric equilibrium urban configurations, where d $(-d)$ represents the right (left) urban fringe distance from the center of the city, 0. Then, in order to assure that the right side of Eq. (3.16) is positive for all $x,y \in [-d,d]$, the following condition must hold

$$2d < 1/\tau \ . \tag{3.17}$$

Given this condition, it is not difficult to show that no amount of agriculture land remains in an equilibrium city, i.e.,

$$g(x) = 1 \quad \text{for} \quad x \in (-d,d) \ . \tag{3.18}$$

Fourth, in order to obtain the explicit analytical solutions, in the construction cost function (2.18) we specify that[5]

$$b = 2 \ . \tag{3.19}$$

For the convenience of the subsequent analysis, we also introduce the following terminologies:

(i) Residential Area: $\mathrm{RA} = \{x : h(x) > 0, f(x) = 0\}$.

(ii) Firm District: $\mathrm{FD} = \{x : h(x) = 0, f(x) > 0\}$.

(iii) Mixed District: $\mathrm{MD} = \{x : h(x) > 0, f(x) > 0\}$.

4. Equilibrium urban configurations

Under the set of assumptions above, we can show that for each given parameters $(M, N, p, c, a, \alpha, \beta, \tau, R_a)$ such that $M > 0$, $N > 0$, $p > c$, $a > 0$, $\alpha > 0$, $\beta > 0$, $\tau > 0$ and

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[4] This implies that $t(x,y) = -(1/\alpha)\log(1-\tau|x-y|)$, which is concave in $|x-y|$. We can conduct similar analyses under other specifications of the trip distribution function. Fujita (1986b) suggests that there is no qualitative difference in the results whether the assume a linear trip distribution or a convex trip distribution, i.e., $e^{-\alpha\tau|x-y|}$.

[5] A similar specification was under by Tabuchi (1986), Liu (1988) and Grimaud (1989). When b is a number other than 2, it is very difficult to obtain the solution in explicit form.

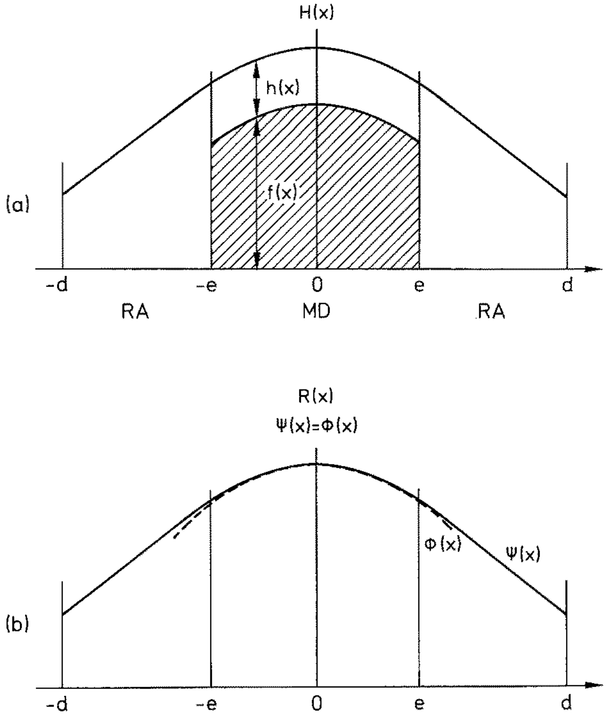

Fig. 2. Equilibrium configuration for mixed pattern I

$R_a > 0$, there always exists a unique land use equilibrium.[6] Each land use equilibrium takes either the form of *the mixed pattern I* or *mixed pattern II*. To show this, first we examine each pattern separately, and then combine the results.

