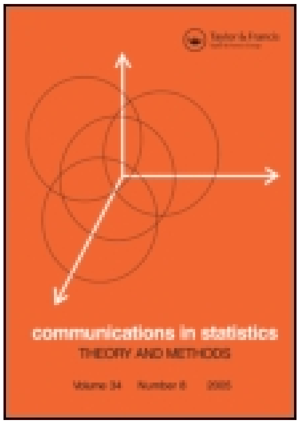

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Fu-Kai Chang [a] & Chao-Ping Ting [b]

[a] Accounting Office , Miaoli County Police Bureau , Miaoli , Taiwan

[b] Department of Statistics, College of Commerce , National Chengchi University , Taipei , Taiwan

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FU-KAI CHANG[1] AND CHAO-PING TING[2]

[1]Accounting Office, Miaoli County Police Bureau, Miaoli, Taiwan
[2]Department of Statistics, College of Commerce, National Chengchi University, Taipei, Taiwan

In two-level fractional factorial designs, homogeneous variance is commonly assumed in analysis of variance. When the variance of the response variable changes when a factor changes from one level to another, we call that factor the dispersion factor. However, the problem of finding optimal designs when dispersion factors are present is relatively unexplored. In this article, we focus on finding optimal designs for the estimation of all location main effects when there are one or two dispersion factors, in the class of regular single replicated two-level fractional factorial designs of resolution III or higher. We show that by appropriate naming of the dispersion factors, D-optimal and A-optimal designs can be identified. Table of D-optimal resolution III designs with two dispersion factors is given.

Keywords *A*-optimality; *D*-optimality; Dispersion effect; Location effects.

Mathematics Subject Classification Primary 62K05; Secondary 62K15.

1. Introduction

The assumption of constant variance is usually made when the analysis is performed on the two-level fractional factorial design. In practice, situations when variance of the response variable differs from one treatment combination to another do happen. Factors that are responsible for such differences are called dispersion factors. Identification of dispersion factors has been extensively studied recently. Box and Meyer (1986) studied the logarithm of the ratio of the residual variance and proposed an informal method to identify dispersion factors. Montgomery (1990) achieved the same goal by plotting these statistics on a normal probability plot. Wiklander (1998) and Wiklander and Holm (2003) combined the ordinary estimators of the two factor interaction to estimate dispersion effects. Wang (1989) developed a large sample test statistic to identify dispersion factors. Bergman and Hynên (1997), Liao (2000), Brenneman and Nair (2001), and McGrath and Lin

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Address correspondence to Chao-Ping Ting, Department of Statistics, College of Commerce, National Changchi University, No. 64, Sec. 2, ZhiNan Rd., Wenshan District, Taipei 11605, Taiwan; E-mail: cpting@nccu.edu.tw

(2001a) developed test procedures to identify dispersion factors in unreplicated regular 2^{n-p} fractional factorial designs. Pan (1999) and McGrath and Lin (2001b) stressed the importance of identifying the location effects before studying the dispersion effects. More recently, van de Ven (2008) showed the estimators of the dispersion effects proposed by Wiklander (1998), Wiklander and Holm (2003), Liao and Iyer (2000), and Brenneman and Nair (2001) are equivalent in a two-level fractional factorial design setting.

All of the aforementioned articles focused on identifying dispersion effects; not until Liao and Iyer (2000) and Liao (2006) has the optimality property for the estimation of dispersion effects been studied. Although there is a growing interest in studying the optimality property for dispersion effects, the optimality property for location effects when dispersion factors are present is relatively unexplored. Lin (2005) formed D-optimal designs for estimating a specific set of location effects with one dispersion factor. Due to the increased theoretical and computational challenges with models when interaction effects are included, the focus of this article is on finding D-optimal and A-optimal designs for estimating all location main effects when one or two dispersion factors are present in the class of regular single replicated 2^{n-p} fractional factorial designs of resolution III or higher. Ting (2010) continued investigating on the D-optimality of resolution III designs when two dispersion factors with equal dispersion main effects are present in the model.

Notation and the information matrix for the estimation of all location main effects are stated in the next section. Section 3 gives the D-optimal and A-optimal designs for the estimation of all location main effects with one dispersion factor. In Sec. 4, D-optimal designs for estimating all location main effects with two dispersion factors are given. Section 5 contains our concluding remarks.

