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Distribution of functionals of a Ferguson–Dirichlet process over an n-dimensional ball

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ABSTRACT

The c-characteristic function has been shown to have properties similar to those of the Fourier transformation. We now give a new property of the c-characteristic function of the spherically symmetric distribution. With this property, we can easily determine whether a distribution is spherically symmetric. The exact probability density function of the random mean of a spherically symmetric Ferguson–Dirichlet process with parameter measure over an n-dimensional spherical surface and that over an n-dimensional ball are given. We further give the exact probability density function of the random mean of a Ferguson–Dirichlet process with parameter measure over an n-dimensional ellipsoidal surface and that over an n-dimensional ellipsoidal solid.

1. Introduction

The Ferguson–Dirichlet process was first introduced and studied by Ferguson [5]. Since then, the random functional of the Ferguson–Dirichlet process has drawn the attention of many researchers. Suppose we are interested in the density estimation. Many density estimators proposed in the literature are of the form of mixtures of densities. Consider the random mixture of the densities over an n-dimensional ball, which has the form

$$f(x) = \int_{\Omega} g(x, y)\, dU(y), \tag{1}$$

where g is a kernel density function and U is a Ferguson–Dirichlet process on the n-dimensional ball Ω. Under the quadratic type loss function, the Bayesian density estimator is then given by the expectation of the random density in (1). See [13] for detailed discussions. The random density (1) is one application for the random functional of the Ferguson–Dirichlet process, which shall be studied in this paper. In general, some other applications of the random functional are for quality control problems (see [3]), polymer chemistry problems, mathematical finance problems (see [4]), and others. More motivations and studies of the random functional of the Ferguson–Dirichlet process can be seen in [15] and references therein, among others.

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Directional data arise naturally in physical world. For example, the time of the day can be expressed in two dimensions, and wind direction can be expressed in three dimensions. Moreover, samples from several different fields, e.g., machine learning, bioinformatics, and data mining, are likely to contain high-dimensional (often with hundreds or thousands of dimensions) data that are also inherently directional in nature. Often such data are L_2 normalized so that it lies on the surface of a unit hypersphere. See [1] and references therein for detailed discussions and applications on high-dimensional spherical data.

In this paper, we shall first study the random functionals of the Ferguson–Dirichlet process over an n-dimensional ball and over its surface. We shall then extend them to those over an ellipsoidal solid in n-space and over its surface.

The organization of this paper is as follows. In Section 2, we give a new property of the c-characteristic function, which can solve many problems that are difficult to manage using the traditional characteristic function (see [9–11]), for a spherically symmetric distribution. This property can be used to easily determine whether a random vector (variable) has a spherically symmetric distribution. In addition, the c-characteristic function of the associated marginal distribution can be easily obtained from the c-characteristic function of the original joint distribution that is spherically symmetric.

In Section 3, we first give the c-characteristic function expression for any bounded functional of a Ferguson–Dirichlet process with parameter measure over Euclidean space. With this expression, we first show that the mean of the random function vector of a Ferguson–Dirichlet process over Euclidean space is the same as the mean of the function vector of the random variable corresponding to the Ferguson–Dirichlet process mean. We then show that the covariance of the random function vector of a Ferguson–Dirichlet process over Euclidean space is proportional to the covariance of the function vector of the random variable corresponding to the Ferguson–Dirichlet process mean. We further show that the random mean of the Ferguson–Dirichlet process with the usual Lebesgue parameter measure over an n-dimensional spherical surface and that over an n-dimensional ball both have spherically symmetric distributions. We then provide the exact probability density functions of these random means. These results generalize those given by Jiang [10], Jiang et al. [11], and Jiang and Kuo [12] over two or three dimensional distributions. Lastly, we further extend the results to the random mean of a spherically symmetric Ferguson–Dirichlet process with parameter measure over an n-dimensional ellipsoidal solid and that over an n-dimensional ellipsoidal surface. Finally, we give conclusions in Section 4.

2. Spherical properties of the c-characteristic function

First, we state the definition of the c-characteristic function.

Definition 2.1 (*Jiang et al. [11]*)**.** If $\boldsymbol{u} = (u_1, \ldots, u_L)'$ is a random vector on a subset of $A = [-a_1, a_1] \times \cdots \times [-a_L, a_L]$, its c-characteristic function is defined as

$$g(\boldsymbol{t}; \boldsymbol{u}, c) = \underset{\boldsymbol{u}}{E}[(1 - i\boldsymbol{t} \cdot \boldsymbol{u})^{-c}], \quad |\boldsymbol{t}| < |\boldsymbol{a}|^{-1},$$

where $c > 0$, $\boldsymbol{a} = (a_1, \ldots, a_L)'$, $\boldsymbol{t} = (t_1, \ldots, t_L)'$, $|\boldsymbol{t}| = \sqrt{\sum_{i=1}^{L} t_i^2}$, and $\boldsymbol{t} \cdot \boldsymbol{u} = \sum_{j=1}^{L} t_j u_j$, the inner product of $\boldsymbol{t}$ and $\boldsymbol{u}$.

Some important properties, e.g., uniqueness and convergence theorems, of the c-characteristic function can be seen in [11]. In the following, we give definitions of a spherically symmetric distribution and an ellipsoidally symmetric distribution.

Definition 2.2. A random vector $\boldsymbol{u}$ is said to be spherically symmetric with center $\boldsymbol{a}$ if $T(\boldsymbol{u} - \boldsymbol{a})$ has the same distribution for every orthogonal matrix T. In particular, we say that $\boldsymbol{u}$ has a spherically symmetric distribution if it is spherically symmetric with center $\boldsymbol{0}$. A random vector $\boldsymbol{v}$ is said to be ellipsoidally symmetric with center $\boldsymbol{b}$ if there exists an invertible matrix P such that $\boldsymbol{v} = P(\boldsymbol{u} - \boldsymbol{a}) + \boldsymbol{b}$ where $\boldsymbol{u}$ has a spherically symmetric distribution with center $\boldsymbol{a}$.

When the c-characteristic function $g(\boldsymbol{t}; \boldsymbol{u}, c)$ is known, the following theorem provides an easy method for determining whether $\boldsymbol{u}$ has a spherically symmetric distribution.