4.1 Mixed pattern I

We first examine the mixed pattern I. In this case, as is depicted in Fig. 2a, there is a MD in the central area of the city, and two RAs situate at the periphery. For this pattern, the equilibrium conditions of the floor space market and land market can be restated as follows (refer to Fig. 2b):

$$R(x) = \Phi(x) = \psi(x) \quad \text{for } -e \le x \le e \; , \tag{4.1}$$

$$R(x) = \psi(x) > \Phi(x) \quad \text{for } -d \le x < -e, \; e < x \le d \; , \tag{4.2}$$

$$L(x) = \Gamma(x) \equiv aH(x)^2 \quad \text{for } -d \le x \le d \; , \tag{4.3}$$

$$L(-d) = L(d) = R_a \; , \tag{4.4}$$

6 The equilibrium is unique subject to the 'symmetric' assumption.

where e represents the fringe distance of the MD such that

$$0<e\leq d \ . \tag{4.5}$$

In the MD, substituting (2.11) and (3.15) into (3.1) and (3.2), we have

$$\psi(x)=\tilde{\gamma}(p)\int_{-e}^{e}(1-\tau|x-y|)f(y)\mathrm{d}y \ - \ U_A^* \ , \tag{4.6}$$

$$\Phi(x)=\tilde{\delta}(p)\left\{\int_{-e}^{e}(1-\tau|x-y|)h(y)\mathrm{d}y+\int_{-d}^{-e}[1-\tau(x-y)]h(y)\mathrm{d}y\right.$$
$$\left.+\int_{e}^{d}[1-\tau(y-x)]h(y)\mathrm{d}y\right\}-\Pi_A^* \ . \tag{4.7}$$

First, we examine the equilibrium spatial configuration in the MD. From the first and second derivatives of (4.6) and (4.7) with respect to x, the bid floor rents in the MD can be seen to be strictly concave and to have their maximum point at $x=0$. Setting $\psi(x)=\Phi(x)$ [by (4.1)], and taking the second derivatives of this equation by using (4.6) and (4.7), we can obtain that

$$\tilde{\gamma}(p)f(x)=\tilde{\delta}(p)h(x) \qquad \text{for } x\in[-e,e] \ , \tag{4.8}$$

which in turn implies that

$$\gamma(p)f(x)=\delta(p)h(x) \qquad \text{for } x\in[-e,e] \ . \tag{4.9}$$

Therefore, from (3.8), (3.18), (4.9) and definitions of $\gamma(p)$ and $\delta(p)$, we can obtain that

$$f(x)=\{\alpha(p-c)/[1+\alpha(p-c)]\}H(x) \qquad \text{for } x\in[-e,e] \ . \tag{4.10}$$

Using (3.13), (3.14) and integrating both sides of (4.9), we have

$$M/N\leq\delta(p)/\gamma(p)\equiv(p-c)\alpha \ \text{ i.e., } p\geq c+(1/\alpha)(M/N) \ . \tag{4.11}$$

From (3.8), (3.18) and (4.9) it follows that

$$H(x)=[h(x)\delta(p)]/\gamma(p)+h(x) \qquad \text{for } x\in[-e,e] \ , \tag{4.12}$$

or

$$H(x)=[f(x)\gamma(p)]/\delta(p)+f(x) \qquad \text{for } x\in[-e,e] \ . \tag{4.13}$$

Substituting (4.13) into (3.7) and using (3.19), it follows that $R(x)=2a[1+(\gamma(p)/\delta(p))]f(x)]$. Therefore, from (4.1) and (4.6) we have

$$\tilde{\gamma}(p)\int_{-e}^{e}(1-\tau|x-y|)f(y)\mathrm{d}y-U^{*}=2a\{[1+(\gamma(p)/\delta(p))]f(x)\}\ . \tag{4.14}$$

Differentiating (4.14) twice with respect to x, and solving the resulting differential equation for $f(x)$ by using the boundary condition $R(e) = V$ (where V is the land rent at e, which will be obtained later), the firm density function $f(x)$ in the MD can be obtained as

$$f(x)=\{[\delta(p)(aV)^{1/2}\cos(Dx)]/[a(\gamma(p)+\delta(p))\cos(De)] \quad \text{for } x\in[-e,e]\ , \tag{4.15a}$$

where

$$D=\{a[1+(\gamma(p)/\delta(p))]\tilde{\gamma}(p)\tau\}^{1/2}/\{a[1+(\gamma(p)/\delta(p))]\}\ . \tag{4.15b}$$