2. Preliminaries

Let $F_1, F_2, \ldots, F_n$ denote the n two-level factors and the main effects of the corresponding factors. Let $F_1^{e_1} F_2^{e_2} \cdots F_n^{e_n}$ denote the general effect with $e_i = 1$ if F_i appears in the effect, and $e_i = 0$, otherwise. Without loss of generality, $F_1, F_2, \ldots,$ and F_a are assumed as the a factors that are responsible for the dispersion effects.

A 2^{n-p} fractional factorial design with $N = 2^{n-p}$ runs is completely determined by appropriately selecting p independent generators and the corresponding defining relation. For example, the treatment combinations of a 2^{6-3} design may be determined when the following generators $F_4 = F_1F_2$, $F_5 = F_1F_3$, and $F_6 = F_2F_3$ are selected, and the corresponding defining relation is $I = F_1F_2F_4 = F_1F_3F_5 = F_2F_3F_6 = F_4F_5F_6 = F_2F_3F_4F_5 = F_1F_3F_4F_6 = F_1F_2F_5F_6$.

The resolution of a design depends on the alias structure. In the defining relation, an effect that is aliased with the general mean is called a word and the number of letters in a word is called the word length. The minimum length of all the words in the defining relation is called the resolution of the design for two-level factional factorial designs. The example above is a design of resolution III and is denoted as 2^{6-3}_{III}.

In a regular single replicated 2^{n-p} fractional factorial design setting, let $\vec{Y}$ be the $N \times 1$ response vector, and the model considered here is the location main effects model, i.e.,

$$\vec{Y} = X\vec{\beta} + \varepsilon,$$

where $\vec{\beta}$ is the $(n+1) \times 1$ vector of the overall mean and all location main effects; $X = [\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_n]$ is the $N \times (n+1)$ model matrix, $\vec{x}_0 = (1, 1, \ldots, 1)'$, and $\vec{x}_j = (x_{1j}, x_{2j}, \ldots, x_{Nj})'$ with $x_{ij} = 1$ or -1 depends on whether factor j appears at its high level or low level in the ith response; and $\vec{\varepsilon}$ is the $N \times 1$ vector of uncorrelated random error with $E(\vec{\varepsilon}) = \vec{0}$ and $V(\vec{\varepsilon}) = \gamma_0 I + \gamma_1 D_1 + \gamma_2 D_2 + \cdots + \gamma_a D_a$ where γ_0 is the dispersion mean, γ_j is the dispersion main effect of factor F_j by Liao and Iyer (2000), and D_j is the $N \times N$ diagonal matrix whose diagonal elements are $x_{1j}, x_{2j}, \ldots,$ and x_{Nj}. γ_0 and the γ_j's are known, and $\sum_{j=1}^{a} |\gamma_j| < \gamma_0$, such that the variances of the response variables are all positive.

To estimate $\vec{\beta}$, the generalized least squares estimator $\hat{\vec{\beta}}$ is used, where $\hat{\vec{\beta}} = (X'V(\vec{Y})^{-1}X)^{-1}X'V(\vec{Y})^{-1}\vec{Y}$, and the corresponding covariance matrix of $\hat{\vec{\beta}}$ is $V(\hat{\vec{\beta}}) = (X'V(\vec{Y})^{-1}X)^{-1}$. Let $M = X'V(\vec{Y})^{-1}X$, and M is called the information matrix for the estimation of $\vec{\beta}$.

3. Optimal 2^{n-p} Fractional Factorial Design with One Dispersion Factor

In this section, we focus on finding optimal designs of resolution III or higher when $a = 1$. Without loss of generality, F_1 is assumed to be responsible for the dispersion effect. Then $V(\vec{Y}) = \gamma_0 I_N + \gamma_1 D_1$, $|\gamma_1| < \gamma_0$, and through direct derivation $V(\vec{Y})^{-1} = m_0 I_N + m_1 D_1$, where $m_0 = \gamma_0/(\gamma_0^2 - \gamma_1^2)$, $m_1 = -\gamma_1/(\gamma_0^2 - \gamma_1^2)$. The information matrix $M = (m_{ij})$, $i, j = 0, \ldots, n$ can be partitioned as

$$M = N \begin{bmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{bmatrix}, \quad \text{where } M_{11} = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_0 \end{bmatrix};$$

M_{12} is a $2 \times (n-1)$ matrix of zeroes; M_{22} is a square matrix of order $n-1$, with $m_{ii} = m_0$, and for $i \neq j$, $m_{ij} = m_1$ if $F_1 F_i F_j$ is a word in the defining relation, otherwise $m_{ij} = 0$. For the derivation of M, see the Appendix.