Theorem 2.1. *Suppose that $\boldsymbol{u}$ is an L-dimensional random vector. Then $\boldsymbol{u}$ has a spherically symmetric distribution if and only if the associated c-characteristic function $g(\boldsymbol{t}; \boldsymbol{u}, c)$ is a function of $|\boldsymbol{t}|$ and c, only.*

Proof. According to Definition 2.2, when $\boldsymbol{u}$ has a spherically symmetric distribution, its probability density function $f(\boldsymbol{u})$ is a function of $|\boldsymbol{u}| = r$ only, and we can write $f(\boldsymbol{u})$ as $p(r)$. The volume element is $d\boldsymbol{u} = r^{L-1}\,dr\,dS$ where S is the surface of the unit sphere in L-dimensions. Also, $\boldsymbol{u} \cdot \boldsymbol{t} = r|\boldsymbol{t}|\cos\theta$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{t}$. Hence, the c-characteristic function of $\boldsymbol{u}$ can be expressed as

$$g(\boldsymbol{t}; \boldsymbol{u}, c) = \int r^{L-1} p(r) \int (1 - ir|\boldsymbol{t}|\cos\theta)^{-c}\, dS\, dr.$$

The integration of the inner integral is with respect to S, whose value is independent of the direction of $\boldsymbol{t}$. Hence, the integrand of the outer integral is a function of $r|\boldsymbol{t}|$ and c, only. Therefore, $g(\boldsymbol{t}; \boldsymbol{u}, c)$ is a function of $|\boldsymbol{t}|$ and c, only.

Conversely, let T be any $L \times L$ orthogonal matrix. Then, by Theorem 4.1 of Jiang et al. [11], we have $g(\boldsymbol{t}; T\boldsymbol{u}, c) = g(T'\boldsymbol{t}; \boldsymbol{u}, c)$ where T' is the transpose of T. Since $g(\boldsymbol{t}; \boldsymbol{u}, c)$ is a function of $|\boldsymbol{t}|$ and c, we have that $g(T'\boldsymbol{t}; \boldsymbol{u}, c)$ is a function of $|T'\boldsymbol{t}| = |\boldsymbol{t}|$ and c. That is, $T\boldsymbol{u}$ and $\boldsymbol{u}$ have the same c-characteristic function. By Lemma 2.2 of Jiang et al. [11], $T\boldsymbol{u}$ and $\boldsymbol{u}$ have the same distribution for any orthogonal matrix T. Therefore, $\boldsymbol{u}$ has a spherically symmetric distribution by Definition 2.2. □

We will call $\boldsymbol{v}$ (k-dimensional vector) a margin of $\boldsymbol{u}$ (L-dimensional vector) and the distribution of $\boldsymbol{v}$ a marginal distribution of the distribution of $\boldsymbol{u}$ if $k \le L$ and $\boldsymbol{v}$ is a linear projection of $\boldsymbol{u}$, that is, $\boldsymbol{v} = D\boldsymbol{u}$, for some matrix D of size $k \times L$, where $DD' = \mathbb{I}_k$ (the identity matrix of dimension k). The following corollary shows that a spherically symmetric distribution and its marginal distributions have the same c-characteristic function expression and so the c-characteristic function in a lower dimension can easily be derived from that in a higher dimension.

Corollary 2.1. *Let $\boldsymbol{u}$ be a random vector having a spherically symmetric distribution and let $\boldsymbol{v}$ be any margin of $\boldsymbol{u}$. Suppose that $g(\boldsymbol{t}; \boldsymbol{u}, c) = g^*(|\boldsymbol{t}|; c)$ for some function g^*. Then $g(\boldsymbol{s}; \boldsymbol{v}, c) = g^*(|\boldsymbol{s}|; c)$.*

Proof. There exists a $k \times L$ matrix D such that $\boldsymbol{v} = D\boldsymbol{u}$ and $DD' = \mathbb{I}_k$. By Theorem 4.1 of Jiang et al. [11], we have

$$g(\boldsymbol{s}; \boldsymbol{v}, c) = g(\boldsymbol{s}; D\boldsymbol{u}, c) = g(D'\boldsymbol{s}; \boldsymbol{u}, c) = g^*(|D'\boldsymbol{s}|; c) = g^*(|\boldsymbol{s}|; c). \quad \square$$

With Theorem 2.1 and Corollary 2.1, we see that any marginal distribution of a spherically symmetric distribution is also spherically symmetric.

3. Random functionals of a Ferguson–Dirichlet process with parameter measure over an n-dimensional ball and its surface

Let μ be a finite non-null measure on $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the σ-field of Borel subsets of Euclidean space Ω; and let U be a stochastic process indexed by elements of $\mathcal{B}$. We say U is a Ferguson–Dirichlet process with parameter μ, denoted by $U \sim D(\mu)$ on Ω, if for every finite measurable partition $\{B_1, \ldots, B_m\}$ of Ω (i.e., the B_i's are measurable, disjoint, and $\bigcup_{i=1}^m B_i = \Omega$), the random vector $(U(B_1), \ldots, U(B_m))$ has a Dirichlet distribution (see [16, Section 7.7]) with parameter vector $(\mu(B_1), \ldots, \mu(B_m))$, where $\sum_{i=1}^m U(B_i) = 1$.

Let $\boldsymbol{X}$ be a N-dimensional random vector defined as

$$\boldsymbol{X} = \int_\Omega \boldsymbol{\ell}(\boldsymbol{y})\, dU(\boldsymbol{y}) = \left(\int_\Omega \ell_1(\boldsymbol{y})\, dU(\boldsymbol{y}), \ldots, \int_\Omega \ell_N(\boldsymbol{y})\, dU(\boldsymbol{y})\right)', \tag{2}$$

where $U \sim D(\mu)$ on Ω, $\mu(\Omega) = c$, $\boldsymbol{y} = (y_1, \ldots, y_L)'$, $\boldsymbol{\ell}(\boldsymbol{y}) = (\ell_1(\boldsymbol{y}), \ldots, \ell_N(\boldsymbol{y}))'$, and the $\ell_n(\boldsymbol{y})$'s are (bounded) measurable real-valued functions defined on Ω. The following lemma, which is proven in Appendix A, gives the c-characteristic function expression for $\boldsymbol{X}$.

Lemma 3.1. *The c-characteristic function of $\boldsymbol{X}$, as in Eq. (2), can be expressed by*

$$g(\boldsymbol{t}; \boldsymbol{X}, c) = \exp\left[-\int_\Omega \ln\left(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}(\boldsymbol{y})\right)\, d\mu(\boldsymbol{y})\right],$$

where $\boldsymbol{t} = (t_1, \ldots, t_N)'$.

The following corollary can easily be obtained by applying the above Lemma 3.1 and Theorem 2.5 of Jiang et al. [11].

