Journal of the Chinese Statistical Association
Vol. 47, (2009) 236–252

OPTIMAL TWO-LEVEL FRACTIONAL FACTORIAL DESIGNS WITH PURE DISPERSION FACTORS

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ABSTRACT

The problem of finding optimal designs when pure dispersion factors are present in the class of regular single replicated two-level fractional factorial design of resolution III and higher is studied. When a single dispersion factor is present, D-optimal and A-optimal designs depend on the number of length three words involving the dispersion factor in the defining relation for resolution III designs. When two dispersion factors with equal dispersion main effect are present, D-optimal designs depend not only on the number of length three and length four words involving the dispersion factors in the defining relation but also on the values of the dispersion mean and main effects and the structures of the words. Tables are given to show how D-optimality ordering of designs changes when the values of the dispersion mean and main effects and the word structure change.

Key words and phrases: A-optimality, D-optimality, location effect, separate dispersion effect.
JEL classification: C13, C21, C90.

1. Introduction

Situations of non-constant variance happen when identifying important location factors in many industrial and business experiments. Factors that are responsible for the changes of the variance of the response variable from one treatment combination to another are called dispersion factors. Identification of dispersion factors has been studied extensively. Box and Meyer (1986) initiated the investigation and proposed an informal method to identify dispersion factors. Montgomery (1990) suggested plotting these statistics on a normal probability plot. Some other papers in this line of studies, for example, can be found in Chang and Ting (2009). More recently, van de Ven (2008) showed the estimators of the dispersion effects proposed by Wiklander (1998), Wiklander and Holm (2003), Liao and Iyer (2000), and Brenneman and Nair (2001) are equivalent.

Not until Lin (2005) has the optimality property for the estimation of location effects when dispersion factors are present in the model been studied. Lin (2005) investigated D-optimal designs for estimating a specific set of location effects when a single dispersion factor is present. Chang and Ting (2009) found D-optimal and A-optimal designs for estimating all location main effects when one or two dispersion factors are present. Ting (2009) focused further on investigating the D-optimality of resolution III designs when two dispersion factors with equal dispersion main effect are present. In all the aforementioned papers the dispersion factors may also have location effects and vice versa. However, there are cases in industrial and business experiments that some factors only have location effects and the others only have dispersion effects in the model. In this paper our interest focuses on finding D-optimal and A-optimal designs for estimating all location main effects when one or two pure dispersion factors are present in the class of single replicated regular 2^{n-p} fractional factorial designs of resolution III or higher.

Model, notation, and the information matrix for the estimation of location main effects are given in section 2. Section 3 investigates the D-optimality and A-optimality of designs for estimating location main effects when one dispersion factor is present in the model. In section 4, D-optimality ordering of designs for estimating location main effects with two dispersion factors having the same dispersion main effect is given. Section 5 contains the concluding remarks.

2. Premilinaries

Let $\vec{Y}$ be the $N \times 1$ response vector, where $N = 2^{n-p}$, and $F_1, F_2, \ldots, F_n$ be the n two-level factors where $F_1, F_2, \ldots, F_a$ denote the a dispersion factors and $F_{a+1}, F_{a+2}, \ldots, F_n$ denote the $n - a$ location factors. Let F_iF_j, be the two-factor interaction, $F_iF_jF_k$, be the three-factor interaction, and so on. Assume that two-factor and higher-order interactions are negligible, the model employed here is

$$\vec{Y} = X\vec{\beta} + \varepsilon\,,$$

where $\vec{\beta}$ is the $(n - a + 1) \times 1$ vector of the overall mean and all location main effects; $X = \left[\vec{1}, \vec{x}_{a+1}, \ldots, \vec{x}_n\right]$ is the $N \times (n - a + 1)$ model matrix, $\vec{1}$ is a vector of ones, and $\vec{x}_j = (x_{1j}, x_{2j}, \ldots, x_{Nj})'$ with $x_{ij} = 1$ if factor j appears at its high level in the ith response or, $x_{ij} = -1$ if factor j appears at its low level; and $\vec{\varepsilon}$ is the vector of uncorrelated random error with $\mathrm{E}(\vec{\varepsilon}) = \vec{0}$ and $\mathrm{V}(\vec{\varepsilon}) = \gamma_0 I_N + \gamma_1 D_1 + \gamma_2 D_2 + \cdots + \gamma_a D_a$, where I_N is the identity matrix of order N, γ_0 is the dispersion mean, γ_j is the dispersion main effect of factor F_j, D_j is the $N \times N$ diagonal matrix with diagonal elements $x_{1j}, x_{2j}, \ldots, x_{Nj}$, and $\sum_{j=1}^{a} |\gamma_j| < \gamma_0$, such that the variances of the response variables are all positive.