Substituting (4.15a) into (4.9), the household density function $h(x)$ in the MD can be obtained as

$$h(x)=[\gamma(p)(aV)^{1/2})\cos(Dx)]/[a(\gamma(p))\cos(De)] \quad \text{for } x\in[-e,e]\ . \tag{4.16}$$

Finally, from (3.8), (3.18), (4.15) and (4.16) the floor space density function $H(x)$ in the MD can be obtained as

$$H(x)=[(aV)^{1/2}\cos(Dx)]/[a\cos(De)] \qquad \text{for } x\in[-e,e]\ . \tag{4.17}$$

Next, we examine the equilibrium spatial configuration in the RAs. Because of the symmetry assumption, it is sufficient to examine the right half of the city. In the right side RA, the floor rent function is given by

$$\psi(x)=\tilde{\gamma}(p)\int_{-e}^{e}[1-\tau(x-y)]f(y)\mathrm{d}y-U_A^{*} \qquad \text{for } x\in(e,d)\ . \tag{4.18}$$

Since the entire floor space is used by households in the RAs, it is clear that $H(x) = h(x)$. Therefore using (2.21), (3.5) and (4.18), we can obtain

$$h(x)=[\tilde{\gamma}(p)(1-\tau x)M-U_A^{*}]/2a \qquad \text{for } x\in(e,d)\ . \tag{4.19}$$

From (4.3), using the boundary condition (3.12), we have

$$U_A^{*}=\tilde{\gamma}(p)(1-\tau d)M\ -\ 2(aR_a)^{1/2}\ . \tag{4.20}$$

Therefore, substituting (4.20) into (4.19) we finally get that

$$h(x)=H(x)=[\tilde{\gamma}(p)\tau M(d-x)+2(aR_a)^{1/2}]/2a \qquad \text{for } x\in(e,d)\ . \tag{4.21}$$

Having obtained the density functions of households and firms, by manipulating the total activity unit constraints (3.13) and (3.14), and using (4.13) and (4.14), we can get the boundary distances d and e, and the land rent $V \equiv L(e)$ at location e, as follows:

$$e = (1/D)\tan^{-1}\{[DaM(\gamma(p)+\delta(p))]/[2\delta(p)(aV)^{1/2}]\} , \tag{4.22}$$

$$V = (1/2)\tilde{\gamma}(p)\tau M[N-(\gamma(p)/\delta(p))M]+2R_a , \tag{4.23}$$

$$d = e+[2(aV)^{1/2}-(aR_a)^{1/2}]/\tilde{\gamma}(p)\tau M . \tag{4.24}$$

Putting household density functions $h(x)$ [which we get from (4.16) and (4.19)] into (4.7), the equilibrium values of Π_A^* can be obtained as follows:

$$\Pi_A^* = \delta(p)\{N-[(\tau e\delta(p))M/\gamma(p)]-[\gamma(p)\tau Md+2(aR_a)^{1/2}]/2a(d^2-e^2)\} \\ -2(aV)^{1/2} . \tag{4.25}$$

Finally, (2.21), (3.9), (4.17) and (4.21) the floor rent function $R(x)$ can be obtained as

$$R(x) = \begin{cases} [2(aV)^{1/2}/\cos(De)]\cos(Dx) & \text{for } x\in[-e,e] \\ \tilde{\gamma}(p)\tau M(d-x)+2(aR_a)^{1/2} & \text{for } x\in(e,d) . \end{cases} \tag{4.26}$$

Therefore, we can conclude that given $(N, M, p, c, a, \alpha, \beta, \tau, R_a)$ *there exists a land use equilibrium of mixed pattern I if and only if condition (4.11) is satisfied.* The floor space density function $H(x)$ is depicted in Fig. 2a. The floor space density is strictly concave everywhere in the MD, and it is linearly decreasing in $|x|$ on the RA. The floor rent configuration associated with this land use pattern is depicted in Fig. 2b. The firms' bid floor rent curve $\Phi(x)$ is strictly concave everywhere, and decreasing in $|x|$. On the MD, the housholds' bid floor rent curve $\psi(x)$ is the same as that of firms; on RAs, it is linearly decreasing in $|x|$.

