

Bayesian Inference for Credit Risk with Serially Dependent Factor Model

Yi-Ping Chang¹, Chih-Tun Yu² and Huimei Liu²

¹Soochow University and ²National Chengchi University

Abstract

Default probability and asset correlation are key factors in determining credit default risk in loan portfolios. Therefore, many articles have been devoted to the study in quantifying default probability and asset correlation. However, the classical estimation methods (e.g. MLE) usually use only historical data and often underestimate the default probability in special situations, such as the occurrence of a financial crisis. By contrast, the Bayesian method is seen to be a more viable alternative to solving such estimation problems. In this paper, we consider the Bayesian approach by applying Markov chain Monte Carlo (MCMC) techniques in estimating default probability and asset correlation under serially dependent factor model. The empirical results and out-of-sample forecasting for S&P default data provide strong evidence to support that the serially dependent factor model is reliable in determining credit default risk.

Keywords: Default probability, asset correlation, serially dependent factor model, Bayesian inference.

1. Introduction

Starting from mid-2007, the subprime credit crisis sweep across global financial markets. The crisis can be attributed to a number of pervasive factors in both the housing and credit derivative markets (Diamond and Rajan [7]), which start to emerge over the past few years. A detailed of the crisis can be found in Soros [27] and Crouhy et al. [4]. This financial crisis results in a significantly higher global default rate, making bond issuers and loan obligors face mounting financial pressure, in the weakening economy. Risk management models that use historical data relationships in assessing the risk assume the risk is driven by a statistical rule. For example, they assume the historical data relationships represent a good basis for forecasting the development of future risks. However, the global financial turmoil has evidently revealed the associated serious flaws when one relies solely on such an approach. Models that use only historical data tend to underestimate the default probability when a special financial issue occurs. Thus,

the negative impacts from a slackening economy can not be timely forecast. With additional subjective judgments, the Bayesian estimator proves to be a feasible choice for the estimation of default probability. Some subjective and professional judgments can compensate the insufficiency of historical data in credit risk modeling. (Kiefer [18, 19, 20])

Over the past decades, several models for measuring credit portfolio risk have been developed. One of them is the independent factor model (Vasicek [29]). This model incorporates the idea that the dependence among individual obligors is driven by a common factor. The independent factor model has been widely used in the credit risk field, as well as been applied in estimating and pricing the economic capital allocations, loss rate distribution on a credit portfolio, and credit derivatives (e.g., Gupton et al. [16], Gordy [13], Schönbucher [26], Liao et al. [22], Wu et al. [31] and others). It is also suggested in KMV's portfolio ManagerTM, CreditMetricsTM and the Basel Committee (BCBS [1]). Another important model in credit default risk modeling is the generalized linear mixed models (GLMMs). McNeil and Wendin [23] and Czado and Pflüger [5] highlight the usefulness of GLMMs in the modelling of portfolio credit risk. The utilities of GLMMs can be maximized by including macroeconomic variable and systematic random effects to construct a credit risk model.

In credit risk modeling, various estimation methods have been proposed for quantifying default probability. These estimation methods can be summed up into two broad categories. First, many studies use the classical (or non-Bayesian) approach. Gordy and Heitfield [12] use maximum likelihood estimation method to estimate default probability and asset correlation. Hanson and Schuermann [17] utilize both parametric and nonparametric bootstrap methods to estimate default probability. The classical method usually only takes historical data into account, which often underestimates default probability during periods of economic recession. On the other hand, McNeil and Wendin [23] consider the Bayesian approach by applying MCMC techniques to model credit risk under the GLMMs. The model was further extended by Czado and Pflüger [5]. McNeil and Wendin [23] and Czado and Pflüger [5] use non-informative priors for the unknown parameter of GLMM. In this paper, we discuss the prior behavior of the prior distribution set-up under GLMMs in McNeil and Wendin [23]. Separately, Gössl [14] and Dwyer [8] also apply the Bayesian method with MCMC techniques under the independent factor model, while Kiefer [18, 19, 20] focus on the Binomial model and incorporate expert information in quantifying default probability. But the fact that Dwyer [8] and Kiefer [18, 19, 20] assume asset correlation as non-random was a key insufficiency. Moreover, Kiefer [21] extend Binomial model into independent factor model. But, the property of autocorrelation of factor have not considered in Kiefer [21].

In this paper, similar to the approach by Gössl [14], we consider the Bayesian approach by applying MCMC technique to estimate default probability and asset correlation under serially dependent factor model. Adopting the serially dependent factor model rather independent factor model is major difference between Gössl [14] and our study. The original set-up in the independent factor model assumes an i.i.d. factor for different time periods. A number of studies (Ebnöther and Vanini [9], McNeil and

Wendin [23], and Czado and Pflüger [5]) suggest that the factor is related with macroeconomic variables. Moreover, they suggest that the common factor might follow a first order autoregressive time series model (AR(1)). Thus, we use the serially dependent factor model instead of the independent factor model with the result of out-of-sample forecasting for supporting the serially dependent factor model is more reliable than the independent factor model. We also provide an empirical study by using Standard and Poor's default data. Furthermore, the capital cushion of 2009 annual S&P portfolio is also predicted in this paper.

This paper is organized as follows. The serially dependent factor model is introduced in Section 2. Bayesian estimation and MCMC techniques are described in Section 3. Section 4 shows the empirical study of Standard and Poor's default data, the result of out-of-sample forecasting, and capital cushion quantifying. The conclusion is summarized in Section 5.

2. The Serially Dependent Factor Model

The serially dependent factor model is introduced in this section. The asset value log-return, $X_{t,k,i}$, in time period t , i th obligor of rating category k , can be written as

$$X_{t,k,i} = \sqrt{\rho_k} Z_t + \sqrt{1 - \rho_k} \varepsilon_{t,k,i}, \quad (1)$$

$$Z_t = \theta Z_{t-1} + \sqrt{1 - \theta^2} \nu_t, \quad i = 1, \dots, n_{t,k}, \quad k = 1, \dots, K, \quad t = 1, \dots, T, \quad (2)$$

where $n_{t,k}$, K , and T are the number of obligors of rating category k in time period t , rating categories, and time period, respectively. Let $\varepsilon_{t,1,k}, \dots, \varepsilon_{t,k,n_{t,k}}$ be i.i.d. $N(0, 1)$, ν_1, \dots, ν_T be i.i.d. $N(0, 1)$, $\varepsilon_{t,k,i}$ independent of ν_t for all t, k , and i , $Z_0 \sim N(0, 1)$, and Z_0 independent of ν_t , $t = 1, \dots, T$. The parameter ρ_k can be referred to the asset correlation of rating category k between $X_{t,k,i}$ and $X_{t,k,j}$ for i and j . The designation factor indicates that there is only one common factor Z_t to all obligors and the factor is assumed to be a first order autoregressive, AR(1), time series. The AR(1) time series is a Markov process that has a Gaussian stationary distribution with mean 0 and variance 1 for $|\theta| < 1$. If θ is positive and close to 1, then the factor Z_{t-1} last time period has a positive and notable influence on the factor Z_t for this time period and vice versa. Moreover, if θ equal to 0, the serially dependent factor model is equivalent to the independent factor model. Hence θ is an important parameter to determine the leverage between Z_t and Z_{t-1} . There is a significant difference between our model and the independent factor model used in Gössl [14], which assumes Z_t independent for all t with standard normal distribution. In this paper, we feel that the dependence between Z_t and Z_{t-1} exists.

The occurrence of default is equivalent to the case when the log return of an obligor's assets of rating category k falls below a threshold value $c_{t,k,i}$, i.e.,

$$Y_{t,k,i} = I(X_{t,k,i} < c_{t,k,i}). \quad (3)$$

Here $Y_{t,k,i}$ is the default indicator. The default probability $p_{t,k,i}$ indicate that the probability of i th obligor of rating category k default in time period t , so that $p_{t,k,i}$ is

$$p_{t,k,i} = P(Y_{t,k,i} = 1)$$

$$=P(X_{t,k,i} < c_{t,k,i}). \quad (4)$$

In this paper, we assume that $p_{t,k,i}$ is the same between every obligor of rating category k for all time periods, i.e., $p_{t,k,i} = p_k$ for all t and i . Similarly, it is assumed that $c_{t,k,i} = c_k$.