Since the variance of the response variables are not all the same, the generalized least squares estimator, $\hat{\vec{\beta}}$, of $\vec{\beta}$ is used. Now $\hat{\vec{\beta}} = (X'\,\mathrm{V}(\vec{Y})^{-1}X)^{-1}X'\,\mathrm{V}(\vec{Y})^{-1}\vec{Y}$ and the variance-covariance matrix of $\hat{\vec{\beta}}$ is $\mathrm{V}\left(\hat{\vec{\beta}}\right) = (X'\,\mathrm{V}(\vec{Y})^{-1}X)^{-1}$. $X'\,\mathrm{V}(\vec{Y})^{-1}X$ is called the information matrix for the estimation of $\vec{\beta}$ and is denoted as M hereafter.

3. Optimal 2^{n-p} fractional factorial design with one dispersion factor

Without loss of generality, F_1 is assumed as the single dispersion factor. Then $\mathrm{V}(\vec{Y}) = \gamma_0 I_N + \gamma_1 D_1$, $|\gamma_1| < \gamma_0$, $\mathrm{V}(\vec{Y})^{-1} = m_0 I_N + mD_1$, where $m_0 = \gamma_0/(\gamma_0^2 - \gamma_1^2)$, $m = -\gamma_1/(\gamma_0^2 - \gamma_1^2)$, and $M = (m_{ij})$, $i, j = 0, 2, 3, \ldots, n$ has the following form,

$$M = N \begin{bmatrix} m_0 & \vec{0}' \\ \vec{0} & M_{11} \end{bmatrix},$$

where $\vec{0}$ is a vector of zeroes; M_{11} is a square matrix of order $n - 1$ whose diagonal elements are m_0 and off-diagonal element $m_{ij} = m$ if $F_1F_iF_j$ is a word in the defining relation, or $m_{ij} = 0$, otherwise. The derivation of M follows a similar procedure as in Chang and Ting (2009).

A design is said to be D-optimal if it minimizes the determinant of M^{-1}, or equivalently maximizes the determinant of M, and is said to be A-optimal if it minimizes the trace of M^{-1}. Through straightforward algebra one can obtain that the determinant of M is $\det(M) = N^n(m_0^2 - m^2)^\theta m_0^{n-2\theta}$, and the trace of M^{-1} is $\text{tr}(M^{-1}) = N^{-1}((n-2\theta)m_0^{-1} + 2\theta m_0(m_0^2 - m^2)^{-1})$, where θ is the number of length three words of the form $F_1F_iF_j$ in the defining relation. In the following, the optimality property of designs is investigated according to design resolution.

(I) Designs of resolution IV or higher. For designs of resolution IV or higher, $\theta = 0$, and they are "robust" against single dispersion factor.

(II) Designs of resolution III. Since $m_0 > |m|$, one can show that $\det(M)$ is decreasing in θ, and $\text{tr}(M^{-1})$ is increasing in θ. The following Theorem 3.1 is a direct consequence.

Theorem 3.1. *Designs having the minimum number of length three words involving the dispersion factor in the defining relation are the D-optimal and A-optimal in 2_{III}^{n-p}.*

4. Optimal 2^{n-p} fractional factorial design with two dispersion factors and equal dispersion main effect

Without loss of generality, F_1 and F_2 are assumed as the two dispersion factors, and $\gamma_1 = \gamma_2 = \gamma$. Then $\text{V}(\vec{Y}) = \gamma_0 I_N + \gamma D_1 + \gamma D_2$, $|\gamma| = \gamma_0/2$, $\text{V}(\vec{Y})^{-1} = m_0 I_N + mD_1 + mD_2 + m_1 D_1 D_2$, where $m_0 = \varphi^{-1}\gamma_0(\gamma_0^2 - 2\gamma^2)$, $m = -\varphi^{-1}\gamma\gamma_0^2$, $m_1 = 2\varphi^{-1}\gamma_0\gamma^2$, and $\varphi = \gamma_0^2(\gamma_0^2 - 4\gamma^2)$; and $M = (m_{ij})$, $i, j = 0, 3, 4, \ldots, n$, has the following form,

$$M = N \begin{bmatrix} m_0 & \vec{\ell}' \\ \vec{\ell} & M_{11} \end{bmatrix},$$

where $\vec{\ell} = (m_{30}, \ldots, m_{n0})'$ and $m_{i0} = m_1$ if $F_1F_2F_i$ is a word in the defining relation, otherwise $m_{i0} = 0$; M_{11} is a square matrix of order $n-2$ whose diagonal elements are m_0 and off-diagonal element $m_{ij} = m$ if either $F_1F_iF_j$ or $F_2F_iF_j$ is a word in the defining relation, or $m_{ij} = m_1$ if $F_1F_2F_iF_j$ is a word in the defining relation, otherwise $m_{ij} = 0$. The derivation of M follows a similar procedure as in Chang and Ting (2009).