4.2 Mixed pattern II

If we interchange the location of households and firms in Fig. 2, we can obtain the mixed pattern II. The equilibrium conditions for these pattern are exactly the same as those for the mixed pattern I, except condition (4.2). In this case, condition (4.2) must be replaced by

$$R(x) = \Phi(x) \geq \psi(x) \qquad \text{for } -d\leq x< -e \ , \ e<x\leq d . \tag{4.27}$$

Then, the argument can be made similarly to the previous case, and we can obtain the equilibrium configuration for pattern II. Since the solution results corresponding to pattern II can be readily inferred from those of pattern I, for brevity we omit them except for noting that condition (4.11) now changes as follows:

$$M/N \geq \delta(p)/\gamma(p) \equiv (p-c)\alpha \text{ i.e., } p\leq [c+(1/\alpha)](M/N) . \tag{4.28}$$

That is, given $(M, N, p, c, a, \alpha, \beta, \tau, R_a)$, *there exists a land use equilibrium of mixed pattern II, if and only if condition (4.28) is satisfied.*

From (4.11) and (4.28), we can see that given any set of parameters, $(M, N, p, c, a, \alpha, \beta, \tau, R_a)$, such that $M>0$, $N>0$, $p \geq c$, $\alpha>0$, $\beta>0$, $\tau>0$ and $R_a>0$, there exists a unique land use equilibrium of either pattern I or pattern II. Conditions (4.11) and (4.28) indicate that given a set of (p, c, a), pattern I (or II) tends to be realized when the number of households, N, is relatively large (or small) compared with the number of firms, M. They also indicate that given a set of (M, N, c, a), pattern I (or II) occurs when p is relatively high (or low). Next, from (4.10) we can see that the relative density, $f(x)/H(x)$, of firms in the MD (for the case of pattern I) continuously increases as p increases. This reflects the fact that as p increases, in the floor space market, firms become relatively more competitive than households (i.e., the ratio $\delta(p)/\gamma(p)$ becomes greater); and hence firms can occupy a greater proportion of floor space in the MD in which floor rents are high because of locational advantages. In particular, under the monopolistic pricing, $p = p_m \equiv c+(1/\alpha)$, we have from (4.10) that $f(x)/H(x) = 1/2$. That is, in the monopolistic market equilibrium, firms and households equally share the floor space in the MD.

Finally, relations (4.22) and (4.24) reveal that both the MD fringe distance, e, and urban fringe distance, d, increase when the construction cost parameter, a, increases. This is because as a increases, firms and households disperse in order to save construction costs.

5. Optimal urban configuration

In the previous section, we have obtained equilibrium urban configurations. In this section we examine the optimal urban configuration.

As an optimization problem, we consider the problem of achieving the given target utility level for all households with the minimum total cost. Let $h(.)$ be a density distribution of households, $f(.)$ that of firms; and let $z(x, .)$ be the demand distribution function of each household at $x \in X$, and $z_0(x)$ the consumption of the imported (numeraire) good by each household at x. Then, the total Cost $\mathbb{C}$ associated with the plan, $\{h(x), f(x), H(x), g(x), z(x,.), z_0(x); x \in X\}$ can be calculated as

$$\mathbb{C} = \int_X \left\{ \int_X z(x,y)[c+t(x,y)]f(y)\,dy \right\} h(x)\,dx + \int_X z_0(x)h(x)\,dx + \mathrm{KM}$$

$$+ a\int_X [H(x)]^2 g(x)\,dx + \int_X R_a g(x)\,dx \,. \tag{5.1}$$

The first two terms on the right-hand side represent the costs of consumption by households, the third term the fixed cost for the firms, and the last two terms the cost for construction. The constraints are:

utility constraint: $\int_X B[z(x,y)]f(y)\,\mathrm{d}y + z_0(x) = \bar{u}$ for $x \in X$, (5.2)

floor space constraint: $h(x)+f(x) = H(x)g(x)$ for $x \in X$, (5.3)

land constraint: $0 \leq g(x) \leq 1$ for $x \in X$, (5.4)

population constraints: $\int_X h(x)\,\mathrm{d}x = N$, $\int_X f(x)\,\mathrm{d}x = M$, (5.5)

and the nonnegativity constraints on all choice-variables.