Let θ be the number of words of the form $F_1 F_i F_j$ in the defining relation. Through row and column operations, M can be transformed into M_T, and

$$M_T = N \begin{bmatrix} I_{\theta+1} \otimes M_{11} & 0 \\ 0 & m_0 I_{n-2\theta-1} \end{bmatrix} = N \cdot \operatorname{diag}(I_{\theta+1} \otimes M_{11}, m_0 I_{n-2\theta-1}),$$

where "$\otimes$" denotes the Kronecker product. The eigenvalues of M are identical to those of M_T, and they are $N(m_0 - m_1)$, $N(m_0 + m_1)$, and $N m_0$, with respective frequencies $\theta + 1$, $\theta + 1$, and $n - 2\theta - 1$.

A design is said to be D-optimal if it minimizes the determinant of M^{-1}, or equivalently maximizes the determinant of M among all designs. A-optimal design minimizes the trace of M^{-1}. Since $m_0^2 > m_1^2$, it can be shown that $\det(M) = N^{n+1}(m_0^2 - m_1^2)^{\theta+1} m_0^{n-2\theta-1}$, and is decreasing in θ; and $\operatorname{tr}(M^{-1}) = (N m_0 (m_0^2 - m_1^2))^{-1} \, (2\theta m_1^2 + (n+1)m_0^2 - (n-1)m_1^2)$, and is increasing in θ. Liao and Iyer (2000) also showed that the $\operatorname{tr}(M^{-1})$ increases as θ increases. One can thus conclude that the smaller the value of θ is, the "better" the corresponding design is. That is, designs having the smallest θ value are D-optimal and A-optimal in 2_{III}^{n-p}. The following Theorem 3.1 is a direct consequence.

Theorem 3.1. *Designs having the minimum number of length three words involving the dispersion factor in the defining relation are D-optimal and A-optimal in estimating all location main effects in* 2^{n-p}_{III}.

Hence, through appropriate "naming" of the dispersion factor, D-optimal and A-optimal designs can be obtained. For example, a 2^{6-2}_{III} design with generators $F_5 = F_2F_3$, $F_6 = F_1F_3F_4$, and defining relation $I = F_2F_3F_5 = F_1F_3F_4F_6 = F_1F_2F_4F_5F_6$. If F_1 is named as the dispersion factor, one can see that there is no length three words involving F_1 in the defining relation, hence $\theta = 0$, and the design is D- and A-optimal in estimating all location main effects in 2^{6-2}_{III}. Also, there are no length three words involving either F_4 or F_6 in the defining relation. Hence, if either F_4 or F_6 is named as the dispersion factor, the resulting design is D-optimal and A-optimal in estimating all location main effects in 2^{6-2}_{III}.

For designs of resolution IV or higher, the shortest word in the defining relation is of length at least four. Hence, resolution IV or higher designs are "robust" against a single dispersion factor when one's interest is to estimate all location main effects.

4. Optimal 2^{n-p} Fractional Factorial Design with Two Dispersion Factors

In this section, finding optimal designs of resolution III or higher with $a = 2$ is our focus. Without loss of generality, F_1 and F_2 are assumed to be responsible for the dispersion effects. Then $V(\vec{Y}) = \gamma_0 I_N + \gamma_1 D_1 + \gamma_2 D_2$, $|\gamma_1| + |\gamma_2| < \gamma_0$, and through direct derivation $V(\vec{Y})^{-1} = m_0 I_N + m_1 D_1 + m_2 D_2 + m_3 D_1 D_2$, where

$$\begin{aligned} m_0 &= \varphi^{-1}\gamma_0(\gamma_0^2 - \gamma_1^2 - \gamma_2^2), \quad m_1 = \varphi^{-1}\gamma_1(\gamma_1^2 - \gamma_0^2 - \gamma_2^2),\\ m_2 &= \varphi^{-1}\gamma_2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2), \quad m_3 = 2\varphi^{-1}\gamma_0\gamma_1\gamma_2, \quad \text{and}\\ \varphi &= \gamma_0^2(\gamma_0^2 - \gamma_1^2 - \gamma_2^2) + \gamma_1^2(\gamma_1^2 - \gamma_0^2 - \gamma_2^2) + \gamma_2^2(\gamma_2^2 - \gamma_0^2 - \gamma_1^2). \end{aligned}$$