Designs having fewer words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation are better in terms of D-optimality and A-optimality. The following Theorem 4.1 is thus stated without proof.

Theorem 4.1. *Designs having no length three words involving either one or both of the dispersion factors and no length four words involving both of the dispersion factors in the defining relation are D-optimal and A-optimal.*

In the following, the optimality property of designs is investigated by design resolution.

(I) Designs of resolution V or higher. By Theorem 4.1, designs of resolution V or higher are robust against two dispersion factors.

(II) Design of resolution IV. The optimality property of designs of resolution IV is similar to that of the optimality property of designs of resolution III with single dispersion factor. The following Theorem 4.2 is a direct consequence

Theorem 4.2. *Designs having the minimum number of length four words involving both of the dispersion factors in the defining relation are D-optimal and A-optimal in 2_{IV}^{n-p}.*

(III) Designs of resolution III. The optimality property is not direct and the optimality ordering of these designs depends not only on the number of words involving the dispersion factors but also on the forms of the words and the values of γ_0 and γ. Some structural characteristics concerning M are stated in the followings.

(1) There is at most one nonzero element in $\vec{\ell}$.

(2) If there is one nonzero element in $\vec{\ell}$, then except for the diagonal element, all the elements in the corresponding row and column in M_{11} are zero, i.e., if $m_{i'0} = m_1$, then $m_{i'j} = m_{ji'} = 0$, $\forall j \neq i'$.

(3) In M_{11}, m appears at most twice in each row and in each column, and m_1 appears at most once in each row and in each column. And if $m_{ij} = m_{ij'} = m$, then $m_{jj'} = m_1$.

(4) Through row and column operations, M can be transformed into a matrix of the form $\mathrm{diag}(I_{\delta_1} \otimes U, I_{\delta_2} \otimes P, I_{\delta_3} \otimes Q, I_{\delta_4} \otimes T, m_0 I_{n-1-4\delta_1-3\delta_2-2(\delta_3+\delta_4)})$, where

$$U = \begin{bmatrix} m_0 & m & m & m_1 \\ m & m_0 & m_1 & m \\ m & m_1 & m_0 & m \\ m_1 & m & m & m_0 \end{bmatrix}, \quad P = \begin{bmatrix} m_0 & m & m \\ m & m_0 & m_1 \\ m & m_1 & m_0 \end{bmatrix},$$
$$Q = \begin{bmatrix} m_0 & m \\ m & m_0 \end{bmatrix}, \quad T = \begin{bmatrix} m_0 & m_1 \\ m_1 & m_0 \end{bmatrix}.$$

Values of the δs depend on the number of words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation. The determinant of M equals to the determinant of the transformed matrix of M above.

(5) U indicates that the defining relation contains words of the forms $F_1F_iF_j$, $F_2F_iF_{j'}$, $F_2F_{i'}F_j$, $F_1F_{i'}F_{j'}$, $F_1F_2F_jF_{j'}$, and $F_1F_2F_iF_{i'}$. For example, $F_1F_3F_4$, $F_2F_3F_5$, $F_2F_4F_6$, $F_1F_5F_6$, $F_1F_2F_4F_5$, and $F_1F_2F_3F_6$, say.

(6) P indicates that the defining relation contains words of the forms $F_1F_iF_j$, $F_2F_iF_{j'}$, and $F_1F_2F_jF_{j'}$. For example, $F_1F_3F_4$, $F_2F_3F_5$, and $F_1F_2F_4F_5$, say.

(7) T indicates that there are words of the forms $F_1F_2F_iF_j$ or $F_1F_2F_i$ in the defining relation, and Q indicates that there are words of the forms $F_1F_iF_j$ or $F_2F_iF_j$ in the defining relation.

An example is given below to show how values of the δs are determined.

Example 1. Consider 2^{n-p}_{III} designs with three words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation. The followings are four possible designs. Design 1 contains $F_1F_2F_3F_4$, $F_1F_2F_5F_6$, and $F_1F_2F_7F_8$ in the defining relation. Design 2 contains $F_1F_3F_4$, $F_2F_3F_5$, and $F_1F_2F_4F_5$. Design 3 contains $F_1F_3F_4$, $F_2F_5F_6$, and $F_1F_2F_7F_8$. Design 4 contains $F_1F_2F_3$, $F_1F_2F_4F_5$, and $F_1F_2F_6F_7$. For Design 1, $\delta_1 =$

$\delta_2 = \delta_3 = 0$, $\delta_4 = 3$; for Design 2, $\delta_1 = \delta_3 = \delta_4 = 0$, $\delta_2 = 1$; for Design 3, $\delta_1 = \delta_2 = 0$, $\delta_3 = 2$, $\delta_4 = 1$; and for Design 4, $\delta_1 = \delta_2 = \delta_3 = 0$, $\delta_4 = 3$.