By serially dependent factor model setup, $Z_1 = \theta Z_0 + \sqrt{1 - \theta^2} \nu_1$ is given. According to $Z_0 \sim N(0, 1)$, $\nu_1 \sim N(0, 1)$, Z_0 independent of ν_1 (model assumptions), and normal additive property of normal random variable, we obtain $Z_1 \sim N(0, 1)$. Again we know $Z_2 = \theta Z_1 + \sqrt{1 - \theta^2} \nu_2$. When ν_2 independent of ν_1 and Z_0 (model assumptions), we obtain ν_2 independent of Z_1 . Then $Z_2 \sim N(0, 1)$ according to Z_1 independent of ν_2 , $Z_1 \sim N(0, 1)$, $\nu_2 \sim N(0, 1)$, and normal additive property of normal random variable. To continue the same process, the $Z_t \sim N(0, 1)$, $t = 1, \dots, T$, is obtained. Then the distribution of $X_{t,k,i}$ follow $N(0, 1)$ by additive property of normal random variable. Thus, we obtain the threshold value c_k equal to $\Phi^{-1}(p_k)$. Given the realization of p_k , ρ_k and Z_t , we assume that the default indicators $Y_{t,k,1}, \dots, Y_{t,k,n_{t,k}}$ are conditionally independent Bernoulli distribution with common conditional default probability can be expressed as

$$\begin{aligned} g_t(p_k, \rho_k, Z_t) &= P(Y_{t,k,i} = 1 | Z_t) \\ &= P(X_{t,k,i} < c_k | Z_t) \\ &= P\left(\varepsilon_{t,k,i} \leq \frac{c_k - \sqrt{\rho_k} Z_t}{\sqrt{1 - \rho_k}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(p_k) - \sqrt{\rho_k} Z_t}{\sqrt{1 - \rho_k}}\right). \end{aligned} \quad (5)$$

For convenience, we denote $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_K)$, each component of that vector is a vector $\mathbf{Y}_k = (\mathbf{Y}_{1,k}, \dots, \mathbf{Y}_{T,k})$, where $\mathbf{Y}_{t,k} = (Y_{t,k,1}, \dots, Y_{t,k,n_{t,k}})$ and $\mathbf{p} = (p_1, \dots, p_K)$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_K)$, and $\mathbf{Z} = (Z_1, \dots, Z_T)$. By assumption, the joint conditional pdf of $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_K)$ is

$$f(\mathbf{y} | \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) = \prod_{t=1}^T \prod_{k=1}^K \left[g_t(p_k, \rho_k, Z_t)^{\sum_{i=1}^{n_{t,k}} y_{t,k,i}} (1 - g_t(p_k, \rho_k, Z_t))^{n_{t,k} - \sum_{i=1}^{n_{t,k}} y_{t,k,i}} \right], \quad (6)$$

here, $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_K)$, each component of that vector is a vector $\mathbf{y}_k = (\mathbf{y}_{1,k}, \dots, \mathbf{y}_{T,k})$, where $\mathbf{y}_{t,k} = (y_{t,k,1}, \dots, y_{t,k,n_{t,k}})$. Notice that the joint conditional pdf of \mathbf{Y} in (6) is a key device for Bayesian estimation in this paper.

3. Bayesian Estimation and MCMC

In the serially dependent factor model, the unknown parameters are \mathbf{p} , $\boldsymbol{\rho}$, \mathbf{Z} , and θ . By Bayes rule, the corresponding joint posterior density function of the unknown parameters is

$$f(\mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}, \theta | \mathbf{y}) = \frac{f(\mathbf{y}, \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}, \theta)}{f(\mathbf{y})}. \quad (7)$$

By assuming the independence of \mathbf{p} , $\boldsymbol{\rho}$ and θ , the joint pdf $f(\mathbf{y}, \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}, \theta)$ can be written as

$$f(\mathbf{y}, \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}, \theta) = f(\mathbf{y}|\mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \times f(\mathbf{p}) \times f(\boldsymbol{\rho}) \times f(\mathbf{Z}|\theta) \times f(\theta), \quad (8)$$

where the joint distribution of K categories default probability is written as

$$f(\mathbf{p}) = \prod_{k=1}^K f(p_k), \quad (9)$$

and the joint distribution of K categories asset correlation is written as

$$f(\boldsymbol{\rho}) = \prod_{k=1}^K f(\rho_k), \quad (10)$$

and

$$\begin{aligned} f(\mathbf{Z}|\theta) &= f(Z_1|\theta) \times f(Z_2|Z_1, \theta) \times f(Z_3|Z_1, Z_2, \theta) \times \cdots \times f(Z_T|Z_1, \dots, Z_{T-1}, \theta) \\ &= f(Z_1|\theta) \times f(Z_2|Z_1, \theta) \times f(Z_3|Z_2, \theta) \times \cdots \times f(Z_T|Z_{T-1}, \theta), \end{aligned} \quad (11)$$

where

$$f(Z_1|\theta) = E_{Z_0} [f(Z_0) \times f(Z_1|Z_0, \theta)]. \quad (12)$$

Note that it is difficult to compute $f(Z_1|\theta)$. For simplicity, we approximate $f(\mathbf{Z}|\theta)$ as

$$f(\mathbf{Z}|\theta) \approx f(Z_2|Z_1, \theta) \times f(Z_3|Z_2, \theta) \times \cdots \times f(Z_T|Z_{T-1}, \theta). \quad (13)$$

Unfortunately, the posterior distribution is generally unobtainable by analytical means, and only its function form is known. A major limitation towards more widespread implementation of Bayesian approaches is that obtaining the posterior distribution often requires the integration of high-dimensional functions. We focus on Markov Chain Monte Carlo method, which attempt to simulate direct draws from some complex distribution. In this paper, we apply the Gibbs sampler that is particularly well-adapted to sample the posterior distribution of a Bayesian approach. It is applicable when the joint distribution is not explicitly known, but the conditional distribution of each variable is already known. The goal of the Gibbs sampling algorithm is to generate a sample from the distribution of each successive variable, based on the current values of the other variables. The purpose of such a sequence is to compute an integral (such as an expected value) in approximating the joint distribution. For more details of the Gibbs sampler, please refer to Robert and Casella [25], Dagpunar [6], Rachev et al. [24] and Greenberg [15]. The full conditional distribution is a key aspect for the application of the Gibbs sampler. By Equation (8), we can obtain the full conditional distribution for each unknown parameter. Here, we treat the components of \mathbf{p} , $\boldsymbol{\rho}$, and \mathbf{Z} sequentially. First, the full conditional distribution of p_k is

$$f(p_k|\mathbf{y}, \mathbf{p}_{-k}, \boldsymbol{\rho}, \mathbf{Z}, \theta) \propto f(\mathbf{y}|\mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \times f(p_k), \quad (14)$$

Table 1: Descriptive basic statistics of default rate in S&P historical default data for each rating category.

	Mean	S.D.	Min	Median	Max
A	0.00068	0.00116	0.00000	0.00000	0.00380
BBB	0.00255	0.00265	0.00000	0.00220	0.01000
BB	0.01097	0.01066	0.00000	0.00760	0.04220
B	0.05012	0.03269	0.00240	0.03269	0.13840
CCC	0.23559	0.12850	0.00000	0.23080	0.48421

Note: S.D. is the standard deviation, Min and Max stand for minimum and maximum.

where $\mathbf{p}_{-k} = (p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_K)$ and the full conditional distribution of ρ_k is

$$f(\rho_k | \mathbf{y}, \mathbf{p}, \boldsymbol{\rho}_{-k}, \mathbf{Z}, \theta) \propto f(\mathbf{y} | \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \times f(\rho_k), \quad (15)$$

where $\boldsymbol{\rho}_{-k} = (\rho_1, \dots, \rho_{k-1}, \rho_{k+1}, \dots, \rho_K)$. Moreover, the full conditional distribution of Z_t is

$$\begin{aligned} f(Z_t | \mathbf{y}, \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}_{-t}, \theta) &\propto f(\mathbf{y} | \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \times f(Z_t | \mathbf{Z}_{-t}, \theta) \\ &\propto f(\mathbf{y} | \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \times f(Z_t | Z_{t-1}, \theta), \end{aligned} \quad (16)$$

where $\mathbf{Z}_{-t} = (Z_1, \dots, Z_{t-1}, Z_{t+1}, \dots, Z_T)$. The last full conditional distribution of the parameter θ is

$$f(\theta | \mathbf{y}, \mathbf{p}, \boldsymbol{\rho}, \mathbf{Z}) \propto f(\mathbf{Z} | \theta) \times f(\theta). \quad (17)$$

We apply the acceptance method (Geweke [11], section 3.2) to simulate the aforementioned univariate full conditional distributions. Booth and Hobert [2] and Wu [30] also use the acceptance method to obtain random samples from the full conditional distribution. We provide the steps of the acceptance method in appendix A. The customized code in C has been used to accomplish the acceptance method in this paper.