The information matrix $M = (m_{ij})$, $i, j = 0, \ldots, n$, for the estimation of $\vec{\beta}$ can be partitioned as

$$M = N\begin{bmatrix} M_{11} & M_{12}\\ M_{12}' & M_{22} \end{bmatrix}, \quad \text{where } M_{11} = \begin{bmatrix} m_0 & m_1 & m_2\\ m_1 & m_0 & m_3\\ m_2 & m_3 & m_0 \end{bmatrix};$$

M_{12} is a $3 \times (n-2)$ matrix and if $F_1F_2F_j$ is a word in the defining relation, $m_{0j} = m_3$, $m_{1j} = m_2$, and $m_{2j} = m_{31}$, otherwise, $m_{ij} = 0$; M_{22} is a square matrix of order $n-2$ whose diagonal elements are m_0 and off-diagonal elements m_{ij} are

$$m_{ij} = \begin{cases} m_1, & \text{if } F_1F_iF_j \text{ is a word in the defining relation},\\ m_2, & \text{if } F_2F_iF_j \text{ is a word in the defining relation},\\ m_3, & \text{if } F_1F_2F_iF_j \text{ is a word in the defining relation},\\ 0, & \text{otherwise.} \end{cases}$$

The derivation of M and its characteristics are given in the Appendix.

Through row and column operations, M can be transformed into M_T, and $M_T = N \cdot \mathrm{diag}(I_{\delta_1} \otimes U, I_{\delta_2} \otimes M_{11}, I_{\delta_3} \otimes Q, I_{\delta_4} \otimes V, I_{\delta_5} \otimes T, m_0 I_{\delta_6})$, where

$$U = \begin{bmatrix} m_0 & m_1 & m_2 & m_3 \\ m_1 & m_0 & m_3 & m_2 \\ m_2 & m_3 & m_0 & m_1 \\ m_3 & m_2 & m_1 & m_0 \end{bmatrix}, \quad Q = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_0 \end{bmatrix}, \quad V = \begin{bmatrix} m_0 & m_2 \\ m_2 & m_0 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} m_0 & m_3 \\ m_3 & m_0 \end{bmatrix}.$$

The δ_i's are functions of the number of words of the forms $F_1F_2F_j$, $F_1F_iF_j$, $F_2F_iF_j$, and $F_1F_2F_iF_j$ in the defining relation, and satisfy $\delta_1 + \delta_2 \geq 1$, and $4\delta_1 + 3\delta_2 + 2\delta_3 + 2\delta_4 + 2\delta_5 + \delta_6 = n + 1$. It is obvious that if design d^* has information matrix $M^* = N \cdot \mathrm{diag}(M_{11}, m_0 I_{n-2})$, then d^* is D-optimal and A-optimal in 2^{n-p}. Theorem 4.1 is stated without proof in the following.

Theorem 4.1. *Designs having no length three words involving either one or both of the dispersion factors and no length four words involving both of the dispersion factors in the defining relation are D-optimal and A-optimal in estimating all location main effects in* 2^{n-p}.

For example, take a 2_{III}^{5-1} design with defining relation $I = F_1F_2F_5$. If F_3 and F_4 are named as the two dispersion factors, the information matrix of this design, through row and column operations, is of the form M^*, hence, it is the D-optimal and A-optimal design in 2^{5-1}.