In the following we investigate how different forms of the words and values of γ_0 and γ affect the order of designs when the total number of words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation is fixed. Our focus is on D-optimality and the total number of words involving the dispersion factors is less than or equal to six.

Case 1. One word in the defining relation involves the dispersion factors.
According to the values of the δs, there are two possible designs and are listed in Table 1. Designs with word of the form $F_1F_2F_iF_j$ or $F_1F_2F_i$ in the defining relation have the same δs' values. Their information matrices through row and column operations are of the same form and are considered as the same design.

Let $\det(d_i)$ denote the determinant of the information matrix of design d_i. One can show that $\det(d_1) > \det(d_2)$, regardless of the values of γ_0 and γ. Hence, the D-optimality ordering of the designs is $d_1 \succ d_2$, where "$\succ$" indicates that d_1 is better than d_2 in terms of D-optimality. That is, when there is one word in the defining relation containing the dispersion factors, the best word structures are either $F_1F_2F_iF_j$, or $F_1F_2F_i$.

Case 2. Two words in the defining relation involve the dispersion factors.
According to the values of the δs, there are three possible designs and are listed in Table 2. $k \times (F_1F_2F_iF_j)$ means that there are k words of the form $F_1F_2F_iF_j$ in the defining relation, and i and j are all distinct. For example, for $k = 2$, the two words are $F_1F_2F_3F_4$ and $F_1F_2F_5F_6$, say.

One can show that $\det(d_1) > \det(d_2) > \det(d_3)$, regardless of the values of γ_0 and γ. Hence, the D-optimality ordering of the designs is $d_1 \succ d_2 \succ d_3$. That is, when there are two words in the defining relation containing the dispersion factors, the best word structures are either $(F_1F_2F_iF_j, F_1F_2F_{i'}F_{j'})$ or $(F_1F_2F_iF_j, F_1F_2F_{i'})$.

When number of words involving the dispersion factors is more than two, the D-optimality ordering is not strict anymore.

Case 3. Three words in the defining relation involve the dispersion factors.
According to the values of the δs, there are five possible designs and are listed in Table 3. P indicates designs having words of the forms as stated in (6).

One can show that $\det(d_1) > \det(d_4) > \det(d_5)$, and $\det(d_2) > \det(d_3) > \det(d_4) > \det(d_5)$, regardless of the values of γ_0 and γ. However, the overall D-optimality ordering of the above five designs depends also on the values of γ_0 and γ. When $0 < |\gamma| < 0.3827\gamma_0$, the ordering is $d_2 \succ d_3 \succ d_1 \succ d_4 \succ d_5$; when $0.3827\gamma_0 < |\gamma| < 0.4370\gamma_0$, the ordering is $d_2 \succ d_1 \succ d_3 \succ d_4 \succ d_5$; and when $0.4370\gamma_0 < |\gamma| < 0.5\gamma_0$, the ordering is $d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_5$. That is, when the absolute value of the common dispersion main effect is large, the best word structures are $(F_1F_iF_j, F_2F_iF_{j'}, F_1F_2F_jF_{j'})$. However, when the absolute value of the common dispersion main effect is moderate or small, the best word structures are either $(F_1F_2F_iF_j, F_1F_2F_{i'}F_{j'}, F_1F_2F_{i''}F_{j''})$ or $(F_1F_2F_i, F_1F_2F_{i'}F_{j'}, F_1F_2F_{i''}F_{j''})$.

Case 4. Four words in the defining relation involve the dispersion factors.
According to the values of the δs, there are seven possible designs and are listed in Table 4. Now

$$\begin{aligned}
&\det(d_1) > \det(d_2),\\
&\det(d_1) > \det(d_5) > \det(d_6) > \det(d_7),\\
&\det(d_2) > \det(d_6) > \det(d_7), \text{ and}\\
&\det(d_3) > \det(d_4) > \det(d_5) > \det(d_6) > \det(d_7),
\end{aligned}$$

regardless of the values of γ_0 and γ. The overall D-optimality ordering of the above designs, for different values of γ_0 and γ, are given in Table 5. When the absolute value of the common dispersion main effect is large, the best word structures are either $(F_1F_iF_j$, $F_2F_iF_{j'}$, $F_1F_2F_jF_{j'}$, $F_1F_2F_{i''}F_{j''})$ or $(F_1F_2F_i$, $F_1F_{i'}F_j$, $F_2F_{i'}F_{j'}$, $F_1F_2F_jF_{j'})$. However, when the absolute value of the common dispersion main effect is moderate or small, the best word structures are either $(F_1F_2F_iF_j, F_1F_2F_{i'}F_{j'}, F_1F_2F_{i''}F_{j''}, F_1F_2F_{i^*}F_{j^*})$ or $(F_1F_2F_i, F_1F_2F_{i'}F_{j'}, F_1F_2F_{i''}F_{j''}, F_1F_2F_{i^*}F_{j^*})$.