4. An Empirical Study of S&P Historical Default Data

The default data used in this paper were published in Standard & Poor's default report (Standard & Poor's [28]). The data covers a 29 year period from 1981 to 2009 in seven rating categories: 'AAA', 'AA', 'A', 'BBB', 'BB', 'B', and 'CCC'. The data includes the number of obligors in a particular rating grade at the beginning of a year, and the number of defaults that occur by year's end. As there was only three default for the top two grades, our analysis does not contain the rating category AAA and AA (Gössl [14], McNeil and Wendin [23]). The basic statistics of the default rate for each rating category are listed in Table 1.

4.1. The Choice of Prior Distribution

In this subsection, we describe the prior distributions for the parameters used in the serially dependent factor model. Without expert information supply, we use most common approach of using non-informative priors for the parameters of our model. The prior distribution of the default probability and asset correlation are given as Uniform distribution restricted to $(0, 1)$ for each rating category, i.e., $p_k \sim U(0, 1)$ and $\rho_k \sim U(0, 1)$, $k = 1, \dots, K$. The prior distribution sets for the autoregressive parameter θ is given as $U(-1, 1)$.

Another issue addressed here is related to the prior distribution set-up discussed in McNeil and Wendin [23]. They consider the Bayesian approach by applying the MCMC technique in modeling the credit risk under GLMMs. They also gave non-informative priors for the intercepts, regression coefficients, and hyperparameters of GLMMs. In this paper, we discuss the prior behavior of the prior distribution set-up under GLMMs in McNeil and Wendin [23]. To address this problem, we derive the joint pdf of default probability and asset correlation implied in McNeil and Wendin [23], where the result is displayed in Figure 1. Note that we describe Figure 1 in one rating category. The detailed derivation for the joint density of default probability and asset correlation implied in McNeil and Wendin [23] is described in the appendix B. From Figure 1, we can see that, given a constant asset correlation, the marginal shape of default probability appears as a “U” shape. More precise, for a fixed asset correlation, the default probability is more likely to happen near 0 or 1. Especially, when the fixed asset correlation is close to either 0 or 1, the phenomenon is more clear. Similar, for a fixed value default probability, the marginal shape of asset correlation also appears as a “U” shape. Both of the “U” shape phenomena are irrational and contradict to practical intuition. Kiefer [18, 19, 20] combines the expert information with the data information to calculate the prior distribution of default probability in Binomial model. Moreover, Kiefer [21] extend this work into independent factor model. The shape of prior distribution of default probability set-up discussed in Kiefer [18, 19, 20, 21] is unimodal. Thus, we feel that the shapes of prior distribution of default probability don’t appear “U” shape. Following this result, we adopt the serially dependent factor model instead of GLMMs in this paper.

4.2. Empirical results

We use the MCMC method to provide iterative procedures to approximate samples from our complicated posterior densities. The trace plots and the plots of autocorrelation functions for \mathbf{p} , $\boldsymbol{\rho}$, and θ under serially dependent factor model are respectively presented in Figures 2 to 6. Obviously, the simulation stabilizes after approximately 10,000 iterations. Thus, we set the burn-in period to be 10,000.

Table 2 summarizes the posterior mean, standard derivation (S.D.), median, 2.5% and 97.5% quantiles for the estimates of \mathbf{p} , $\boldsymbol{\rho}$, and θ under serially dependent factor model. Table 3 also lists the estimates of \mathbf{p} and $\boldsymbol{\rho}$ under independent factor model (Gössl [14]). One can see that our posterior estimates for \mathbf{p} are higher than that under independent factor model for all rating categories. The phenomena may be due to the

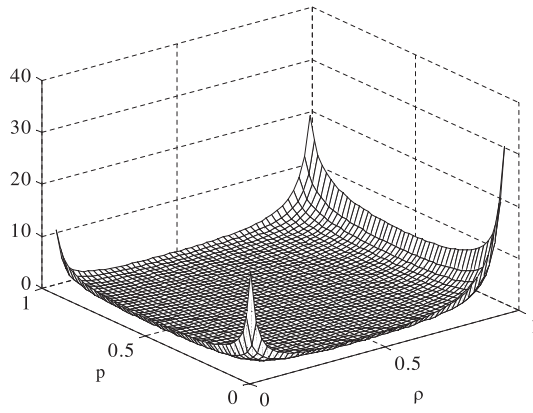


Figure 1: Joint pdf of default probability p and asset correlation ρ corresponding to McNeil and Wendin [23].

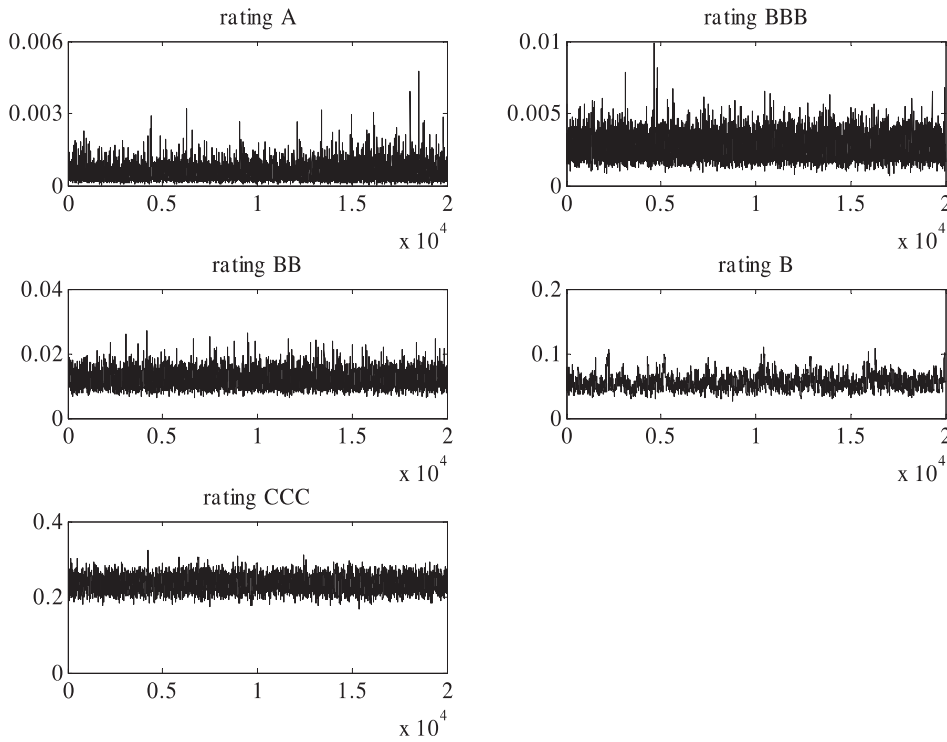


Figure 2: The trace plots for p under serially dependent factor model.

serial dependence of S&P default data captured by serially dependent factor model. The histograms of p , ρ , and θ under serially dependent factor model for each rating category are presented in Figures 7, 8, and 9. For the histograms of p , this may not be surprising, as the mode of density is close to 0 for the investment rating category. The shapes of the histograms of p and ρ are both skewed to the right for each rating category.

