Joint Inventory and Pricing Decisions for Perishable Products with Two-Period Lifetime

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Abstract: We consider a periodic review model over a finite horizon for a perishable product with fixed lifetime equal to two review periods. The excess demand in a period is backlogged. The optimal replenishment and demand management (using price) decisions for such a product depend on the relative order of consumption of fresh and old units. We obtain insights on the structure of these decisions when the order of consumption is first-in, first-out and last-in, first-out. For the FIFO system, we also obtain bounds on both the optimal replenishment quantity as well as expected demand. We compare the FIFO system to two widely analyzed inventory systems that correspond to nonperishable and one-period lifetime products to understand if demand management would modify our understanding of the relationship among the three systems. In a counterintuitive result, we find that it is more likely that bigger orders are placed in the FIFO system than for a nonperishable product when demand is managed. Naval Research Logistics 60: 343–366, 2013

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1. INTRODUCTION

In this article, we examine the joint replenishment and demand management decisions for a perishable product with fixed lifetime equal to two review periods. A perishable product is characterized by its usefulness over a limited period of time. Once its lifetime is over, the usefulness of the product declines rapidly. The cost impact of spoilage due to perishability is massive. For example, the \$1.7-billion apple industry in the US loses as much as \$300 million every year to spoilage [24]. Similarly, it is estimated that the top 40 retailers in the US dump as much as 500 million pounds of food every year due to spoilage [13]. However, spoilage is not limited to produce or consumer goods alone; several industrial products also have a limited lifetime. For example, Chen [7] and Karaesmen et al. [15] mention that adhesive materials used for plywood panels lose their strength within 7 days.

Obviously, spoilage is a loss, and the bottom line of a firm can improve significantly if some of this spoilage is prevented, that is, if the perishable nature of products is managed properly. A mechanism by which this may be achieved is demand management using price. (Throughout this article, we consider only price as a lever to modulate demand. As a result, the terms "demand management" and "dynamic pricing" mean the same in the context of this work.) Through an appropriate selection of price, demand can be modulated to improve profit. The modulation of demand can not only increase revenue but also reduce shortage, holding, and spoilage costs. Potential spoilage due to limited lifetime of perishable products is the main reason demand management for them is even more important than for nonperishable products. For nonperishable products, the only cost of unsold inventory is the cost of holding inventory. For perishable products, the unsold inventory not only incurs holding cost but, in addition, with increasing age of the inventory, the risk of it remaining unsold by the end of its lifetime increases.

For such a product, it is well-understood that the profit during the planning horizon (and hence the optimal replenishment policy) depends on the relative order in which units arrive and are consumed [20]. In the existing literature on perishable products with multiperiod lifetime, two scenarios are commonly modeled. In the first scenario, inventory is consumed in the order of first-in, first-out (FIFO). This scenario is primarily realized when the manufacturer manipulates or controls the order of inventory consumption so that older units are consumed before new units. The condition that the manufacturer determines the order of inventory consumption may be satisfied in both business-to-business (B2B)

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as well as business-to-consumer (B2C) settings. This condition is satisfied in a B2B setting if the manufacturer selects the units to ship to a customer. For example, in a vendor managed inventory system, the manufacturer selects which units to deliver to a customer. Similarly, an example of a B2C setting is online grocery stores, such as NetGrocer.com and bigbasket.com, where the grocer picks inventory and delivers it to consumers' homes.

The second scenario is the opposite of the first scenario. In the second scenario, inventory is consumed in the order of last-in, first-out (LIFO). This scenario occurs when customers prefer (and are able to obtain) the freshest units available. Customers may prefer to obtain the freshest unit available because such a unit has the most time-to-expiry, which reduces the risk of spoilage. The risk of spoilage before sale, however, increases for the manufacturer in this scenario. In a B2B setting, inventory may be consumed in the LIFO order if customers pick up inventory from the manufacturer's warehouse themselves. The LIFO consumption order may be realized even when the manufacturer is responsible for delivering inventory if the manufacturer is contractually required to ship the freshest units available. Erhun et al. [10] gives an example of such an arrangement. Stanford Blood Center, which is a nonprofit organization, delivers only fresh platelet inventory to four small hospital customers, even if it has older inventory in stock.

In this article, we develop and analyze a periodic review model over finite horizon with backlogging for both the FIFO and LIFO scenarios to obtain insights on the nature of the optimal replenishment and demand decisions. For both scenarios, we derive structural results on the optimal policy and profit function and extend them to infinite horizon. For the FIFO scenario, we also derive bounds on the optimal replenishment quantity and expected demand when the planning horizon is finite.

Our analysis reveals several interesting insights. In one interesting result, we find that the likelihood of the optimal order-quantity for a two-period lifetime product being greater than that of a nonperishable product increases for the FIFO scenario when demand is managed. When demand is not managed, the risk of spoilage forces an inventory manager to usually order in smaller quantity when the product has a two-period lifetime compared to a nonperishable product. Our analysis reveals that demand management causes this relationship to breakdown more often. On the other hand, demand management does not change the relationship between the optimal order-quantities for one-period and two-period lifetime products. With or without demand management, the order quantity for the two-period lifetime product remains greater than the one-period lifetime product.