Corollary 3.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_N)'$ and $\boldsymbol{\ell}(\boldsymbol{y}) = (\ell_1(\boldsymbol{y}), \ldots, \ell_N(\boldsymbol{y}))'$, which are defined in Eq. (2). Then, for $1 \le n \le N$,*

$$E(X_n) = \frac{1}{c}\int_\Omega \ell_n(\boldsymbol{y})\, d\mu(\boldsymbol{y}),$$

and, for $1 \le n, m \le N$,

$$\mathrm{Cov}(X_n, X_m) = \frac{1}{c+1}\left[\frac{1}{c}\int_\Omega \ell_n(\boldsymbol{y})\ell_m(\boldsymbol{y})\, d\mu(\boldsymbol{y}) - \left(\frac{1}{c}\int_\Omega \ell_n(\boldsymbol{y})\, d\mu(\boldsymbol{y})\right)\left(\frac{1}{c}\int_\Omega \ell_m(\boldsymbol{y})\, d\mu(\boldsymbol{y})\right)\right].$$

The following corollary is obtained immediately from Corollary 3.1.

Corollary 3.2. *Let $\boldsymbol{Y}$ be a random vector associated with the probability measure μ/c. Then the mean vector and the variance–covariance matrix of $\boldsymbol{X}$, Eq. (2), can be expressed as $E(\boldsymbol{X}) = E(\boldsymbol{\ell}(\boldsymbol{Y}))$ and $\mathrm{Cov}(\boldsymbol{X}) = \mathrm{Cov}(\boldsymbol{\ell}(\boldsymbol{Y}))/(c+1)$, respectively.*

In this article, we want to study the distributions of the random functionals,

$$\widetilde{\boldsymbol{S}}_{n,r,c} = \int_{S_{n,r}} \boldsymbol{y}\, dU_{n,r,c}(\boldsymbol{y}), \quad U_{n,r,c} \sim D(\mu_{n,r,c}), \tag{3}$$

and

$$\widetilde{\boldsymbol{B}}_{n,r,c} = \int_{B_{n,r}} \boldsymbol{y}\, dV_{n,r,c}(\boldsymbol{y}), \quad V_{n,r,c} \sim D(\nu_{n,r,c}) \tag{4}$$

where n is any positive integer, $S_{n,r} = \{\boldsymbol{y} \in \mathbb{R}^n : |\boldsymbol{y}| = r\}$ is the spherical surface of the n-dimensional ball $B_{n,r} = \{\boldsymbol{y} \in \mathbb{R}^n : |\boldsymbol{y}| \le r\}$, $r > 0$, $\mu_{n,r,c}$ is the usual Lebesgue measure (i.e., usual rotation-invariant measure) on $S_{n,r}$ with total measure c, and $\nu_{n,r,c}$ is the usual Lebesgue measure on $B_{n,r}$ with total measure c. Specifically, $d\mu_{n,r,c}(\boldsymbol{y}) = c\Gamma(\frac{n}{2})/(2r^{n-1}\pi^{n/2})d\boldsymbol{y}$ and $d\nu_{n,r,c}(\boldsymbol{y}) = cn\Gamma(\frac{n}{2})/(2r^n\pi^{n/2})d\boldsymbol{y}$. In particular, it can be shown that $\widetilde{\boldsymbol{S}}_{n,r,c} = r\widetilde{\boldsymbol{S}}_{n,1,c}$ and $\widetilde{\boldsymbol{B}}_{n,r,c} = r\widetilde{\boldsymbol{B}}_{n,1,c}$ by Lemma 3.3, which will be given later.

Just as the parameter measures $\mu_{n,r,c}$ and $\nu_{n,r,c}$ are spherically symmetric, the Ferguson–Dirichlet processes $U_{n,r,c} \sim D(\mu_{n,r,c})$ and $V_{n,r,c} \sim D(\nu_{n,r,c})$ will be said to be spherically symmetric.

First, we give the c-characteristic function expressions for $\widetilde{\boldsymbol{S}}_{n,r,c}$ and $\widetilde{\boldsymbol{B}}_{n,r,c}$.

Theorem 3.1. *The c-characteristic function of $\widetilde{\boldsymbol{S}}_{n,r,c}$ in Eq. (3) and that of $\widetilde{\boldsymbol{B}}_{n,r,c}$ in Eq. (4) can be expressed as*

$$g(\boldsymbol{t}; \widetilde{\boldsymbol{S}}_{n,r,c}, c) = \exp\left\{\sum_{k=1}^{\infty} \frac{c\left(\frac{1}{2}, k\right)}{2k\left(\frac{n}{2}, k\right)}[-r^2(t_1^2 + \cdots + t_n^2)]^k\right\},$$

and

$$g(\boldsymbol{t}; \widetilde{\boldsymbol{B}}_{n,r,c}, c) = \exp\left\{\sum_{k=1}^{\infty} \frac{c\left(\frac{1}{2}, k\right)}{2k\left(\frac{n}{2}+1, k\right)}[-r^2(t_1^2 + \cdots + t_n^2)]^k\right\},$$

respectively, where (a, k) denotes the Appell's notation that is defined as $(a, k) = a(a + 1)\cdots(a + k - 1)$.

The proof of Theorem 3.1 is given in Appendix B. Theorem 3.1 shows that the c-characteristic function of $\widetilde{\boldsymbol{S}}_{n,r,c}$ is a function of $|\boldsymbol{t}|$ and c, only. In accordance with Theorem 2.1, $\widetilde{\boldsymbol{S}}_{n,r,c}$ has a spherically symmetric distribution. Moreover, by Corollary 2.1, the c-characteristic function of any one-dimensional margin of $\widetilde{\boldsymbol{S}}_{n,r,c}$, say $\widetilde{S}_{n,r,c}$, is expressed as

$$g(t; \widetilde{S}_{n,r,c}, c) = \exp\left\{\sum_{k=1}^{\infty} \frac{c\left(\frac{1}{2}, k\right)}{2k\left(\frac{n}{2}, k\right)}(-r^2t^2)^k\right\}. \tag{5}$$

Similarly, $\widetilde{\boldsymbol{B}}_{n,r,c}$ also has a spherical symmetric distribution and its one-dimensional margin $\widetilde{B}_{n,r,c}$ is

$$g(t; \widetilde{B}_{n,r,c}, c) = \exp\left\{\sum_{k=1}^{\infty} \frac{c\left(\frac{1}{2}, k\right)}{2k\left(\frac{n}{2}+1, k\right)}(-r^2t^2)^k\right\}. \tag{6}$$

By comparing Eqs. (5) and (6), it can be seen that the c-characteristic function of $\widetilde{S}_{n,r,c}$ and $\widetilde{B}_{n,r,c}$ are related to each other in the sense of $\widetilde{S}_{n+2,r,c} = \widetilde{B}_{n,r,c}$.