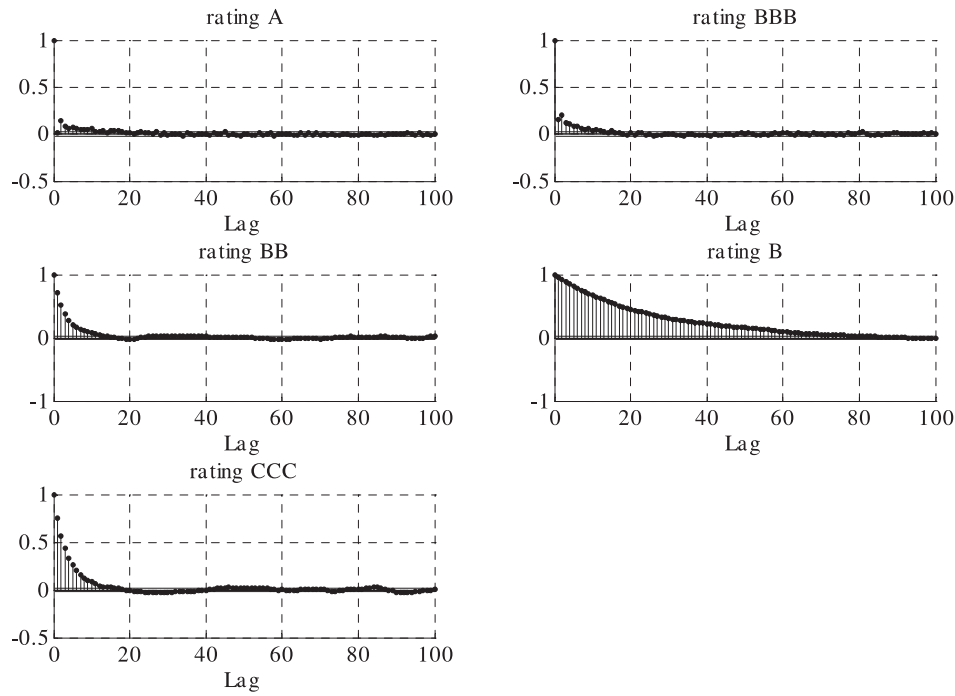
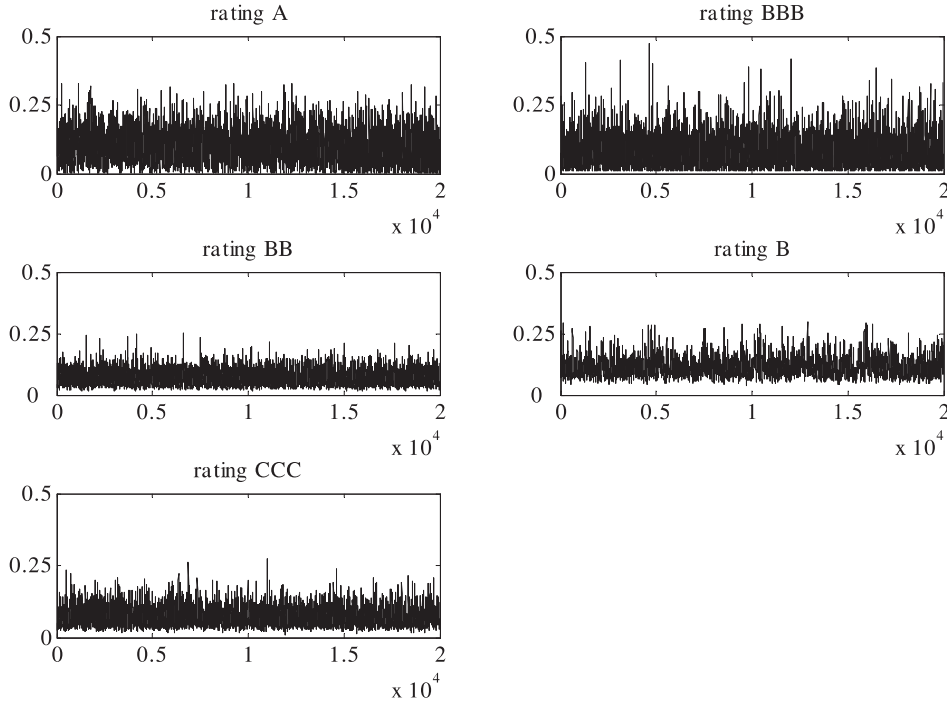


Figure 3: The plots of autocorrelation functions for \mathbf{p} under serially dependent factor model.

Table 2: Posterior mean, standard deviation, median, 2.5% and 97.5% quantiles of \mathbf{p} , $\boldsymbol{\rho}$, and θ based on S&P default data 1981-2009 under serially dependent factor model.

Rating	\mathbf{p}				
	Mean	S.D.	2.5%	Median	97.5%
A	0.0005	0.0003	0.0001	0.0005	0.0013
BBB	0.0027	0.0007	0.0017	0.0026	0.0046
BB	0.0121	0.0023	0.0094	0.0119	0.0185
B	0.0561	0.0110	0.0390	0.0545	0.0824
CCC	0.2348	0.0176	0.2140	0.2344	0.2914
	$\boldsymbol{\rho}$				
	Mean	S.D.	2.5%	Median	97.5%
A	0.0877	0.0505	0.0136	0.0832	0.2129
BBB	0.0768	0.0461	0.0112	0.0690	0.1823
BB	0.0696	0.0256	0.0309	0.0655	0.1296
B	0.1164	0.0374	0.0639	0.1096	0.2078
CCC	0.0724	0.0266	0.0339	0.0682	0.1416
	θ				
	Mean	S.D.	2.5%	Median	97.5%
All rating	0.4987	0.1609	0.1455	0.5143	0.7678

Moreover, the left skewness is observed in the histogram of θ . This is the evidence of the serial dependence existing in the S&P historical default data.

Figure 4: The trace plots for ρ under serially dependent factor model.Table 3: Posterior mean, standard deviation, median, 2.5% and 97.5% quantiles of \mathbf{p} and $\boldsymbol{\rho}$ based on S&P default data 1981-2009 under independent factor model.

Rating	\mathbf{p}				
	Mean	S.D.	2.5%	Median	97.5%
A	0.0004	0.0002	0.0001	0.0004	0.0011
BBB	0.0022	0.0005	0.0015	0.0021	0.0042
BB	0.0100	0.0019	0.0084	0.0098	0.0172
B	0.0500	0.0080	0.0378	0.0491	0.0803
CCC	0.1954	0.0167	0.2011	0.1948	0.2720
	$\boldsymbol{\rho}$				
	Mean	S.D.	2.5%	Median	97.5%
A	0.0859	0.0442	0.0111	0.0811	0.2085
BBB	0.0779	0.0423	0.0119	0.0714	0.1840
BB	0.0713	0.0247	0.0312	0.0677	0.1306
B	0.1096	0.0315	0.0622	0.1048	0.2052
CCC	0.0668	0.0248	0.0326	0.0624	0.1375

4.3. Out-of-sample forecasts

In this subsection, we give the out-of-sample forecasting to support the serially dependent factor model is more reliable than the independent factor model. We use S&P default data from 1981 to 2008 to compute how likely the observed 2009 annual S&P

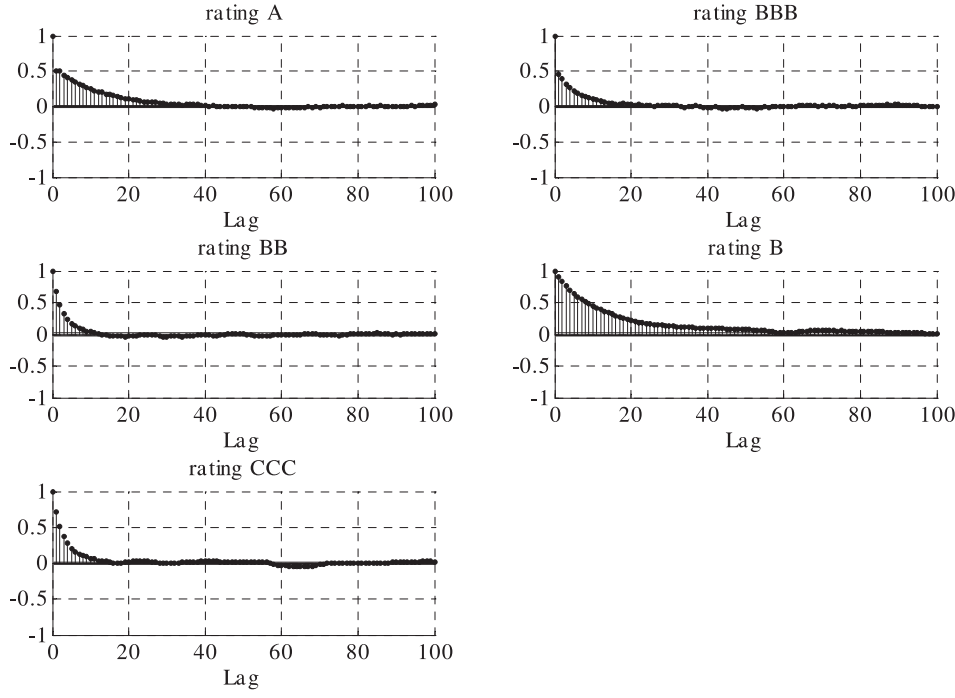


Figure 5: The plots of autocorrelation functions for ρ under serially dependent factor model.