To obtain the optimality conditions, it is convenient to rewrite the problem in terms of surplus. Let Y be the income of each household, which is exogenously given and fixed as before. Then, the minimization of $\mathbb{C}$ is equivalent to the maximization of surplus defined as

$$S = NY - \mathbb{C} \ . \tag{5.6}$$

If we solve (5.2) for $z_0(x)$ and substitute it into (5.1), and use population constraints, then we can obtain that

$$S = \int_X \left\{ \int_X [B(z(x,y)) - (c+t(x,y))z(x,y)]f(y)\,\mathrm{d}y \right\} h(x)\,\mathrm{d}x - \bar{U}N - \mathrm{KM}$$
$$- \int_X R_a g(x)\,\mathrm{d}x - a \int_X [H(x)]^2 g(x)\,\mathrm{d}x \tag{5.7}$$

where $\bar{U} \equiv \bar{u} - Y$.

Hence, our problem now is to choose a plan $\{h(x), f(x), H(x), g(x), z(x,.); x \in X\}$ so as to maximize (5.7) subject to (5.3) to (5.5) (with additional nonnegativity constraints).

The term inside the braces of the objective function can be maximized with respect to each $z(x,y)$, independently of all other variables. And under the benefit function (2.4), using definitions (2.11) and (2.15), for each $x, y \in X$ we have that

$$\begin{aligned} \operatorname{Max} B[z(x,y)] - [c+t(x,y)]z(x,y) &= \beta \exp\{-\alpha[t(x,y)+c]\} \\ &\equiv \gamma(c) z^0(x,y) \ , \end{aligned} \tag{5.8}$$

where $z^0(x,y)$ is given by (2.11), and the optimal $z(x,y)$ is given by

$$\hat{z}(x,y) = \alpha\beta \exp\{-\alpha[t(x,y)+c]\} \equiv \alpha\gamma(c) z^0(x,y) \ . \tag{5.9}$$

Comparing (2.12) and (5.9), we can see that the optimal consumption pattern, $z(x,y)$, coincides with the equilibrium demand distribution under $p = c$. Substituting (5.8) into (5.7), we have

$$S = \gamma(c) \int_X \int_X h(x) z^0(x,y) f(y)\,\mathrm{d}y\,\mathrm{d}x - \int_X R_a g(x)\,\mathrm{d}x - a \int_X [H(x)]^2 g(x)\,\mathrm{d}x$$
$$- \bar{U}N - \mathrm{KM} \ . \tag{5.10}$$

Next, we shall choose density distribution $h(.)$, $f(.)$, $H(.)$ and $g(x)$ so as to maximize (5.10) subject to (5.2)–(5.5).

Let $\{\hat{h}(x), \hat{F}(x), \hat{H}(x), \hat{g}(x); x \in X\}$ be a solution to this maximization problem. To state the optimality conditions, let us introduce the following (shadow) bid rent functions associated with this allocation:

$$\hat{\psi}(x) \equiv \gamma(c) \int_X z^0(x, y) f(y)\,\mathrm{d}y - \hat{U} \ , \tag{5.11}$$

$$\hat{\Phi}(x) \equiv \gamma(c) \int_X z^0(y, x) h(y)\,\mathrm{d}y - \hat{\Pi} \ , \tag{5.12}$$