Our aim now is to find the probability density functions of $\widetilde{\boldsymbol{S}}_{n,r,c}$ and $\widetilde{\boldsymbol{B}}_{n,r,c}$. Lord [14] demonstrated that a spherically symmetric distribution can be determined by its marginal distribution. Therefore, if the probability density function of the marginal random variable $\widetilde{S}_{n,r,c}$ (or $\widetilde{B}_{n,r,c}$) is known, then we can obtain the probability density function of the random vector $\widetilde{\boldsymbol{S}}_{n,r,c}$ (or $\widetilde{\boldsymbol{B}}_{n,r,c}$). Before giving the probability density functions of $\widetilde{S}_{n,r,c}$ and $\widetilde{B}_{n,r,c}$, we first show that an interesting random mean of a Ferguson–Dirichlet process has the same c-characteristic function of $\widetilde{S}_{n,r,c}$ in the following lemma.

Lemma 3.2. *Let n be any positive integer and $c > 0$. Suppose that $\widetilde{R}_{n,r,c} = \int_{-r}^{r} y\, dW_{n,r,c}(y)$ where $W_{n,r,c} \sim D(\alpha_{n,r,c})$ and let $\alpha_{n,r,c}$ (having total measure c) be the measure on $\{-r, r\}$ with $\alpha_{1,r,c}(\{-r\}) = \alpha_{1,r,c}(\{r\}) = c/2$ if $n = 1$, and be the measure on the open interval $(-r, r)$ with density*

$$d\alpha_{n,r,c}(y) = \frac{c(r^2 - y^2)^{(n-3)/2}}{B\left(\frac{1}{2}, \frac{n-1}{2}\right) r^{n-2}}\, dy$$

if $n > 1$ where $B(\alpha, \beta)$ denotes the complete beta function. Then the c-characteristic function of $\widetilde{R}_{n,r,c}$ can be expressed as

$$g(t; \widetilde{R}_{n,r,c}, c) = \exp\left\{\sum_{k=1}^{\infty} \frac{c\left(\frac{1}{2}, k\right)}{2k\left(\frac{n}{2}, k\right)}(-r^2t^2)^k\right\}.$$

In particular, $g(t; \widetilde{R}_{1,r,c}, c) = (1 + r^2t^2)^{-c/2}$.

Proof. By Lemma 3.1, the c-characteristic function of $\widetilde{R}_{1,r,c}$ is

$$g(t;\widetilde{R}_{1,r,c},c)=\exp\left\{-\frac{c}{2}\ln(1-it(-r))-\frac{c}{2}\ln(1-itr)\right\}=(1+r^2t^2)^{-c/2}.$$

By Lemma 3.1 again, the c-characteristic function of $\widetilde{R}_{n,r,c}$, $n>1$, is

$$\begin{aligned}g(t;\widetilde{R}_{n,r,c},c)&=\exp\left\{-\int_{-r}^{r}\ln(1-ity)\frac{c(r^2-y^2)^{(n-3)/2}}{B\left(\frac{1}{2},\frac{n-1}{2}\right)r^{n-2}}\,dy\right\}\\&=\exp\left\{\frac{c}{B\left(\frac{1}{2},\frac{n-1}{2}\right)r^{n-2}}\sum_{k=1}^{\infty}\frac{i^kt^k}{k}\int_{-r}^{r}y^k(r^2-y^2)^{(n-3)/2}\,dy\right\}\\&=\exp\left\{\sum_{k=1}^{\infty}\frac{c\left(\frac{1}{2},k\right)}{2k\left(\frac{n}{2},k\right)}(-r^2t^2)^k\right\}.\end{aligned}$$

The last identity can be obtained by Eq. 8.380.1 of Gradshteyn and Ryzhik [6, p. 898]. □

By Eq. (5) and Lemma 3.2, the c-characteristic functions of $\widetilde{S}_{n,r,c}$ and $\widetilde{R}_{n,r,c}$ are the same. Similarly, by Eq. (6) and Lemma 3.2, $\widetilde{B}_{n,r,c}$ and $\widetilde{R}_{n+2,r,c}$ have the same c-characteristic function. The following theorem can then be obtained by Lemma 2.2 of Jiang et al. [11].

Theorem 3.2. *$\widetilde{S}_{n,r,c}$ and $\widetilde{B}_{n,r,c}$ have the same distribution as $\widetilde{R}_{n,r,c}$ and $\widetilde{R}_{n+2,r,c}$, respectively.*

Applying Proposition 9 of Regazzini et al. [15], we can derive the probability density function of $\widetilde{R}_{n,r,c}$, denoted by $f(x;n,r,c)$. We consider the case of $c=1$ in the following example.

Example 1. Let $f(x;n,r,1)$ be the probability density function of $\widetilde{R}_{n,r,1}$. By Proposition 9(iii) of Regazzini et al. [15], we have

(i) $f(x;1,r,1)=\frac{1}{\pi\sqrt{r^2-x^2}}$, that is, $\widetilde{R}_{1,r,1}$ is distributed as $-ru_1+ru_2$ where (u_1,u_2) has a Dirichlet distribution with parameter vector $(1/2,1/2)$, and $\widetilde{R}^2_{1,r,1}/r^2$ is distributed as the beta distribution Beta$(1/2,1/2)$;

(ii) $f(x;2,r,1)=\frac{2\sqrt{r^2-x^2}}{r^2\pi}$, that is, $\widetilde{R}^2_{2,r,1}/r^2$ is distributed as Beta$(1/2,3/2)$;

(iii) $f(x;3,r,1)=\frac{e}{\pi}(r+x)^{-(r+x)/(2r)}(r-x)^{-(r-x)/(2r)}\cos\frac{\pi x}{2r}$;

(iv) $f(x;4,r,1)=\frac{2}{r\pi}e^{1/2-x^2/r^2}\sin\left(\frac{x\sqrt{r^2-x^2}}{r^2}+2\arcsin\sqrt{\frac{r+x}{2r}}\right)$.