default data is under both models. The observed 2009 annual S&P default data is given in Table 4. One can see that the annual default rates in 2009 are larger than average default rate from 1981 to 2008 in each rating category except rating BB. In this paper, we use conditional predictive ordinate (CPO) (Gelfand [10]) to achieve the out-of-sample forecasting. Czado and Pflüger [5] also use CPO to compare the forecasting accuracy of the credit risk model. The CPO of rating category k for 2009 is defined as

$$\text{CPO}_{2009,k} := P\left(\mathbf{Y}_{2009,k}^{(\text{obs})} | \mathbf{Y}_{t,k}^{(\text{obs})}, t \neq 2009\right), \quad (18)$$

where $\mathbf{Y}_{t,k}^{(\text{obs})} = (Y_{t,k,1}^{(\text{obs})}, Y_{t,k,2}^{(\text{obs})}, \dots, Y_{t,k,n_{t,k}}^{(\text{obs})})$. The conditional predictive ordinate suggests how likely the joint observation $\mathbf{Y}_{2009,k}^{(\text{obs})}$ is, when the model is fitted to all observations from 1981 to 2008 except $\mathbf{Y}_{2009,k}^{(\text{obs})}$. Obviously a good model should have large $\text{CPO}_{2009,k}$, $k = 1, \dots, K$. Note that $\text{CPO}_{2009,k}$, $k = 1, \dots, K$ can be calculated from MCMC output. The relevant summary statistics are listed in Table 5. From Table 5, the serially dependent factor model have larger $\log(\text{CPO}_{2009,k})$ for all rating categories. Obviously the serially dependent factor model outperform than the independent factor model. This is strong evidence supporting the serially dependent factor model is more reliable than the independent factor model.

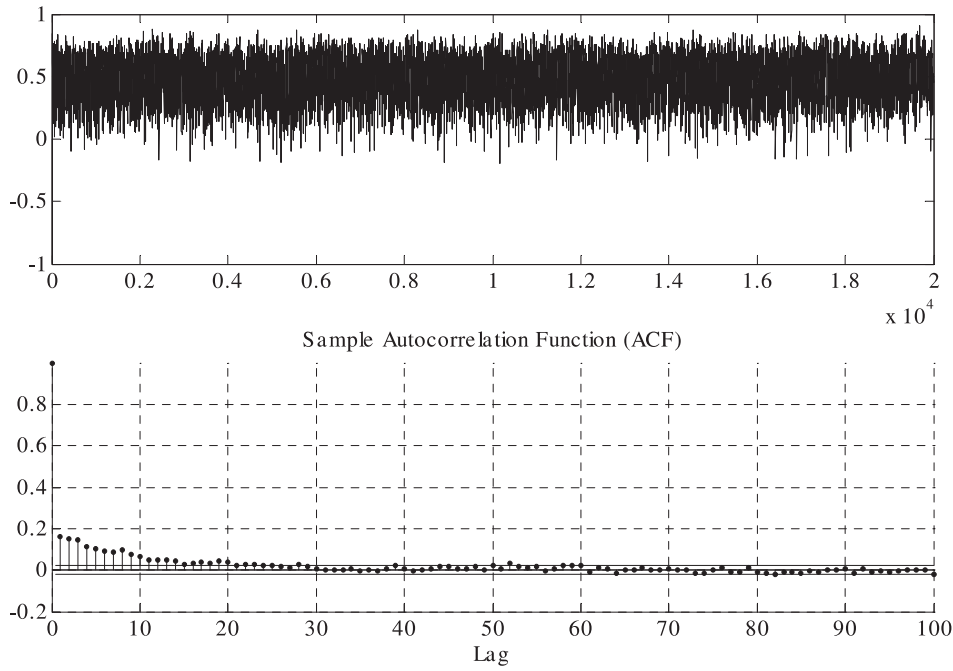


Figure 6: The trace plot and plot of autocorrelation function for θ under serially dependent factor model.

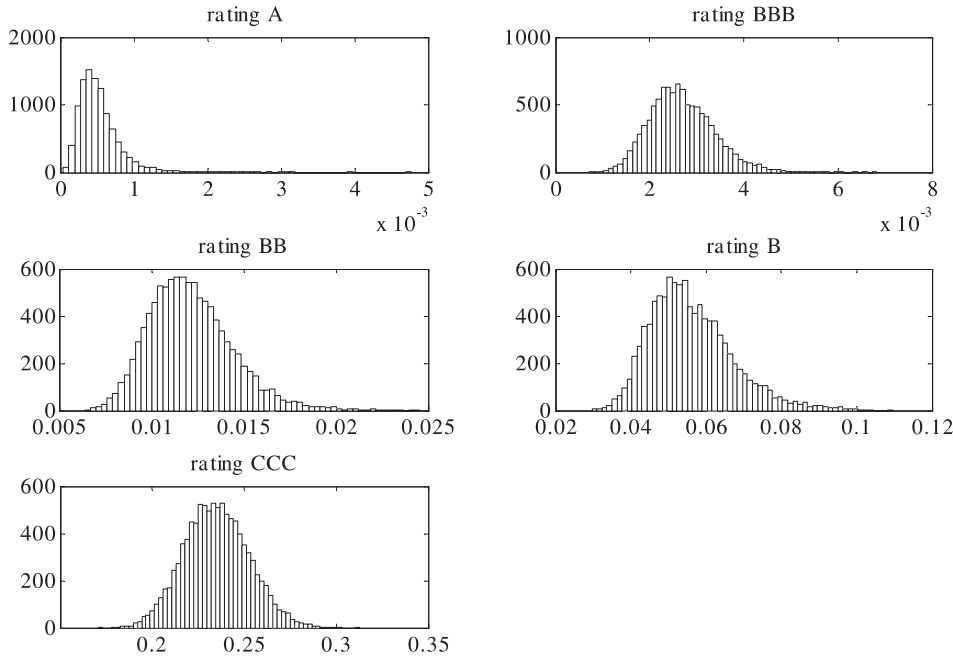


Figure 7: The histograms for p under serially dependent factor model.

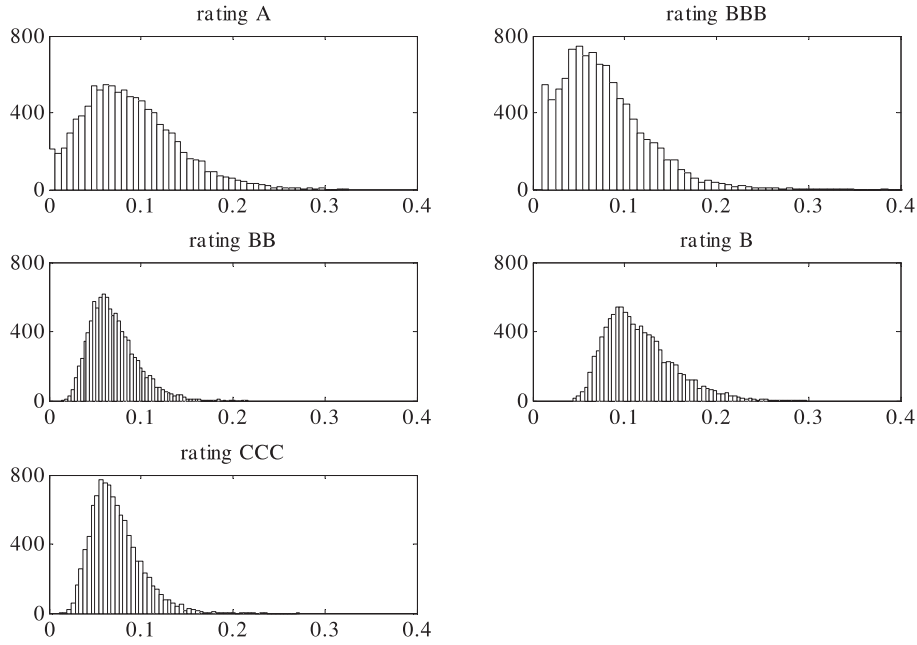


Figure 8: The histograms for ρ under serially dependent factor model.

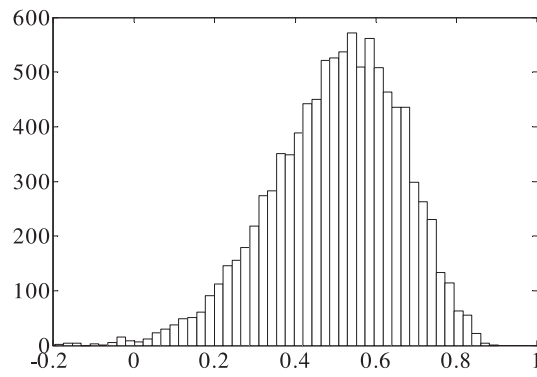


Figure 9: The histogram for θ under serially dependent factor model.