$$\hat{\Gamma}(x) \equiv aH(x)^2 \ , \tag{5.13}$$

where $\hat{U}$ and $\hat{\Pi}$ represent the (shadow) utility level and profit level. [Notice that unlike (3.2), we have $\gamma(c)$ in (5.12) instead of $\delta(c)$.] Then, applying optimal control theory, we can show that if the allocation, $\{\hat{h}(x), \hat{f}(x), \hat{H}(x), \hat{g}(x); x \in X\}$, is optimal, then there exists a set of multipliers, $\{\hat{R}(x), \hat{L}(x), \hat{U}, \hat{\Pi}; x \in X\}$, under which the same set of conditions with (3.4) to (3.15) is satisfied.[7] Here, of course, each function without '$\wedge$' must be replaced with the corresponding function with '$\wedge$'.[8] Hence, we can see that the only difference between the two set of conditions is the definition of bid rent functions. Namely, *if we replace* $\gamma(p)$ *with* $\gamma(c)$ *in* (3.1), *then we have* (5.11). Similarly, *if we replace* $\delta(p)$ *with* $\gamma(c)$ *in* (3.2), *we have* (5.12).

Hence, utilizing the previous results for equilibrium configurations, we can readily obtain optimal configurations. (To avoid repetition, hereafter we omit details of calculations.) As before, optimal configurations can be classified into two patterns, I and II. First, we examine the optimal land use pattern of type I. For this pattern, in the manner similar to the equilibrium solution, the optimal density function of firm $\hat{f}(x)$ in the MD can be obtained as

$$\hat{f}(x) = [(a\hat{V})^{1/2} \cos(\hat{D}x)]/[2a \cos(\hat{D}\hat{e})] \qquad x \in [-\hat{e}, \hat{e}] \ . \tag{5.14}$$

The optimal density function of household $\hat{h}(x)$ in the MD can be obtained as

$$\hat{h}(x) = [(a\hat{V})^{1/2} \cos(\hat{D}x)]/[2a \cos(\hat{D}\hat{e})] \qquad x \in [-\hat{e}, \hat{e}] \ . \tag{5.15}$$

The optimal density function of floor space $\hat{H}(x)$ in the MD can be obtained as follows:

$$\hat{H}(x) = [(a\hat{V})^{1/2} \cos(\hat{D}x)]/a \cos(\hat{D}\hat{e}), \qquad x \in [-\hat{e}, \hat{e}] \tag{5.16}$$

where

$$\hat{D} = [(2a\tau\hat{\gamma}(c)]^{1/2}/(2a) \ , \tag{5.17}$$

[7] This can be shown in a manner similar to Appendix 4 of Fujita (1986b).

[8] For example, condition (3.4) must now be replaced by the condition, $\hat{R}(x) = \mathrm{Max}\{\hat{\psi}(x), \hat{\Phi}(x), 0\}$.

$$\hat{V} = (1/2)\tilde{y}(c)\tau M(N-M)+2R_a \ , \tag{5.18}$$

$$\hat{e} = (1/\hat{D}) \tan^{-1} [(2\hat{D}aM)/(a\hat{V})^{1/2}] \ . \tag{5.19}$$

On the RA's, the optimal density function of floor space $\hat{H}(x)$ equals that of household density, $\hat{h}(x)$, and we have

$$\hat{H}(x) = \hat{h}(x) = [\tilde{y}(c)\tau M(\hat{d}-x)+2(aR_a)^{1/2}]/(2a) \qquad x \in [\hat{e}, \hat{d}] \ , \tag{5.20}$$

where the city boundary, $\hat{d}$, is given by

$$\hat{d} = \hat{e}+[2(a\hat{V})^{1/2}-(aR_a)^{1/2}]/[\tilde{y}(c)\tau M] \ . \tag{5.21}$$

Finally the (shadow) floor rent is given by

$$\hat{R}(x) = \begin{cases} 2a(2\hat{V})^{1/2}\cos(\hat{D}x)/\cos(\hat{D}\hat{e}) & x \in [-\hat{e}, \hat{e}] \\ \tilde{y}(c)\tau M(\hat{d}-x)+2(aR_a)^{1/2} & x \in [\hat{e}, \hat{d}] \ . \end{cases} \tag{5.22}$$