For designs of resolution IV, there are no length three words in the defining relation; hence, the transformed information matrix $M_T = N \cdot \mathrm{diag}(M_{11}, I_\delta \otimes T, m_0 I_{n-2\delta-2})$, where δ is the number of length four words involving both of the dispersion factors in the defining relation. Let λ_1, λ_2, and λ_3 be the eigenvalues of M_{11}, then the eigenvalues of M are $N\lambda_1$, $N\lambda_2$, $N\lambda_3$, $N(m_0 - m_3)$, $N(m_0 + m_3)$, and Nm_0, with respective frequencies 1, 1, 1, δ, δ, and $n - 2\delta - 2$. Since $m_0^2 > m_3^2$, it can be shown that $\det(M) = N^{n+1}(m_0^2 - m_3^2)^\delta m_0^{n-2\delta-2}\lambda_1\lambda_2\lambda_3$ is decreasing in δ, and $\mathrm{tr}(M^{-1}) = N^{-1}((m_0(m_0^2 - m_3^2))^{-1}(2\delta m_3^2 + (n-2)(m_0^2 - m_3^2)) + \lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1})$ is increasing in δ. Hence, D-optimal and A-optimal designs are designs having the smallest δ value among all designs in 2_{IV}^{n-p}. The following Theorem 4.2 is a direct consequence. Through appropriate naming of the two dispersion factors, D-optimal and A-optimal designs can easily be obtained.

Theorem 4.2. *Designs having the minimum number of length four words involving both of the dispersion factors are D-optimal and A-optimal in estimating all location main effects in* 2_{IV}^{n-p}.

Resolution V or higher designs are robust against two dispersion factors, if one's interest is to estimate all location main effects.

As to designs of resolution III, values of $\det(M)$ and $\mathrm{tr}(M^{-1})$ depend not only on the number of length three words involving either one or both of the dispersion factors, and the number of length four words involving both of the dispersion factors in the defining relation but also on the values of the dispersion mean and dispersion main effects. Due to the complexity in calculating $\det(M)$ and $\mathrm{tr}(M^{-1})$ for an arbitrary design, Ting (2010) investigated the case when the two dispersion main effects are equal, that is, $\gamma_1 = \gamma_2 = \gamma$, and gives the best naming of the two dispersion factors, in terms of maximizing $\det(M)$, for 16-run and 32-run 2_{III}^{n-p} designs.