Case 5. Five words in the defining relation involve the dispersion factors.
According to the values of the δs, there are nine possible designs and are listed in

Table 6. Now

$$\det(d_1) > \det(d_2) > \det(d_3),$$
$$\det(d_1) > \det(d_6) > \det(d_7) > \det(d_8) > \det(d_9),$$
$$\det(d_2) > \det(d_7) > \det(d_8) > \det(d_9),$$
$$\det(d_3) > \det(d_8) > \det(d_9), \text{ and}$$
$$\det(d_4) > \det(d_5) > \det(d_6) > \det(d_7) > \det(d_8) > \det(d_9),$$

regardless of the values of γ_0 and γ. The overall D-optimality ordering of the above designs, for different values of γ_0 and γ, are given in Table 7. When the absolute value of the common dispersion main effect is large, the best word structures are either ($F_1F_iF_j$, $F_2F_iF_{j'}$, $F_1F_2F_jF_{j'}$, $F_1F_2F_{i''}F_{j''}$, $F_1F_2F_{i^*}F_{j^*}$) or ($F_1F_2F_i$, $F_1F_{i'}F_j$, $F_2F_{i'}F_{j'}$, $F_1F_2F_jF_{j'}$, $F_1F_2F_{i''}F_{j''}$). However, when the absolute value of the common dispersion main effect is moderate or small, the best word structures are either ($F_1F_2F_iF_j, F_1F_2F_{i'}F_{j'}, F_1F_2F_{i''}F_{j''}, F_1F_2F_{i^*}F_{j^*}, F_1F_2F_{i'''}F_{j'''}$) or ($F_1F_2F_i$, $F_1F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}F_{j''}$, $F_1F_2F_{i^*}F_{j^*}$, $F_1F_2F_{i'''}F_{j'''}$).

Case 6. Six words in the defining relation involve the dispersion factors.
According to the values of the δs, there are 13 possible designs and are listed in Table 8. U indicates designs having words of the forms as stated in (5). Now

$$\det(d_1) > \det(d_2) > \det(d_5) > \det(d_6) > \det(d_{12}) > \det(d_{13}),$$
$$\det(d_3) > \det(d_4) > \det(d_5) > \det(d_6),$$
$$\det(d_3) > \det(d_9) > \det(d_{10}) > \det(d_{11}) > \det(d_{12}) > \det(d_{13}),$$
$$\det(d_4) > \det(d_{10}) > \det(d_{11}) > \det(d_{12}) > \det(d_{13}),$$
$$\det(d_5) > \det(d_{11}) > \det(d_{12}) > \det(d_{13}), \text{ and}$$
$$\det(d_7) > \det(d_8) > \det(d_9) > \det(d_{10}) > \det(d_{11}) > \det(d_{12}) > \det(d_{13}),$$

regardless of the values of γ_0 and γ. The overall D-optimality ordering of the above designs, for different values of γ_0 and γ, are given in Table 9. When the absolute value of the common dispersion main effect is large, the best word structures are ($F_1F_iF_j$, $F_2F_iF_{j'}$, $F_1F_{i'}F_{j'}$, $F_2F_{i'}F_j$, $F_1F_2F_jF_{j'}$, $F_1F_2F_iF_{i'}$). However, when the absolute value of the common dispersion main effect is moderate or small, the best word structures are either $(F_1F_2F_iF_j,\ F_1F_2F_{i'}F_{j'},\ F_1F_2F_{i''}F_{j''},\ F_1F_2F_{i^*}F_{j^*},\ F_1F_2F_{i'''}F_{j'''},\ F_1F_2F_{i^\wedge}F_{j^\wedge})$ or $(F_1F_2F_i,\ F_1F_2F_{i'}F_{j'},\ F_1F_2F_{i''}F_{j''},\ F_1F_2F_{i^*}F_{j^*},\ F_1F_2F_{i'''}F_{j'''},\ F_1F_2F_{i^\wedge}F_{j^\wedge})$.

Example 2. Consider the following four 2^{6-2}_{III} designs. Design d_A has $I = F_1F_2F_5 = F_4F_5F_6 = F_1F_2F_4F_6$, d_B has $I = F_1F_3F_5 = F_1F_2F_4F_6 = F_2F_3F_4F_5F_6$, d_C has $I = F_1F_3F_5 = F_2F_4F_6 = F_1F_2F_3F_4F_5F_6$, and d_D has $I = F_1F_2F_5 = F_1F_3F_6 = F_2F_3F_5F_6$. The numbers of words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relations are all two. The corresponding $(\delta_1, \delta_2, \delta_3, \delta_4)$ values for the four designs are $d_A : (0,0,0,2)$, $d_B : (0,0,1,1)$, $d_C : (0,0,2,0)$, and $d_D : (0,0,1,1)$, respectively. One can see from Table 2 that d_A is of type d_1, d_B and d_D are of type d_2, and d_C is of type d_3. According to the results in Case 2, $d_A \succ d_B = d_D \succ d_C$. That is, regardless of the values of γ_0 and γ, design d_A is D-optimal in estimating the location main effects among all designs having two words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation.