Table 4: The observed 2009 annual S&P default data and default rate.

Rating	Obligors	Defaults	Default Rate
A	1396	3	0.0021
BBB	1498	8	0.0053
BB	1002	7	0.0070
B	1223	124	0.1014
CCC	190	92	0.4842
Total	5309	265	0.0499

Table 5: $\log(\text{CPO}_{2009,k})$ using 1981-2008 S&P default data.

	$\log(\text{CPO}_{2009,k})$				
	A	BBB	BB	B	CCC
Independent factor model	-1.4244	-1.7162	-1.1984	-2.5912	-3.0120
Serially dependent factor model	-1.3880	-1.6551	-1.1570	-2.5211	-2.9287

4.4. Quantify capital cushion

In this paper, we also use S&P default data from 1981 to 2008 to predict the capital cushion of 2009 annual S&P portfolio. The most common way to quantify capital cushion is the concept of economic capital (EC). For this purpose we assume that the 2009 annual S&P portfolio loss can be defined as the random variable

$$\tilde{L}_{2009} = \sum_{k=1}^K \sum_{i=1}^{n_{2009,k}} \text{EAD}_{2009,k,i} \times \text{LGD}_{2009,k,i} \times Y_{2009,k,i}, \quad (19)$$

where $\text{EAD}_{2009,k,i}$, $\text{LGD}_{2009,k,i}$, and $Y_{2009,k,i}$ represent the “Exposure at Default”, “Loss given Default”, and the default indicator of obligor i of rating category k in time period 2009, respectively as well as $n_{2009,k}$ is number of obligors of rating category k in 2009 annual S&P portfolio. Then the expected loss (EL) of the portfolio is defined as

$$\text{EL} = \sum_{k=1}^K \sum_{i=1}^{n_{2009,k}} \text{EAD}_{2009,k,i} \times \text{LGD}_{2009,k,i} \times E[Y_{2009,k,i}], \quad (20)$$

where $E[Y_{2009,k,i}] = p_k$. The α -quantile of \tilde{L}_{2009} is called Value-at-Risk (VaR) of the portfolio, i.e.,

$$\text{VaR} = \inf\{\text{VaR} > 0 | P(\tilde{L}_{2009} \leq \text{VaR}) \geq \alpha\}. \quad (21)$$

Note that the VaR is a value of assessing the maximum probable loss of the investment portfolio over a specific period of time under certain level of confidence (Chang et al. [3]). Then EC is defined as the VaR minus the EL of the portfolio, determined by

$$\text{EC} = \text{VaR} - \text{EL}. \quad (22)$$

One can see that EL, VaR, and EC are based on the portfolio loss variable \tilde{L}_{2009} that can be estimated using Monte Carlo simulations with MCMC outputs. The detailed simulation steps are shown in the appendix C. Under each MCMC iteration, we obtain one EL, one VaR, and one EC. In this paper, the number of EC simulation is 10000. Thus, an posterior distribution of EC can be obtained from 10000 EC. Note that, the relevant parameters are $\alpha = 99.9\%$, $\text{EAD}_{2009,k,i} = 1.0$, and $\text{LGD}_{2009,k,i} = 1.0$ for all i and k in these simulations. In Figure 10, we plot the estimated quantile of posterior distribution of EC under different models and 2009 annual S&P portfolio total loss. The x-axis of Figure 10 is the quantile level β and the y-axis is the β -quantile of EC.

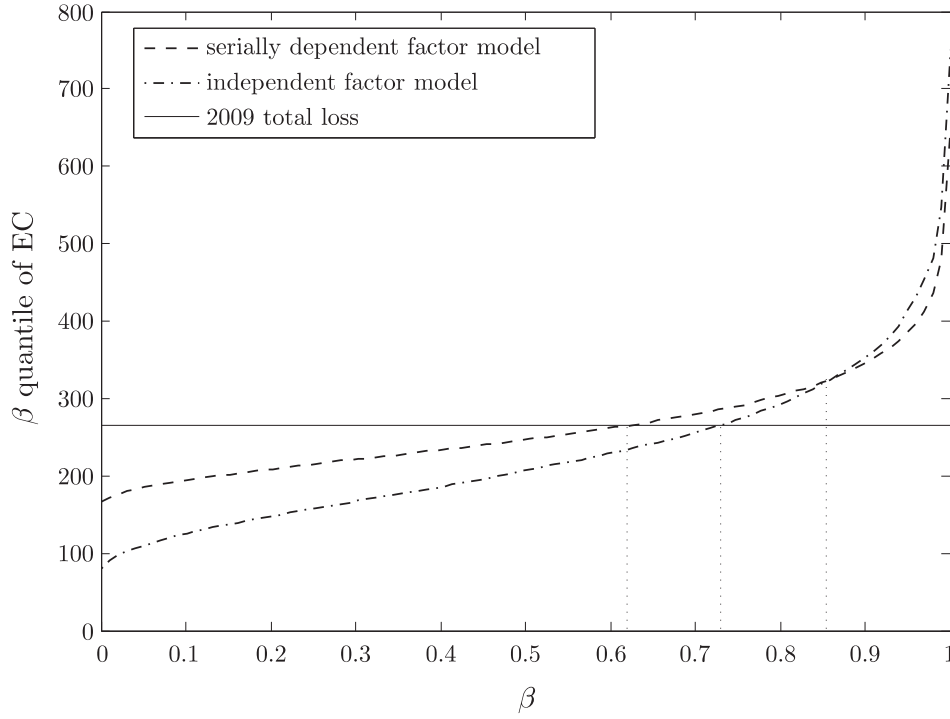


Figure 10: The estimated quantile of posterior distribution of EC ($EC_{\beta}^{(\cdot)}$) under different model and 2009 annual S&P portfolio total loss.

Under an estimated quantile of posterior distribution of EC, if banks or the supervisors of banks choose the higher quantile level β as criterion, it means they are conservative about the economic capital. On the contrary, if the banks or the supervisors of banks choose the lower quantile level β as criterion, it means the banks or the supervisors of banks have liberal view about the economic capital. Moreover, we can obtain different estimated quantile of posterior distribution of EC from different models. Under the fixed quantile level β , we compare the β -quantile of EC of different models. The lower β -quantile of EC means that the banks take more default risk. That is, if the β -quantile of EC can't cover the portfolio total loss, the banks go bankrupt. But, the lower β -quantile of EC leads the banks increase the profit from investment. On the contrary, the higher β -quantile of EC means that the banks decrease the circulation of capital assets and the profit from investment. But it also leads the banks suffer less default risk even if the sudden excess loss happen.

We use three dotted lines to separate Figure 10 into four parts. Note that the x-axis coordinates of the three dotted line are 0.64, 0.74, and 0.88 respectively. For simplicity, the estimated β -quantile of EC under serially dependent factor model is denoted as $EC_{\beta}^{(D)}$. Similarly, the estimated β -quantile of EC under independent factor model is denoted as $EC_{\beta}^{(I)}$. We are aware of at least four distinct facts from Figure 10.