More generally, we have

(v) when n is an odd integer greater than 3,

$$\begin{aligned}f(x;n,r,1)=\frac{1}{r\pi}\exp&\left\{\frac{-1}{B\left(\frac{1}{2},\frac{n-1}{2}\right)}\sum_{k=0}^{(n-3)/2}\binom{\frac{n-3}{2}}{k}\frac{(-1)^k}{(2k+1)}\right.\\&\times\left[\left(1+\frac{x^{2k+1}}{r^{2k+1}}\right)\ln\left(1+\frac{x}{r}\right)+\left(1-\frac{x^{2k+1}}{r^{2k+1}}\right)\ln\left(1-\frac{x}{r}\right)\right.\\&\left.\left.-2\sum_{m=0}^{k}\frac{x^{2k-2m}}{(2m+1)r^{2k-2m}}\right]\right\}\sin\left(\int_{-r}^{x}\pi\,d\alpha_{n,r,1}(y)\right);\end{aligned}$$

(vi) when n is an even integer greater than 4,

$$f(x;n,r,1)=\frac{2}{r\pi}\exp\left[\frac{2^{3-n}\pi}{B\left(\frac{1}{2},\frac{n-1}{2}\right)}\sum_{k=0}^{(n-4)/2}\binom{n-2}{k}\frac{\cos\left[(n-2-2k)\arcsin\frac{x}{r}\right]}{n-2-2k}\right]\sin\left(\int_{-r}^{x}\pi\,d\alpha_{n,r,1}(y)\right).$$

All above probability density functions have the same support $(-r,r)$. □

Using Eq. (30) of Lord [14] and the probability density function $f(x;n,r,c)$ of $\widetilde{R}_{n,r,c}$, we have the following theorem.

Theorem 3.3. *Let $h_{\widetilde{\mathbf{S}}}(\boldsymbol{x};n,r,c)$ and $h_{\widetilde{\mathbf{B}}}(\boldsymbol{x};n,r,c)$ be the probability density functions of $\widetilde{\boldsymbol{S}}_{n,r,c}$ and $\widetilde{\boldsymbol{B}}_{n,r,c}$, respectively, where $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $n\geq 2$.*

(i) *When n is odd,*

$$h_{\widetilde{\boldsymbol{S}}}(\boldsymbol{x}; n, r, c) = \left(\frac{-1}{2\pi t}\frac{d}{dt}\right)^m f(t; n, r, c), \quad |\boldsymbol{x}| < r,$$

and

$$h_{\widetilde{\boldsymbol{B}}}(\boldsymbol{x}; n, r, c) = \left(\frac{-1}{2\pi t}\frac{d}{dt}\right)^m f(t; n+2, r, c), \quad |\boldsymbol{x}| < r,$$

with $t = \sqrt{x_1^2 + \cdots + x_n^2}$ *and* $m = (n-1)/2$*;*

(ii) *when n is even,*

$$h_{\widetilde{\boldsymbol{S}}}(\boldsymbol{x}; n, r, c) = \left(\frac{-1}{2\pi s}\frac{d}{ds}\right)^m \left(\frac{-1}{\pi}\int_s^r \frac{f'(t; n, r, c)}{\sqrt{t^2 - s^2}}\, dt\right), \quad |\boldsymbol{x}| < r,$$

and

$$h_{\widetilde{\boldsymbol{B}}}(\boldsymbol{x}; n, r, c) = \left(\frac{-1}{2\pi s}\frac{d}{ds}\right)^m \left(\frac{-1}{\pi}\int_s^r \frac{f'(t; n+2, r, c)}{\sqrt{t^2 - s^2}}\, dt\right), \quad |\boldsymbol{x}| < r,$$

with $s = \sqrt{x_1^2 + \cdots + x_n^2}$ *and* $m = (n-2)/2$*,*

where

$$\left(\frac{-1}{2\pi t}\frac{d}{dt}\right)^m f(t; n, r, c) = \frac{-1}{2\pi t}\frac{d}{dt}\left[\left(\frac{-1}{2\pi t}\frac{d}{dt}\right)^{m-1} f(t; n, r, c)\right]$$

for $m \geq 1$*, and* $\left(\frac{-1}{2\pi t}\frac{d}{dt}\right)^0 \equiv 1$*.*

Example 2 (Example 1 *Continue*)**.** The probability density functions of $\widetilde{\boldsymbol{S}}_{n,r,1}$ and $\widetilde{\boldsymbol{B}}_{n,r,1}$, for $n = 1$, $n = 2$ and $n = 3$, are as follows.

(i) $h_{\widetilde{\boldsymbol{S}}}(x_1; 1, r, 1) = \frac{1}{\pi\sqrt{r^2 - x_1^2}}, \ -r < x_1 < r$;

(ii) $h_{\widetilde{\boldsymbol{S}}}(x_1, x_2; 2, r, 1) = \frac{1}{r^2\pi}, \ x_1^2 + x_2^2 < r^2$, so $\widetilde{\boldsymbol{S}}_{2,r,1}$ has a uniform distribution on the disk of radius r;

(iii) $h_{\widetilde{\boldsymbol{S}}}(x_1, x_2, x_3; 3, r, 1) = \frac{e}{4r\pi^2 s}(r+s)^{-(r+s)/(2r)}(r-s)^{-(r-s)/(2r)}\left(\cos\frac{\pi s}{2r}\ln\frac{r+s}{r-s} + \pi\sin\frac{\pi s}{2r}\right)$, where $s = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $x_1^2 + x_2^2 + x_3^2 < r^2$;

(iv) $h_{\widetilde{\boldsymbol{B}}}(x_1; 1, r, 1) = \frac{e}{\pi}(r+x_1)^{-(r+x_1)/(2r)}(r-x_1)^{-(r-x_1)/(2r)}\cos\frac{\pi x_1}{2r}$, for $-r < x_1 < r$;

(v) $h_{\widetilde{\boldsymbol{B}}}(x_1, x_2; 2, r, 1) = \frac{4}{r^2\pi^2}\int_{\sqrt{x_1^2+x_2^2}}^r \frac{e^{1/2 - t^2/r^2}}{\sqrt{t^2 - x_1^2 - x_2^2}}\sin\left(\frac{t\sqrt{r^2-t^2}}{r^2} + 2\arcsin\sqrt{\frac{r+t}{2r}} - \arcsin\frac{\sqrt{r^2-t^2}}{r}\right) dt, \ x_1^2 + x_2^2 < r^2$;

(vi) $h_{\widetilde{\boldsymbol{B}}}(x_1, x_2, x_3; 3, r, 1) = \frac{-3e^{4/3 - t^2/(2r^2)}}{8tr^3\pi^2}(r+t)^{-(2r-t)(r+t)^2/(4r^3)}(r-t)^{-(r-t)^2(2r+t)/(4r^3)}\left[\pi(r^2 - t^2)\cos\frac{\pi(2r-t)(r+t)^2}{4r^3} + \left(-2rt + (r^2 - t^2)\ln\frac{r-t}{r+t}\right)\sin\frac{\pi(2r-t)(r+t)^2}{4r^3}\right]$, where $t = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $x_1^2 + x_2^2 + x_3^2 < r^2$. □

Notice that when $r = 1$, the probability density functions from Example 2(ii) and from Example 2(iii) are consistent with those given by Jiang [10] and by Jiang and Kuo [12], respectively.