Example 3. Consider the following two 2^{9-4}_{III} designs. Design d_A has generators $F_6 = F_2F_3$, $F_7 = F_1F_3$, $F_8 = F_2F_4$, $F_9 = F_1F_2F_3F_4F_5$, and the corresponding defining relation up to words of length four is $I = F_2F_3F_6 = F_1F_3F_7 = F_2F_4F_8 = F_1F_2F_6F_7 = F_3F_4F_6F_8 = F_5F_7F_8F_9$. Design d_B has generators $F_6 = F_1F_3$, $F_7 = F_2F_4$, $F_8 = F_1F_2F_5$, $F_9 = F_1F_2$, and the corresponding defining relation up to words of length four is $I = F_1F_3F_6 = F_2F_4F_7 = F_1F_2F_9 = F_5F_8F_9 = F_1F_2F_5F_8 = F_1F_4F_7F_9 = F_2F_3F_6F_9$. The numbers of words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relations are all four. The corresponding $(\delta_1, \delta_2, \delta_3, \delta_4)$ values for the two designs are $d_A : (0,1,1,0)$, and $d_B : (0,0,2,2)$. From Table 4 one can see that d_A is of type d_2, and d_B is of type d_5. By Table 5, d_B is better than d_A in terms of D-optimality when $0 < |\gamma| < 0.3827\gamma_0$, otherwise d_A is better among all designs having four words of the forms $F_1F_iF_j$, $F_2F_iF_j$, $F_1F_2F_i$, and $F_1F_2F_iF_j$ in the defining relation.

5. Concluding remarks

D-optimal designs for the estimation of location main effects when two dispersion factors are present in the model depend not only on the number of length three words containing either one or both of the dispersion factors and length four words containing both of the dispersion factors, but also on the word structures in the defining relation, and the values of γ_0 and γ. It seems to the author that when the absolute value of the common dispersion main effect is moderate or small, minimum aberration works well in distinguishing designs. That is, designs having more length four words involving both of the dispersion factors are better. However, when the absolute value of common dispersion main effect is large, having words of the specific structures as stated in (5) and (6) in section 4 results in a better design. That is, having more factors in common for the length three and length four words that contain the dispersion factors gives a better design.

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[Received September 2009; accepted November 2009.]

Table 1 Values of the δs and word structures with one word involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(0, 0, 0, 1)$	$F_1F_2F_iF_j$
		$F_1F_2F_i$
d_2	$(0, 0, 1, 0)$	$F_1F_jF_j$ or $F_2F_iF_j$

Table 2 Values of the δs and word structures with two words involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(0, 0, 0, 2)$	$2 \times (F_1F_2F_iF_j)$
		$F_1F_2F_iF_j$, $F_1F_2F_{i'}$
d_2	$(0, 0, 1, 1)$	$F_1F_2F_iF_j$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$
		$F_1F_iF_j$ or $F_2F_iF_j$, $F_1F_2F_{i'}$
d_3	$(0, 0, 2, 0)$	$2 \times (F_1F_iF_j)$ or $F_2F_iF_j)$

Table 3 Values of the δs and word structures with three words involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(0, 1, 0, 0)$	P
d_2	$(0, 0, 0, 3)$	$3 \times (F_1F_2F_iF_j)$
		$2 \times (F_1F_2F_iF_j)$, $F_1F_2F_{i'}$
d_3	$(0, 0, 1, 2)$	$2 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}(F_2F_{i'}F_{j'})$
		$F_1F_2F_iF_j$, $F_1F_{i'}F_{j'}(F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_4	$(0, 0, 2, 1)$	$F_1F_2F_iF_j$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$2 \times (F_1F_iF_j)$, $F_1F_2F_{i'}$
d_5	$(0, 0, 3, 0)$	$3 \times (F_1F_iF_j$ or $F_2F_iF_j)$

Table 4 Values of the δs and word structures with four words involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(0, 1, 0, 1)$	P, $F_1F_2F_{i'}F_{j'}$
		P, $F_1F_2F_{i'}$
d_2	$(0, 1, 1, 0)$	P, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$
d_3	$(0, 0, 0, 4)$	$4 \times (F_1F_2F_iF_j)$
		$3 \times (F_1F_2F_iF_j)$, $F_1F_2F_{i'}$
d_4	$(0, 0, 1, 3)$	$3 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$
		$2 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}$
d_5	$(0, 0, 2, 2)$	$2 \times (F_1F_2F_iF_j)$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$F_1F_2F_iF_j$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_6	$(0, 0, 3, 1)$	$F_1F_2F_iF_j$, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$3 \times (F_1F_iF_j$ or $F_2F_iF_j)$, $F_1F_2F_{i'}$
d_7	$(0, 0, 4, 0)$	$4 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$

Table 5 D-optimality ordering with four words involving the dispersion factors.