1. When β belongs to $(0, 0.64)$, the $EC_{\beta}^{(D)}$ is larger than $EC_{\beta}^{(I)}$. Both values are smaller than 2009 total loss. This reflects both β -quantile of ECs can't cover the 2009 annual S&P portfolio total loss. Even so, β -quantile of EC under serially dependent factor model is closer to the 2009 annual S&P portfolio total loss. Thus, we realize that the serially dependent factor model provide better estimation for EC than the independent factor model in this β interval.
2. When β belongs to $(0.64, 0.74)$. The $EC_{\beta}^{(D)}$ is also larger than $EC_{\beta}^{(I)}$. The $EC_{\beta}^{(I)}$ is still smaller than 2009 total loss. Obviously, the serially dependent factor model provides a precise and close estimation for the 2009 annual S&P portfolio total loss in this situation. But, banks will be exposed to the huge default risk under independent factor model. Thus, the serially dependent factor model also has better prediction ability than the independent factor model in this β interval.
3. When β belongs to $(0.74, 0.88)$, the $EC_{\beta}^{(D)}$ is larger than $EC_{\beta}^{(I)}$. Both values are larger than 2009 total loss. It means that both β -quantile of ECs can total cushion the 2009 annual S&P portfolio total loss. It seems that the independent factor model has better prediction ability than the serially dependent factor model in this situation. However, we observe that β -quantile of EC under independent factor model just cover the 2009 annual S&P portfolio total loss. If the 2009 annual S&P portfolio total loss increases 10% or 20% due to unexpected bankruptcy, most of β -quantile of ECs under independent factor model can't cover the excess loss. Note that all β -quantile of ECs under serially dependent factor model could cover the excess loss. Furthermore, as β increases, the difference of β -quantile of ECs between the serially factor model and the independent factor model become smaller. Thus, the serially dependent factor model could supply good estimations of EC to banks.
4. When β belongs to $(0.88, 1.0)$, the $EC_{\beta}^{(I)}$ is larger than $EC_{\beta}^{(D)}$. Both values are larger than 2009 total loss, indicating banks would overestimate the 2009 annual S&P portfolio total loss under both models. It leads to banks to hold more capital and decrease the circulations of capital assets and the profit from investment. Moreover, the serially dependent factor model would provide better estimation for EC than the independent factor model.

In above statements, we list the predicting scenarios under each model in the different β intervals. Obviously, the serially dependent factor model has better prediction ability than the independent factor model in most quantiles level β . This is another evidence supporting the serially dependent factor model is more reliable than the independent factor model.

Typically, the banks wish to hold smaller capital cushion and increase the circulations of capital assets. Then banks will choose a small β -quantile of EC to estimate the 2009 annual S&P portfolio total loss. On the contrary, the supervisors of banks try to regulate that banks should hold more capital to undertake the credit risk, which means that the supervisors of banks would like to choose a high β -quantile of EC to cover the 2009

annual S&P portfolio total loss. Thus, the choice of quantile level β depends on position of chooser.

5. Conclusions

Considering the severe impact that the subprime credit crisis has caused on global financial markets, it highlights the fact that many studies seriously underestimate the risk of obligors, especially for those with high quality rating. A key issue is that only historical data are used in estimating default probability and asset correlation. Finding other methods in assigning the default probability and asset correlation to the particular high quality ratings is necessary. In this paper, we chose the Bayesian approach and show the feasibility of the Bayesian method in estimating the serially dependent factor model. Several interesting findings were made from our empirical study. First, it shows the evidence of profound serial dependence in S&P historical default data. Second, comparisons of the empirical results between Gössl [14] and our research reveals that the default probability are underestimated when the serially dependence caused from the cyclical state of economic, which can not be covered in independent factor model. Moreover, the out-of-sample forecasting for S&P default data provide strong evidence to support that the serially dependent factor model is more reliable than the independent factor model. Furthermore, we provide the results of quantify capital cushion under serially dependent factor model and independent factor model for banks and their supervisors. Using more general prior distributions in estimating the default probability is an important issue for further work.

Acknowledgement

We sincerely express our thankfulness to the anonymous referee for the valuable comments. This research was supported in part by the National Science Council, Taiwan, ROC, under NSC 100-2118-M-031-001.

Appendix A.

In Section 3, we claim that the acceptance method can simulate the random samples from the full conditional distributions. For example, suppose that we want to generate random samples from $f(p_k|\mathbf{y}, \mathbf{p}_{-k}, \boldsymbol{\rho}, \mathbf{Z}, \theta)$, i.e., the full conditional distribution of p_k . Note that,

$$f(p_k|\mathbf{y}, \mathbf{p}_{-k}, \boldsymbol{\rho}, \mathbf{Z}, \theta) \propto f(\mathbf{y}_{\mathbf{k}}|p_k, \rho_k, \mathbf{Z}) \times f(p_k). \quad (\text{A1})$$

We denote $h(p_k) = f(\mathbf{y}_{\mathbf{k}}|p_k, \rho_k, \mathbf{Z})$. According to Geweke [11], a random sample from $f(p_k|\mathbf{y}, \mathbf{p}_{-k}, \boldsymbol{\rho}, \mathbf{Z}, \theta)$ can be obtained as follows.

Step 1: Sampling $p_k^{(*)}$ from $f(p_k)$, and independently, sample w from the uniform distribution on the interval $(0, 1)$.

Step 2: If $w \leq \frac{h(p_k^{(*)})}{\sup_{p_k} h(p_k)}$ then accept $p_k^{(*)}$, otherwise, go to Step 1.

Samples from the other full conditional distributions can be obtained in a similar way.

Appendix B.

In section 4.1, we claim the joint density of default probability and asset correlation implied in McNeil and Wendin [23]. In appendix B, the detailed derivation has been described. First of all, we show the conversion between the independent factor model and the GLMM. In order to simplify the notation we describe the conversion between the independent factor model and the GLMM in one rating category. Specifically, in the independent factor model,

$$X_{ti} = \sqrt{\rho}Z_t + \sqrt{1-\rho}\epsilon_{ti}, \quad i = 1, \dots, n_t, \quad t = 1, \dots, T. \quad (\text{B1})$$

By dividing both sides by $\sqrt{1-\rho}$, we have

$$X_{ti}^* = wZ_t + \epsilon_{ti}, \quad (\text{B2})$$

where $X_{ti}^* = X_{ti}/\sqrt{1-\rho}$ and $w = \sqrt{\rho/(1-\rho)}$. This model (B2) is a special case of GLMMs and corresponds to model 1 discussed in McNeil and Wendin [23]. Next, we derive the joint distribution of default probability p and asset correlation ρ from (B2). McNeil and Wendin [23] assume both p and ρ depend on parameter w and the threshold parameter c . The relationship can be described as

$$w = \sqrt{\frac{\rho}{1-\rho}} \quad \text{and} \quad c = \sqrt{1+w^2}\Phi^{-1}(p). \quad (\text{B3})$$

McNeil and Wendin [23] consider that the prior distribution of the parameter w owns a inverse-gamma distribution with parameter v and η . Moreover, the threshold parameters c used in their paper is given a zero-mean Gaussian distribution with variance τ^2 , where $\tau = 100$. At the same time, w is independent of c . Thus, the joint pdf for w and c becomes

$$f(w, c) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1}{2\tau^2}c^2\right) \times \frac{v^\eta}{\Gamma(\eta)} w^{-\eta-1} \exp\left(\frac{-v}{w}\right), \quad (\text{B4})$$

for a transformation of (w, c) to (p, ρ) , we have $f(p, \rho)$ as

$$\begin{aligned} f(p, \rho) = & \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{1}{2\tau^2}\left(\Phi^{-1}(p)\sqrt{\frac{1}{1-\rho}}\right)^2\right) \times \frac{v^\eta}{\Gamma(\eta)} \left(\sqrt{\frac{\rho}{1-\rho}}\right)^{-\eta-1} \\ & \times \exp\left(-v\sqrt{\frac{1-\rho}{\rho}}\right) \times \frac{1}{1-\rho^2} \left(\frac{\pi}{2\rho}\right)^{-1/2} \exp\left(\frac{1}{2}(\Phi^{-1}(p))^2\right). \end{aligned} \quad (\text{B5})$$

McNeil and Wendin [23] choose the value of parameter (v, η) as $(0, 0)$. In Figure 1, the joint pdf $f(p, \rho)$ is calculated by choosing parameter $(v, \eta) = (10^{-4}, 10^{-4})$.

Appendix C.

In section 4.4, we discuss that EL, VaR, and EC can be estimated using Monte Carlo simulations with MCMC outputs. In the appendix C, the detail Monte Carlo simulations steps have been described. Firstly, we define some symbols. The MCMC samples using 1981-2008 S&P default data under serially dependent factor model are denoted as $(p_k^{(j)}, \rho_k^{(j)}, \theta^{(j)}, Z_{1981}^{(j)}, \dots, Z_{2008}^{(j)} : 1 \leq j \leq G \text{ and } 1 \leq k \leq K)$, where G is the number of MCMC iterations. For simplicity, the MCMC samples under independent factor model is also denoted as $(p_k^{(j)}, \rho_k^{(j)}, Z_{1981}^{(j)}, \dots, Z_{2008}^{(j)} : 1 \leq j \leq G \text{ and } 1 \leq k \leq K)$. Then the simulation steps of EL, VaR, and EC are shown in the following.