Finally, we extend Theorem 3.3 to an n-dimensional ellipsoidal surface and to an n-dimensional ellipsoidal solid. Before giving the theorem, we need the following lemma which may be proved by applying Eq. (1.1) of Hjort and Ongaro [8] twice.

Lemma 3.3. *Let* $U_1 \sim D(\mu)$ *on* Ω_1 *and* $U_2 \sim D(\mu \circ h^{-1} \circ g)$ *on* Ω_2 *where* $\Omega_2 = (g^{-1} \circ h)(\Omega_1)$ *and h and g are measurable functions defined on* Ω_1 *and* Ω_2*, respectively. Suppose that* $\zeta_1 = \int_{\Omega_1} h(y)\, dU_1(y)$ *and* $\zeta_2 = \int_{\Omega_2} g(y)\, dU_2(y)$*. Then* $\zeta_1 = \zeta_2$*.*

Let $S_n^E = \{P\boldsymbol{x} + \boldsymbol{b} \mid \boldsymbol{x} \in S_{n,1}\}$ be the surface of the n-dimensional ellipsoidal solid with center $\boldsymbol{b}$, $B_n^E = \{P\boldsymbol{x} + \boldsymbol{b} \mid \boldsymbol{x} \in B_{n,1}\}$, where P is an invertible matrix and $n \geq 2$. Suppose that $U_{n,c}^E \sim D(\mu_{n,c}^E)$ on S_n^E and $V_{n,c}^E \sim D(\nu_{n,c}^E)$ on B_n^E, where $\mu_{n,c}^E$ and $\nu_{n,c}^E$ are the regular Lebesgue measures having total measure c on S_n^E and B_n^E, respectively.

Theorem 3.4. *Let* $h_{\widetilde{\boldsymbol{S}}_{n,c}^E}(\boldsymbol{x})$ *and* $h_{\widetilde{\boldsymbol{B}}_{n,c}^E}(\boldsymbol{x})$ *denote the probability density functions of* $\widetilde{\boldsymbol{S}}_{n,c}^E = \int_{S_n^E} \boldsymbol{y}\, dU_{n,c}^E(\boldsymbol{y})$ *and* $\widetilde{\boldsymbol{B}}_{n,c}^E = \int_{B_n^E} \boldsymbol{y}\, dV_{n,c}^E(\boldsymbol{y})$*, respectively. Then,*

$$h_{\widetilde{\boldsymbol{S}}_{n,c}^E}(\boldsymbol{x}) = |\det(P)|^{-1} h_{\widetilde{\boldsymbol{S}}}(P^{-1}(\boldsymbol{x} - \boldsymbol{b}); n, 1, c), \quad \boldsymbol{x} \in S_n^E,$$

and

$$h_{\widetilde{\boldsymbol{B}}^E_{n,c}}(\boldsymbol{x}) = |\det(P)|^{-1} h_{\widetilde{\boldsymbol{B}}}(P^{-1}(\boldsymbol{x}-\boldsymbol{b}); n, 1, c), \quad \boldsymbol{x} \in B^E_n,$$

where $h_{\widetilde{\boldsymbol{S}}}$ *and* $h_{\widetilde{\boldsymbol{B}}}$ *are given in Theorem 3.3.*

Proof. By Lemma 3.3, we have

$$\widetilde{\boldsymbol{S}}^E_{n,c} = \int_{S^E_n} \boldsymbol{y}\, dU^E_{n,c}(\boldsymbol{y}) = P\int_{S_{n,1}} \boldsymbol{y}\, dU_{n,1,c}(\boldsymbol{y}) + \boldsymbol{b} = P\widetilde{\boldsymbol{S}}_{n,1,c} + \boldsymbol{b}.$$

Similarly, $\widetilde{\boldsymbol{B}}^E_{n,c} = P\widetilde{\boldsymbol{B}}_{n,1,c} + \boldsymbol{b}$. □

The following corollary is a special case of Theorem 3.4 when P is an invertible diagonal matrix and $\boldsymbol{b}$ is the zero vector.

Corollary 3.3. *Let* $P = \operatorname{diag}(a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = \boldsymbol{0}$*, where each* $a_j \neq 0$ *and* $n \geq 2$*. Then,*

$$h_{\widetilde{\boldsymbol{S}}^E_{n,c}}(x_1, \ldots, x_n) = \frac{h_{\widetilde{\boldsymbol{S}}}(x_1/a_1, \ldots, x_n/a_n; n, 1, c)}{\prod_{j=1}^{n} |a_j|}, \qquad \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} < 1,$$

and

$$h_{\widetilde{\boldsymbol{B}}^E_{n,c}}(x_1, \ldots, x_n) = \frac{h_{\widetilde{\boldsymbol{B}}}(x_1/a_1, \ldots, x_n/a_n; n, 1, c)}{\prod_{j=1}^{n} |a_j|}, \qquad \frac{x_1^2}{a_1^2} + \cdots + \frac{x_n^2}{a_n^2} < 1.$$

We provide some exact probability density function expressions for $h_{\widetilde{\boldsymbol{S}}^E_{n,1}}$ and $h_{\widetilde{\boldsymbol{B}}^E_{n,1}}$ in the next example by using Corollary 3.3 and Example 2. It will be seen that the probability density function (i) of the next example is the same as that given by Theorem 6.3 of Jiang et al. [11] when $c = 1$.

Example 3. (i) $h_{\widetilde{\boldsymbol{S}}^E_{2,1}}(x_1, x_2) = \frac{1}{a_1 a_2 \pi}$, $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} < 1$, so $\widetilde{\boldsymbol{S}}^E_{2,1}$ has a uniform distribution on the central elliptical region with respective horizontal and vertical axes a_1, a_2;

(ii) $h_{\widetilde{\boldsymbol{S}}^E_{3,1}}(x_1, x_2, x_3) = \frac{e}{4a_1a_2a_3\pi^2 s}(1+s)^{-\frac{1+s}{2}}(1-s)^{-\frac{1-s}{2}}\left(\cos\frac{\pi s}{2}\ln\frac{1+s}{1-s} + \pi\sin\frac{\pi s}{2}\right)$, where $s = \sqrt{x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2}$ and $s < 1$;

(iii) $h_{\widetilde{\boldsymbol{B}}^E_{2,1}}(x_1, x_2) = \frac{4}{a_1a_2\pi^2}\int_{\sqrt{x_1^2/a_1^2+x_2^2/a_2^2}}^{1} \frac{e^{1/2-t^2}}{\sqrt{t^2-x_1^2/a_1^2-x_2^2/a_2^2}} \sin\left(t\sqrt{1-t^2} + 2\arcsin\sqrt{\frac{1+t}{2}} - \arcsin\sqrt{1-t^2}\right) dt$, $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} < 1$;

(iv) $h_{\widetilde{\boldsymbol{B}}^E_{3,1}}(x_1, x_2, x_3) = \frac{-3e^{4/3-t^2/2}}{8ta_1a_2a_3\pi^2}(1+t)^{-(2-t)(1+t)^2/4}(1-t)^{-(1-t)^2(2+t)/4}\left[\pi(1-t^2)\cos\frac{\pi(2-t)(1+t)^2}{4} + \left(-2t + (1-t^2)\ln\frac{1-t}{1+t}\right)\sin\frac{\pi(2-t)(1+t)^2}{4}\right]$, where $t = \sqrt{x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2}$ and $t < 1$.