Interval	D-optimality ordering
$0< \lvert\gamma\rvert <0.3827\gamma_0$	$d_3 \succ d_4 \succ d_1 \succ d_5 \succ d_2 \succ d_6 \succ d_7$
$0.3827\gamma_0< \lvert\gamma\rvert <0.4370\gamma_0$	$d_3 \succ d_1 \succ d_4 \succ d_2 \succ d_5 \succ d_6 \succ d_7$
$0.4370\gamma_0< \lvert\gamma\rvert <0.4611\gamma_0$	$d_1 \succ d_3 \succ d_2 \succ d_4 \succ d_5 \succ d_6 \succ d_7$
$0.4611\gamma_0< \lvert\gamma\rvert <0.5\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_5 \succ d_6 \succ d_7$

Table 6 Values of the δs and word structures with five words involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(0, 1, 0, 2)$	P, $2 \times (F_1F_2F_{i'}F_{j'})$
		P, $F_1F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}$
d_2	$(0, 1, 1, 1)$	P, $F_1F_2F_{i'}F_{j'}$, $F_1F_{i''}F_{j''}$ or $F_2F_{i''}F_{j''}$
		P, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}$
d_3	$(0, 1, 2, 0)$	P, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
d_4	$(0, 0, 0, 5)$	$5 \times (F_1F_2F_iF_j)$
		$4 \times (F_1F_2F_iF_j)$, $F_1F_2F_{i'}$
d_5	$(0, 0, 1, 4)$	$4 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$
		$3 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}$
d_6	$(0, 0, 2, 3)$	$3 \times (F_1F_2F_iF_j)$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$2 \times (F_1F_2F_iF_j)$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_7	$(0, 0, 3, 2)$	$2 \times (F_1F_2F_iF_j)$, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$F_1F_2F_iF_j$, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_8	$(0, 0, 4, 1)$	$F_1F_2F_iF_j$, $4 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$4 \times (F_1F_iF_j$ or $F_2F_iF_j)$, $F_1F_2F_{i'}$
d_9	$(0, 0, 5, 0)$	$5 \times (F_1F_iF_j$ or $F_2F_iF_j)$

Table 7 D-optimality ordering with five words involving the dispersion factors.

Interval	D-optimality ordering
$0 < \lvert\gamma\rvert < 0.3827\gamma_0$	$d_4 \succ d_5 \succ d_1 \succ d_6 \succ d_2 \succ d_7 \succ d_3 \succ d_8 \succ d_9$
$0.3827\gamma_0 < \lvert\gamma\rvert < 0.4370\gamma_0$	$d_4 \succ d_1 \succ d_5 \succ d_2 \succ d_6 \succ d_3 \succ d_7 \succ d_8 \succ d_9$
$0.4370\gamma_0 < \lvert\gamma\rvert < 0.4611\gamma_0$	$d_1 \succ d_4 \succ d_2 \succ d_5 \succ d_3 \succ d_6 \succ d_7 \succ d_8 \succ d_9$
$0.4611\gamma_0 < \lvert\gamma\rvert < 0.4744\gamma_0$	$d_1 \succ d_2 \succ d_4 \succ d_3 \succ d_5 \succ d_6 \succ d_7 \succ d_8 \succ d_9$
$0.4744\gamma_0 < \lvert\gamma\rvert < 0.5\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_5 \succ d_6 \succ d_7 \succ d_8 \succ d_9$

Table 8 Values of the δs and word structures with six words involving the dispersion factors.