Step 1: Firstly, we generate $Y_{2009,k,i}^{(j)}, \quad i = 1, \dots, n_{2009,k}, \quad k = 1, \dots, K$ from $Bernoulli(g_{2009}(p_k^{(j)}, \rho_k^{(j)}, Z_{2009}^{(j)}))$, where

$$g_{2009}(p_k^{(j)}, \rho_k^{(j)}, Z_{2009}^{(j)}) = \Phi\left(\frac{\Phi^{-1}(p_k^{(j)}) - \sqrt{\rho_k^{(j)}} Z_{2009}^{(j)}}{\sqrt{1 - \rho_k^{(j)}}}\right). \quad (C1)$$

The value of $Z_{2009}^{(j)}$ can be generated as following

$$Z_{2009}^{(j)} = \theta^{(j)} \times Z_{2008}^{(j)} + \sqrt{1 - (\theta^{(j)})^2} \nu^{(j)}, \quad (C2)$$

under serially dependent factor model and $Z_{2009}^{(j)} = \nu^{(j)}$ under independent factor model, where random sample $\nu^{(j)}$ generate from $N(0, 1)$.

Step 2: Compute

$$\tilde{L}_{2009}^{(j)} = \sum_{k=1}^K \sum_{i=1}^{n_{2009,k}} \text{EAD}_{2009,k,i} \times \text{LGD}_{2009,k,i} \times Y_{2009,k,i}^{(j)}. \quad (C3)$$

Step 3: Repeat Step 1 to Step 2 N times, we obtain $N \tilde{L}_{2009}^{(j)}$ under j th MCMC iteration.

Note that $N \tilde{L}_{2009}^{(j)}$ become an empirical loss distribution. In this paper, we take $N = 10000$.

Step 4: Compute the VaR, the α -quantile of the empirical loss distribution and EL, the mean of the empirical loss distribution.

Step 5: The economic capital under j th MCMC iteration can then be obtained by

$$\text{EC}^{(j)} = \text{VaR}^{(j)} - \text{EL}^{(j)}. \quad (C4)$$

References

- [1] Basel Committee on Banking Supervision., *International convergence of capital measurement and capital standards: A revised framework – comprehensive version*, Consultative Document, Bank for International Settlements, 2006.
<http://www.bis.org/publ/bcbs128.pdf>.
- [2] Booth, J. G. and Hobert, J. P., *Maximizing generalized linear mixed models likelihoods with an automated Monte Carlo EM algorithm*, Journal of the Royal Statistical Society Series B, Vol.61, No.1, pp.265-285, 1999.
- [3] Chang, Y. P., Hung, M. C., Wang, S. F. and Yu, C. T., *An EM algorithm for multivariate NIG distribution and its application to Value-at-Risk*, International Journal of Information and Management Sciences, Vol.21, No.3, pp.265-283, 2010.
- [4] Crouhy, M. G., Jarrow, R. A. and Turnbull, S. M., *The subprime credit crisis of 2007*, The Journal of Derivatives, Vol.16, No.1, pp.81-110, 2008.
- [5] Czado, C. and Pflüger, C., *Modeling dependencies between rating categories and their effects on prediction in a credit risk portfolio*, Applied Stochastic Models in Business and Industry, Vol.24, No.3, pp.237-259, 2008.
- [6] Dagpunar, J. S., *Simulation and Monte Carlo – with Applications in Finance and MCMC*, Wiley, New York, 2007.
- [7] Diamond, D. W. and Rajan, R. G., *The credit crisis: Conjectures about causes and remedies*, American Economic Review, Vol.99, No.2, pp.606-610, 2009.
- [8] Dwyer, D. W., *The distribution of defaults and Bayesian model validation*, Journal of Risk Model Validation, Vol.1, No.1, pp.23-53, 2007.
- [9] Ebnother, S. and Vanini, P., *Credit portfolios: What defines risk horizons and risk measurement?*, Journal of Banking & Finance, Vol.31, No.12, pp.3663-3679, 2007.
- [10] Gelfand, A., *Model determination using sampling-based methods*, in: W. Gilks, S. Richardson, and D. Spiegelhalter, (eds.), *Markov Chain Monte Carlo in Practice*, Chapman & Hall, London, pp.145-161, 1996.
- [11] Geweke, J., *Monte carlo simulation and numerical integration*, in: H. M. Amman, D. A. Kendrick, and J. Rust, (eds.), *Handbook of Computational Economics*, North-Holland, Amsterdam, pp.731-800, 1996.
- [12] Gordy, M. and Heitfield, E., *Estimating default correlation from short panels of credit rating*, Working Paper, Federal Reserve Board, 2002.
- [13] Gordy, M. B., *A risk-factor model foundation for ratings-based capital rules*, Journal of Financial Intermediation, Vol.12, No.3, pp.199-232, 2003.
- [14] Gössl, M., *Predictions based on certain uncertainties – a Bayesian credit portfolio approach*, Discuss Paper, HypoVereinsbank, 2005.
- [15] Greenberg, E., *Introduction to Bayesian Econometrics*, Cambridge University Press, New York, 2007.
- [16] Gupton, G., Finger, C. and Bhatia, M., *CreditMetricsTM, technical document*, CreditMetrics, 1997.
- [17] Hanson, S. and Schuermann, T., *Confidence intervals for probabilities of default*, Journal of Banking & Finance, Vol.30, No.8, pp.2281-2301, 2006.
- [18] Kiefer, N. M., *The probability approach to default probabilities*, Risk, Vol.20, No.7, pp.146-150, 2007.
- [19] Kiefer, N. M., *Default estimation for low-default portfolios*, Journal of Empirical Finance, Vol.16, No.1, pp.164-173, 2009.
- [20] Kiefer, N. M., *Default estimation and expert information*, Journal of Business and Economic Statistics, Vol.28, No.2, pp.320-328, 2010.
- [21] Kiefer, N. M., *Default estimation, correlated defaults, and expert information*, Journal of Applied Econometrics, Vol.26, No.2, pp.173-192, 2011.
- [22] Liao, S. L., Chen M. S. and Li, F. C., *A factor-copula based valuation of synthetic CDO-squared under a stochastic intensity*, International Journal of Information and Management Sciences, Vol.20, No.1, pp.103-120, 2009.
- [23] McNeil, A. J. and Wendin, J. P., *Bayesian inferences for generalized linear mixed models of portfolio credit risk*, Journal of Empirical Finance, Vol.14, No.2, pp.131-149, 2007.

- [24] Rachev, S. T., Hsu, J. S. J., Bagasheva, B. S. and Fabozzi, F. J., *Bayesian Methods in Finance*, Wiley, New York, 2008.
- [25] Robert, C. and Casella, G., *Monte Carlo Statistical Models*, Springer, New York, 2004.
- [26] Schönbucher, P., *Factor models: Portfolio credit risks when defaults are correlated*, *Journal of Risk Finance*, Vol.3, No.1, pp.45-56, 2001.
- [27] Soros, G., *The New Paradigm for Financial Markets: The Credit Crisis of 2008 and What it Means*, PublicAffairs, New York, 2008.
- [28] Standard & Poor's., *Default, transition, and recovery: 2009 annual global corporate default study and rating transitions*, Technical Report, Global Fixed Income Research, 2009.
- [29] Vasicek, O., *Loan portfolio value*, *Risk*, Vol.15, No.12, pp.160-162, 2002.
- [30] Wu, L., *Non-linear mixed-effect models with non-ignorably missing covariates*, *The Canadian Journal of Statistics*, Vol.32, No.1, pp.27-37, 2004.
- [31] Wu, P. C., Kao, L. J. and Lee, C. W., *How issuer default risk affects basket credit linked note coupon rate*, *International Journal of Information and Management Sciences*, Vol.22, No.1, pp.59-71, 2011.

Department of Financial Engineering and Actuarial Mathematics, Soochow University, 56, Kueiyang St., Sec. 1, Taipei, Taiwan 100, R.O.C.

E-mail: ypchang@scu.edu.tw

Major area(s): Financial risk management, quantitative finance, and statistical inference.

Department of Statistics, National Chengchi University, 64, Sec. 2, ZhiNan Road, Wenshan District, Taipei, Taiwan 116, R.O.C.

E-mail: 95354501@nccu.edu.tw

Major area(s): Financial risk management, quantitative finance, and statistical inference.

Department of Statistics, National Chengchi University, 64, Sec. 2, ZhiNan Road, Wenshan District, Taipei, Taiwan 116, R.O.C.

E-mail: hliu@nccu.edu.tw

Major area(s): Financial risk management, quantitative finance, and statistical inference.

(Received January 2010; accepted May 2011)