4. Conclusions

Through the c-characteristic function, we have given a new approach for studying spherically symmetric distributions. The c-characteristic function expression of any functional of any Ferguson–Dirichlet process is given. With this expression, we can easily obtain first two moments of any random functional. In addition, we give the exact n-dimensional probability density function of the random mean of a symmetric Ferguson–Dirichlet process with parameter measure over any spherical surface, spherical solid, ellipsoidal surface, and ellipsoidal solid.

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Appendix A. Proof of Lemma 3.1

For any $k \geq 2$, let $\{B_{k1}, \ldots, B_{kk}\}$ be a partition of Ω. Then $(U(B_{k1}), \ldots, U(B_{kk}))$ follows a Dirichlet distribution with parameter $(\mu(B_{k1}), \ldots, \mu(B_{kk}))$. So, $\sum_{j=1}^{k} U(B_{kj}) = 1$ and $\sum_{j=1}^{k} \mu(B_{kj}) = c$, for all $k \geq 2$. Define $\boldsymbol{\ell}_k(\boldsymbol{y}) = \sum_{j=1}^{k} \boldsymbol{\ell}(b_{kj})\delta_{\boldsymbol{y}}(B_{kj})$

and $\boldsymbol{X}_k = \int_\Omega \boldsymbol{\ell}_k(\boldsymbol{y})\, dU(\boldsymbol{y})$, where $b_{kj} \in B_{kj}$, and where $\delta_{\boldsymbol{y}}(B_{kj})$ is 1, for $\boldsymbol{y} \in B_{kj}$, and is 0 otherwise, for $1 \le j \le k$. Assume $\max_{1\le j\le k} \mu(B_{kj}) \to 0$ as $k \to \infty$. Then $\boldsymbol{\ell}_k(\boldsymbol{y}) \to \boldsymbol{\ell}(\boldsymbol{y})$ as $k \to \infty$ for all $\boldsymbol{y} \in \Omega$, and $\boldsymbol{X}_k = \sum_{j=1}^{k} \boldsymbol{\ell}_k(b_{kj}) U(B_{kj})$. The c-characteristic function of $\boldsymbol{X}_k$, by Definition 2.1, can be expressed as

$$\begin{aligned} g(\boldsymbol{t}; \boldsymbol{X}_k, c) &= E(1 - i\boldsymbol{t} \cdot \boldsymbol{X}_k)^{-c} \\ &= E\left[1 - i \sum_{j=1}^{k} [\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{kj})] U(B_{kj})\right]^{-c} \\ &= E\left[\sum_{j=1}^{k} U(B_{kj})[1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{kj})]\right]^{-c} \\ &= \mathcal{R}_{-c}\left(\mu(B_{k1}), \ldots, \mu(B_{kk}); 1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{k1}), \ldots, 1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{kk})\right) \\ &= \prod_{j=1}^{k} \left(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{kj})\right)^{-\mu(B_{kj})}, \end{aligned}$$

where $\mathcal{R}$ denotes Carlson's $\mathcal{R}$ function [2], and the last identity follows from Eq. (6.6-5) of [2, p. 175]. Then the limit of c-characteristic functions of $\boldsymbol{X}_k$'s, as k approaches ∞, is

$$\begin{aligned} \lim_{k\to\infty} g(\boldsymbol{t}; \boldsymbol{X}_k, c) &= \exp\left[\lim_{k\to\infty} \sum_{j=1}^{k} -\mu(B_{kj}) \ln\left(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}_k(b_{kj})\right)\right] \\ &= \exp\left[-\int_\Omega \ln\left(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}(\boldsymbol{y})\right)\, d\mu(\boldsymbol{y})\right]. \end{aligned}$$

Note that the last identity is the transition from a Riemann sum to an integral. By the Lebesgue Dominated Convergence Theorem, we have $\lim_{k\to\infty} \boldsymbol{X}_k = \boldsymbol{X}$. Hence, by Theorem 2.4 of Jiang et al. [11], the c-characteristic function of $\boldsymbol{X}$ is

$$g(\boldsymbol{t}; \boldsymbol{X}, c) = \exp\left[-\int_\Omega \ln\left(1 - i\boldsymbol{t} \cdot \boldsymbol{\ell}(\boldsymbol{y})\right)\, d\mu(\boldsymbol{y})\right].$$

Appendix B. Proof of Theorem 3.1

First, we list some equations which are useful in the proof. The following two equations, concerning Appell's notation, $(a, k) = a(a + 1) \cdots (a + k - 1)$, can easily be shown.

$$\Gamma(a + n) = \Gamma(a)(a, n), \tag{B.1}$$

$$(a, 2n) = 2^{2n} \left(\frac{a}{2}, n\right) \left(\frac{a + 1}{2}, n\right). \tag{B.2}$$

From Gröbner and Hofreiter [7, p. 105], we have

$$\int_0^{2\pi} (a \cos x + b \sin x)^n\, dx = \begin{cases} \dfrac{(1/2, n/2)2(a^2 + b^2)^{n/2}\pi}{(n/2)!}, & n \text{ is even}, \\ 0, & n \text{ is odd}, \end{cases} \tag{B.3}$$

where a and b are real numbers. The following equation follows from Eq. 3.621.5 of Gradshteyn and Ryzhik [6, p. 389],

$$\int_0^{\pi} \sin^{a-1} x \cos^{b-1} x\, dx = \begin{cases} \dfrac{B(a/2, b/2)}{2}, & a, b > 0 \text{ and } b \text{ is odd}, \\ 0, & a, b > 0 \text{ and } b \text{ is even}, \end{cases} \tag{B.4}$$

where $B(\alpha, \beta)$ denotes the complete beta function.