Design	$(\delta_1, \delta_2, \delta_3, \delta_4)$	Word Structures
d_1	$(1, 0, 0, 0)$	U
d_2	$(0, 2, 0, 0)$	$2 \times P$
d_3	$(0, 1, 0, 3)$	P, $3 \times (F_1F_2F_{i'}F_{j'})$
		P, $2 \times (F_1F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_4	$(0, 1, 1, 2)$	P, $2 \times (F_1F_2F_{i'}F_{j'})$, $F_1F_{i''}F_{j''}$ or $F_2F_{i''}F_{j''}$
		P, $F_1F_2F_{i'}F_{j'}$, $F_1F_{i''}F_{j''}$ or $F_2F_{i''}F_{j''}$, $F_1F_2F_{i^*}$
d_5	$(0, 1, 2, 1)$	P, $F_1F_2F_{i'}F_{j'}$, $2 \times (F_1F_{i''}F_{j''}$ or $F_2F_{i''}F_{j''})$
		P, $2 \times (F_1F_{i'}F_{j'}$ or $F_1F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_6	$(0, 1, 3, 0)$	P, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
d_7	$(0, 0, 0, 6)$	$6 \times (F_1F_2F_iF_j)$
		$5 \times (F_1F_2F_iF_j)$, $F_1F_2F_{i'}$
d_8	$(0, 0, 1, 5)$	$5 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$
		$4 \times (F_1F_2F_iF_j)$, $F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'}$, $F_1F_2F_{i''}$
d_9	$(0, 0, 2, 4)$	$4 \times (F_1F_2F_iF_j)$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$3 \times (F_1F_2F_iF_j)$, $2 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_{10}	$(0, 0, 3, 3)$	$3 \times (F_1F_2F_iF_j)$, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$2 \times (F_1F_2F_iF_j)$, $3 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_{11}	$(0, 0, 4, 2)$	$2 \times (F_1F_2F_iF_j)$, $4 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$F_1F_2F_iF_j$, $4 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$, $F_1F_2F_{i''}$
d_{12}	$(0, 0, 5, 1)$	$F_1F_2F_iF_j$, $5 \times (F_1F_{i'}F_{j'}$ or $F_2F_{i'}F_{j'})$
		$5 \times (F_1F_iF_j$ or $F_1F_iF_j)$, $F_1F_2F_{i'}$
d_{13}	$(0, 0, 6, 0)$	$6 \times (F_1F_iF_j$ or $F_2F_iF_j)$

Table 9 D-optimality ordering with six words involving the dispersion factors.

Interval	D-optimality ordering
$0< \|\gamma\| <0.2692\gamma_0$	$d_7 \succ d_8 \succ d_3 \succ d_9 \succ d_4 \succ d_{10} \succ d_1 \succ d_2 \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.2692\gamma_0< \|\gamma\| <0.3059\gamma_0$	$d_7 \succ d_8 \succ d_3 \succ d_9 \succ d_4 \succ d_1 \succ d_{10} \succ d_2 \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.3059\gamma_0< \|\gamma\| <0.3173\gamma_0$	$d_7 \succ d_8 \succ d_3 \succ d_9 \succ d_1 \succ d_4 \succ d_{10} \succ d_2 \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.3173\gamma_0< \|\gamma\| <0.3383\gamma_0$	$d_7 \succ d_8 \succ d_3 \succ d_9 \succ d_1 \succ d_4 \succ d_2 \succ d_{10} \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.3383\gamma_0< \|\gamma\| <0.3743\gamma_0$	$d_7 \succ d_8 \succ d_3 \succ d_1 \succ d_9 \succ d_4 \succ d_2 \succ d_{10} \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.3743\gamma_0< \|\gamma\| <0.3771\gamma_0$	$d_7 \succ d_8 \succ d_1 \succ d_3 \succ d_9 \succ d_4 \succ d_2 \succ d_{10} \succ d_5 \succ d_{11} \succ d_6 \succ d_{12} \succ d_{13}$
$0.3771\gamma_0< \|\gamma\| <0.3827\gamma_0$	$d_7 \succ d_1 \succ d_8 \succ d_3 \succ d_9 \succ d_4 \succ d_2 \succ d_{10} \succ d_5 \succ d_6 \succ d_{11} \succ d_{12} \succ d_{13}$
$0.3827\gamma_0< \|\gamma\| <0.4025\gamma_0$	$d_7 \succ d_1 \succ d_3 \succ d_8 \succ d_2 \succ d_4 \succ d_9 \succ d_5 \succ d_{10} \succ d_6 \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4025\gamma_0< \|\gamma\| <0.4163\gamma_0$	$d_1 \succ d_7 \succ d_3 \succ d_8 \succ d_2 \succ d_4 \succ d_9 \succ d_5 \succ d_{10} \succ d_6 \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4163\gamma_0< \|\gamma\| <0.4370\gamma_0$	$d_1 \succ d_7 \succ d_3 \succ d_2 \succ d_8 \succ d_4 \succ d_9 \succ d_5 \succ d_6 \succ d_{10} \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4370\gamma_0< \|\gamma\| <0.4611\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_7 \succ d_4 \succ d_8 \succ d_5 \succ d_6 \succ d_9 \succ d_{10} \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4611\gamma_0< \|\gamma\| <0.4744\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_7 \succ d_5 \succ d_8 \succ d_6 \succ d_9 \succ d_{10} \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4744\gamma_0< \|\gamma\| <0.4825\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_5 \succ d_7 \succ d_6 \succ d_8 \succ d_9 \succ d_{10} \succ d_{11} \succ d_{12} \succ d_{13}$
$0.4825\gamma_0< \|\gamma\| <0.5\gamma_0$	$d_1 \succ d_2 \succ d_3 \succ d_4 \succ d_5 \succ d_6 \succ d_7 \succ d_8 \succ d_9 \succ d_{10} \succ d_{11} \succ d_{12} \succ d_{13}$