Table 1
16-run D-optimal 2_{III}^{n-p} designs with two dispersion factors

Design	Generators	Dispersion factors	Remark
5-1.3	$F_5 = F_1F_2$	(F_3, F_4)	
6-2.2	$F_5 = F_1F_2,\ F_6 = F_1F_3F_4$	$(F_3, F_j),\ j = 4, 6;\ (F_4, F_6)$	
7-3.2	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3F_4$	(F_4, F_7)	
8-4.6	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3$	$(F_i, F_4),\ 1 \le i \le 3;\ (F_4, F_j),\ 5 \le j \le 8$	$\lvert\gamma\rvert/\gamma_0 \ge 0.428$
8-4.4	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3F_4$	(F_4, F_8)	$\lvert\gamma\rvert/\gamma_0 < 0.428$
9-5.2	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_4,\ F_8 = F_3F_4,\ F_9 = F_1F_2F_3F_4$	$(F_1, F_j),\ j = 2, 3, 5, 6;\ (F_2, F_j),\ j = 4, 5, 7;\ (F_3, F_j),\ j = 4, 6, 8;$ $(F_4, F_j),\ 7 \le j \le 8;\ (F_5, F_j),\ 8 \le j \le 9;\ (F_6, F_j),\ j = 7, 9;$ (F_8, F_9)	$\lvert\gamma\rvert/\gamma_0 \ge 0.437$
9-5.1	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_1F_4,\ F_8 = F_2F_3F_4,\ F_9 = F_1F_2F_3F_4$	$(F_i, F_j),\ 2 \le i < j \le 9$	$\lvert\gamma\rvert/\gamma_0 < 0.437$
9-5.3	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_4,\ F_9 = F_2F_3F_4$	(F_8, F_9)	$\lvert\gamma\rvert/\gamma_0 < 0.437$
9-5.5	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3,\ F_9 = F_1F_4$	$(F_3, F_j),\ 8 \le j \le 9;\ (F_4, F_j),\ 6 \le j \le 7;\ (F_6, F_9);\ (F_7, F_8)$	$\lvert\gamma\rvert/\gamma_0 < 0.437$
10-6.1	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_4,\ F_9 = F_2F_3F_4,$ $F_{10} = F_1F_2F_3F_4$	$(F_i, F_j),\ 2 \le i < j \le 10,\ i, j \ne 7$	
10-6.2	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_4,\ F_9 = F_2F_4,$ $F_{10} = F_1F_3F_4$	$(F_7, F_j),\ j = 9, 10;\ (F_9, F_{10})$	
10-6.4	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3,\ F_9 = F_1F_4,$ $F_{10} = F_2F_4$	$(F_4, F_j),\ 9 \le j \le 10;\ (F_9, F_{10})$	
11-7.1	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_4,\ F_9 = F_2F_4,$ $F_{10} = F_1F_3F_4,\ F_{11} = F_2F_3F_4$	$(F_1, F_j),\ j = 2, 6, 7, 8, 9, 10, 11;\ (F_2, F_j),\ 6 \le j \le 11;$ $(F_6, F_j),\ 7 \le j \le 11;\ (F_7, F_j),\ 8 \le j \le 11;$ $(F_8, F_j),\ 9 \le j \le 10;\ (F_i, F_j),\ 9 \le i < j \le 11$	
11-7.2	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3,\ F_9 = F_1F_4,$ $F_{10} = F_2F_4,\ F_{11} = F_3F_4$	$(F_4, F_j),\ 8 \le j \le 11;\ (F_i, F_j),\ 8 \le i < j \le 11$	
11-7.3	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3,\ F_9 = F_1F_4,$ $F_{10} = F_2F_4,\ F_{11} = F_1F_2F_4$	$(F_i, F_j),\ 3 \le i < j \le 11,\ i, j \ne 5$	
12-8.1	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_4,\ F_9 = F_2F_4,$ $F_{10} = F_1F_3F_4,\ F_{11} = F_2F_3F_4,\ F_{12} = F_1F_2F_3F_4$	$(F_i, F_j),\ 1 \le i < j \le 12$, except for (F_6, F_8)	
12-8.2	$F_5 = F_1F_2,\ F_6 = F_1F_3,\ F_7 = F_2F_3,\ F_8 = F_1F_2F_3,\ F_9 = F_1F_4,$ $F_{10} = F_2F_4,\ F_{11} = F_1F_2F_4,\ F_{12} = F_3F_4$	$(F_i, F_j),\ 3 \le i < j \le 12,\ i, j \ne 5$	

Based on Table 1 in Ting (2010), 16-run D-optimal 2_{III}^{n-p} designs for $5 \leq n \leq 12$ can be found and are listed in the following Table 1. For 2_{III}^{5-1}, regardless of the values of γ_0 and γ, design 5-1.3 is D-optimal when F_3 and F_4 are named as the two dispersion factors. For 2_{III}^{8-4} and $|\gamma|/\gamma_0 \geq 0.428$, design 8-4.6 is D-optimal when F_4 is named as one of the two dispersion factors. And for $|\gamma|/\gamma_0 < 0.428$, design 8-4.4 is D-optimal when F_4 and F_8 are named as the two dispersion factors. For 2_{III}^{10-6}, regardless of the values of γ_0 and γ, designs 10-6.1, 10-6.2, and 10-6.4 are all D-optimal when the two factors as listed in Table 1 are named as the dispersion factors.

5. Concluding Remarks

The commonly used criteria in selecting designs, for example, highest resolution and minimum aberration, are inappropriate when dispersion factors are present in the model and our interest is on the estimation of all location main effects. Current work is focusing on the establishment of a criterion to distinguish among 2_{III}^{n-p} designs when two dispersion factors are present in the model. It should be noted that if model under consideration is not of main effects only, D-optimal designs for the estimation of effects of interest may be different, and the highest resolution and minimum aberration criteria may still be appropriate in ranking designs.