By Lemma 3.1, the c-characteristic functions of $\widetilde{\boldsymbol{S}}_{n,r}$ and $\widetilde{\boldsymbol{B}}_{n,r}$ are

$$\begin{aligned} g(\boldsymbol{t}; \widetilde{\boldsymbol{S}}_{n,r}, c) &= \exp\left\{-\int_{S_{n,r}} \ln(1 - i\boldsymbol{t} \cdot \boldsymbol{y})\, d\mu_{n,r}(\boldsymbol{y})\right\} \\ &= \exp\left\{\frac{-c\Gamma(n/2)}{2r^{n-1}\pi^{n/2}} \int_{S_{n,r}} \ln(1 - i\boldsymbol{t} \cdot \boldsymbol{y})\, d\boldsymbol{y}\right\}, \end{aligned}$$

and

$$g(\boldsymbol{t};\widetilde{\boldsymbol{B}}_{n,r},c) = \exp\left\{-\int_{B_{n,r}} \ln(1-i\boldsymbol{t}\cdot\boldsymbol{y})\,d\nu_{n,r}(\boldsymbol{y})\right\}$$
$$= \exp\left\{\frac{-cn\Gamma(n/2)}{2r^n\pi^{n/2}}\int_{B_{n,r}} \ln(1-i\boldsymbol{t}\cdot\boldsymbol{y})\,d\boldsymbol{y}\right\},$$

respectively. Consider the following transformation:

$$\begin{aligned}
y_1 &= r\cos\theta_1,\\
y_2 &= r\sin\theta_1\cos\theta_2,\\
y_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\vdots\\
y_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1},\\
y_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1},
\end{aligned}$$

where $\theta_1,\theta_2,\ldots,\theta_{n-2}$ run from 0 to π, and θ_{n-1} from 0 to 2π. We then have

$$\begin{aligned}
\int_{S_r^n}\ln(1-i\boldsymbol{t}\cdot\boldsymbol{y})\,d\boldsymbol{y} &= \int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\ln(1-it_1r\cos\theta_1-\cdots-it_nr\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1})\\
&\quad\times r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,d\theta_{n-1}d\theta_{n-2}\cdots d\theta_1\\
&= \sum_{k=1}^{\infty}\frac{-i^k}{k}\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}(t_1\cos\theta_1+\cdots+t_n\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1})^k\\
&\quad\times r^{n+k-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,d\theta_{n-1}d\theta_{n-2}\cdots d\theta_1\\
&= \sum_{k=1}^{\infty}\frac{-i^k}{k}\sum_{\substack{m_1=0\\ m_1\text{ is even}}}^{k}\binom{k}{m_1}\frac{(1/2,m_1/2)2(t_{n-1}^2+t_n^2)^{m_1/2}\pi}{(m_1/2)!}\\
&\quad\times\int_0^\pi\cdots\int_0^\pi(t_1\cos\theta_1+\cdots+t_{n-2}\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2})^{k-m_1}\\
&\quad\times r^{n+k-1}\sin^{m_1+n-2}\theta_1\sin^{m_1+n-3}\theta_2\cdots\sin^{m_1+1}\theta_{n-2}\,d\theta_{n-2}\cdots d\theta_1\\
&= \sum_{k=1}^{\infty}\frac{-i^k}{k}\sum_{\substack{m_1=0\\ m_1\text{ is even}}}^{k}\sum_{\substack{m_2=0\\ m_2\text{ is even}}}^{k-m_1}\cdots\sum_{\substack{m_{n-2}=0\\ m_{n-2}\text{ is even}}}^{k-m_1-\cdots-m_{n-3}}\binom{k}{m_1}\binom{k-m_1}{m_2}\cdots\binom{k-m_1-\cdots-m_{n-3}}{m_{n-2}}\\
&\quad\times B\left(\frac{m_1+2}{2},\frac{m_2+1}{2}\right)B\left(\frac{m_1+m_2+3}{2},\frac{m_3+1}{2}\right)\cdots\\
&\quad B\left(\frac{m_1+\cdots+m_{n-3}+n-2}{2},\frac{m_{n-2}+1}{2}\right)\\
&\quad\times B\left(\frac{m_1+\cdots+m_{n-2}+n-1}{2},\frac{k-m_1-\cdots-m_{n-2}+1}{2}\right)\\
&\quad\times\frac{(1/2,m_1/2)2(t_{n-1}^2+t_n^2)^{m_1/2}\pi}{(m_1/2)!}r^{n+k-1}t_{n-2}^{m_2}t_{n-3}^{m_3}\cdots t_2^{m_{n-2}}t_1^{k-m_1-\cdots-m_{n-2}}\\
&= -\sum_{k=1}^{\infty}\frac{(1/2,k)\pi^{n/2}r^{n-1}}{k\Gamma(n/2)(n/2,k)}[-r^2(t_1^2+\cdots+t_n^2)]^k.
\end{aligned}$$

The third identity above can be obtained by the following expression

$$\begin{aligned}
&(t_1\cos\theta_1+\cdots+t_n\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1})^k\\
&\quad=\sum_{m_1=0}^{k}\binom{k}{m_1}[(\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2})^{m_1}(t_{n-1}\cos\theta_{n-1}+t_n\sin\theta_{n-1})^{m_1}\\
&\qquad\times(t_1\cos\theta_1+\cdots+t_{n-2}\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2})^{k-m_1}],
\end{aligned}$$

and Eq. (B.3). The fourth identity follows by using Eq. (B.4) and the binomial identity repeatedly. The fifth identity can be obtained by Eqs. (B.1) and (B.2).

Similarly,

$$\begin{aligned}\int_{B_r^n} \ln(1 - i\boldsymbol{t}\cdot\boldsymbol{y})\,d\boldsymbol{y} &= \int_0^r\int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \ln(1 - it_1\gamma\cos\theta_1 - \cdots - it_n\gamma\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}) \\ &\quad \times \gamma^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\,d\theta_{n-1}d\theta_{n-2}\cdots d\theta_1 d\gamma \\ &= -\sum_{k=1}^{\infty}\frac{(1/2,k)\pi^{n/2}}{k\Gamma(n/2)(n/2,k)}[-(t_1^2+\cdots+t_n^2)]^k\int_0^r \gamma^{2k+n-1}\,d\gamma \\ &= -\sum_{k=1}^{\infty}\frac{(1/2,k)\pi^{n/2}r^n}{kn\Gamma(n/2)(n/2+1,k)}[-r^2(t_1^2+\cdots+t_n^2)]^k.\end{aligned}$$

Therefore, the identities of the theorem hold.

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