From (5.14) and (5.15), we can see that, at optimal, the *firms and households equally share the floor space* in the MD. Therefore, the following relation must hold in order to have pattern I:

$$N \geq M \ . \tag{5.23}$$

In Fujita (1988), it was shown that the optimal land use pattern coincides with the market equilibrium pattern under the monopolistic price of c-goods. To examine whether the same result holds in our model (with variable density), let us compare the optimal solution and the equilibrium solution under the monopolistic price $p_m \equiv c+(1/\alpha)$. For this, by setting $p = p_m \equiv c+(1/\alpha)$ in (2.16) we have

$$\gamma(p_m) = \delta(p_m) = (1/\alpha)\{\exp[-(1+\alpha c)]\} \ . \tag{5.24}$$

Hence, setting $p \equiv p_m$ in (4.15) to (4.17) and in (4.21) to (4.24) we have equilibrium distributions of households and firms under price p_m (for clarity, here all variables of this urban configuration will be marked superscript *). Setting $p = p_m \equiv c+(1/\alpha)$ in (4.15b) and (4.23), we have

$$D^* = \{[2a\tau\tilde{y}(p_m)^{1/2}\}/(2a) \ , \tag{5.25}$$

$$V^* = (1/2)\tilde{y}(p_m)\tau M(N-M)2R_a \ . \tag{5.26}$$

Note that $\gamma(c) > \gamma(p_m)$ and $\tilde{y}(c) > \tilde{y}(p_m)$. Hence, in comparison of (5.17) and (5.18) with (5.25) and (5.26), we have $\hat{D} > D^*$ and $\hat{V} > V^*$ which imply that

$$\hat{e} < e^*, \ \hat{d} < d^* \text{ and } (\hat{d}-\hat{e}) < (d^*-e^*) \ . \tag{5.27}$$

Therefore, for type I, we can conclude that *the equilibrium urban configuration under the monopolistic price is more dispersed than the optimal configuration.*

We can obtain the same conclusion for type II by provided that $M \geq N$. We can also readily see that the optimal land use pattern never coincides with the equilibrium land use pattern under any value of p.

6. Conclusion

In this paper, we have developed a monopolistic competition model of spatial agglomeration with variable density, and compared the equilibrium urban configurations with optimal configurations. As in Fujita (1988), both equilibrium configurations and optimal configurations take two types of mixed land use patterns depending on parameters. However, unlike Fujita (1988), we found that when the land-use density is a variable, the optimal land use pattern do not coincide with the equilibrium land use pattern under the monopolistic equilibrium price p_m. This difference can be explained as follows. When the land-use density is kept constant (as in Fujita 1988), the equality of $\gamma(p_m)$ and $\delta(p_m)$ as expressed by (5.24) is necessary and sufficient for an equilibrium land-use pattern to be an optimal pattern. However, when the land-use density is a variable (as in the present paper), this equality is necessary but not sufficient for an equilibrium pattern to be optimal. That is, under the monopolistic price p_m (which is higher than the socially optimal price, $\hat{p} = c$), the bid rent levels of households and firms (in the floor space market) are lower than those under the optimal price $\hat{p} = c$. This in turn makes the equilibrium floor density at the central area to be lower than the optimal floor density there. Therefore, we can conclude that the variable density model is not only more realistic but also provides qualitatively different results from the fixed density model.

To conclude this paper, we may note four possible extensions of our model. First, although we have considered spatial interactions between households and firms (that provide local consumer goods), we can develop similar models for studying spatial interactions among different types of industries. For example, if we replace the consumers with an export-good industry, and the consumer-service industry with a producer-service, then we can develop a model of spatial agglomeration due to product variety in producer services. As the export industry, we may consider the headquarters of multiregional and multinational firms (e.g., New York), high-technology firms (e.g., Silicon Valley), or (low-technology) manufacturing firm (e.g., old Pittsburgh). Therefore, with appropriate specification of industries, our model can explain a variety of spatial agglomeration phenomena. For an initial study in this direction, see Fujita (1990). Second, we have assumed that each household (and each firm) consumes a fixed amount of floor space. It is, however, more desirable to generalize it so that households and firms can choose optimal amounts of their floor space. Third, although (2.1) represents an additive utility function, it is desirable to replace it with a more general utility function (or production function) such as a CES function. Finally, in this paper we did not consider the role of households in providing labor to firms.

A more complete model of cities should conside both the consumption aspect and the labor-supply aspect of households.

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