Appendix

A.1. Derivation of the Information Matrix with One Dispersion Factor

$M = (m_{ij}) = X'V(\vec{Y})^{-1}X = X'(m_0 I_N + m_1 D_1)X$, $i, j = 0, \ldots, n$, where $m_{ij} = m_0(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0) + m_1(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1)$, and "$\circ$" denote the general inner product of vectors, i.e., $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = \sum_{k=1}^{N} x_{ik} x_{jk} x_{1k}$. Now:

(i) for $i = j$, $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_0 = N$, and $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_1 = 0$;
(ii) for $i = 0$, $j = 1$, $\vec{x}_0 \circ \vec{x}_1 \circ \vec{x}_0 = 0$, and $\vec{x}_0 \circ \vec{x}_1 \circ \vec{x}_1 = N$;
(iii) for $i = 0$, $j \geq 2$, $\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_0 = 0$, and $\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_1 = 0$;
(iv) for $i = 1$, $j \geq 2$, $\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_0 = 0$, and $\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_1 = 0$;
(v) for $2 \leq i < j \leq n$, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0 = 0$, and $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = N$, if $F_1 F_i F_j$ is a word, otherwise all are zeroes,

M is thus derived.

Since the designs we consider here are of resolution III or higher, there is at most one nonzero off-diagonal entry in each row and column of M; that is, it is not possible to have two words of the forms $F_1 F_i F_j$, $F_1 F_{i'} F_j$, or of the forms $F_1 F_i F_j$, $F_1 F_i F_{j'}$, respectively, in the defining relation.

A.2. Derivation of the Information Matrix with Two Dispersion Factors

$M = (m_{ij}) = X'(m_0 I_N + m_1 D_1 + m_2 D_2 + m_3 D_1 D_2)X$, where $m_{ij} = m_0(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0) + m_1(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1) + m_2(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_2) + m_3(\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2)$. Now:

(i) for $0 \leq i = j \leq n$, $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_0 = N$, $\vec{x}_i \circ \vec{x}_i \circ \vec{x}_1 = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_2 = \vec{x}_i \circ \vec{x}_i \circ \vec{x}_1 \circ \vec{x}_2 = 0$;
(ii) for $0 \leq i < j \leq 2$, only $\vec{x}_0 \circ \vec{x}_1 \circ \vec{x}_1 = \vec{x}_0 \circ \vec{x}_2 \circ \vec{x}_2 = \vec{x}_1 \circ \vec{x}_2 \circ \vec{x}_1 \circ \vec{x}_2 = N$, all the others are zeroes.

(iii) for $0 \leq i \leq 2 < j \leq n$, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_0 = 0$, and if $F_1F_2F_j$ is a word, $\vec{x}_1 \circ \vec{x}_j \circ \vec{x}_2 = N$, $\vec{x}_2 \circ \vec{x}_j \circ \vec{x}_1 = N$, $\vec{x}_0 \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2 = N$, otherwise all are zeroes. The designs under consideration are of resolution III or higher; hence, it is not possible to have two length three words of the form $F_1F_2F_j$ in the defining relation. Therefore, there is at most one such j.

(iv) for $3 \leq i < j \leq n$, if $F_1F_iF_j$ is a word, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 = N$, and all the others are zeroes; if $F_2F_iF_j$ is a word, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_2 = N$, and all the others are zeroes; and if $F_1F_2F_iF_j$ is a word, $\vec{x}_i \circ \vec{x}_j \circ \vec{x}_1 \circ \vec{x}_2 = N$, all the others are zeroes. Since designs of resolution III or higher are considered, it is not possible to have two length three words of the following forms $(F_1F_iF_j, F_1F_iF_{j'})$, $(F_1F_iF_j, F_1F_{i'}F_j)$, $(F_2F_iF_j, F_2F_iF_{j'})$, or $(F_2F_iF_j, F_2F_{i'}F_j)$, and two length four words of the following forms $(F_1F_2F_iF_j, F_1F_2F_iF_{j'})$, or $(F_1F_2F_iF_j, F_1F_2F_{i'}F_j)$ in the defining relation. Hence, each of the m_0, m_1, m_2, and m_3 appears at most once in each row and each column.

Some characteristics concerning M are listed below.

1. There is at most one j in M_{12} such that $m_{ij} \neq 0$. That is, M_{12} is either a matrix of zeroes, or a matrix with exactly one column of the form $[m_3, m_2, m_1]'$ and all the other entries are zeroes.
2. In M_{22}, the number of appearances of m_i, $i = 1, 2, 3$, is at most one in each row and each column.
3. In M_{22}, if $m_{ij} = m_1$ (or m_2), $m_{ik} = m_2$ (or m_1), then $m_{jk} = m_3$.

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