The rest of this article is organized as follows. In Section 3, we discuss the basic assumptions and notation. In Sections 4 and 5, we develop and analyze periodic review models for the FIFO and LIFO scenarios, respectively. For the FIFO scenario, we compare the optimal policy for a two-period lifetime system with that of a one-period lifetime system and an infinite period lifetime system. For the same scenario, in Subsection 4.2, we also develop upper and lower bounds on the optimal order quantity and expected demand. In Section 6, we discuss computational experiments. Finally, we conclude in Section 7. We begin by positioning our work in the existing literature in the following section.

2. LITERATURE REVIEW

Because this article looks at the joint replenishment and pricing decisions for a perishable product, two streams of literature are relevant. The first stream corresponds to inventory control decisions for perishable products, and the second stream consists of papers that examine joint replenishment and pricing decisions.

The literature on the inventory control of perishable products can be classified into two categories depending on how the perishability of a product is modeled. Papers in the first category seek to model products with random lifetimes such as meat and vegetable produce. On the other hand, papers in the second category consider products whose lifetime is fixed and completely known. Once the product reaches the end of its usable lifetime, it becomes unfit for consumption and must be discarded (perhaps for a cost) or salvaged. This approach is motivated by products whose lifetime is predictable such as packaged and processed food products.

Among the papers in the first category, a popular way to model random lifetimes is by assuming that the lifetime of each unit is an exponentially distributed random variable. (This leads to the model being called an exponential decay model.) Given this approach, numerous models have been developed using an approach similar to the Economic Order Quantity model; see Dave [9], Goyal and Giri [14], and Raafat [22] for a review.

On the other hand, most of the papers in the second category use periodic review models with random demand. As we noted in the Introduction, the optimal replenishment policy depends on the relative order of inventory arrival and consumption. Assuming that inventory is consumed in the order of FIFO, Fries [12] and Nahmias [18] characterize the form of the optimal policy for the lost-sales and backlogging cases, respectively, with general lifetime. Using the special characteristics of the optimal solution, many papers have developed myopic or near-myopic policies that ignore the age distribution of on-hand inventory (e.g., Nahmias [19]); see review papers by Nahmias [20] and Karaesmen et al. [15] for a summary of these papers. The analysis of the optimal inventory policy for the LIFO rule is difficult using standard techniques, so research on this topic is limited. As an example of research in this area, Cohen and Pekelman [8] develop age distributions in a periodic review inventory system with lost sales to determine the order policy. Once again, Nahmias [20] and Karaesmen et al. [15] summarize many of these papers. Research has also been conducted to compare a LIFO system to a FIFO system. In a recent paper, Parler et al. [21] compare the cost of a LIFO system to that of a FIFO system when both supply and demand processes are independent Poisson processes. See Parler et al. [21] for more information on this topic.

Naturally, our work also contributes to the second category of literature given our consideration of a product with fixed and multiperiod lifetime. To the best of our knowledge, only Li et al. [17] have analyzed a setting in which demand is price dependent for a perishable product with fixed and multiperiod lifetime. They consider both two-period and arbitrary lifetime systems in which the order of inventory consumption is FIFO. For the two-period lifetime case, they derive the structure of the optimal policy. With respect to their work, we contribute in several ways. One, we present an alternative proof and additional results on the structure of the optimal policy for a FIFO system. Two, we derive bounds on the optimal demand and order quantity for the same system. Three, we derive insights on whether demand management alters our understanding of how the optimal order quantity in a two-period lifetime FIFO system compares with two other widely analyzed systems that correspond to nonperishable and one-period lifetime products. Four, we characterize the structure of the optimal policy for a LIFO system. Five and final, we computationally derive insights on the relative effect of demand management on the FIFO and LIFO systems.

We would like to point out that Chande et al. [2, 3], and Chandrashekar [4] also consider replenishment and demand management decisions for perishable products, but these papers model only a single promotion (the only demand-related decision) during the horizon. Relative to these papers, our contribution is to optimize demand dynamically. We also develop analytical results to derive insights, whereas these papers use a computational approach to obtain insights.

We would also like to mention the stream of literature on the coordination of demand (or price) and production/ordering decisions for nonperishable products. For instance, Chen and Simchi-Levi [5] and Federgruen and Heching [11] develop optimal policies using periodic review models with backlogging with and without fixed cost, respectively. On the other hand, Chen et al. [6] consider the problem with fixed cost, but assume that excess demand is lost. Finally, we note that excellent reviews of literature on the coordination of price and inventory decisions are provided by Federgruen and Heching [11], Chen et al. [6], and Yano and Gilbert [25].

3. NOTATION AND COMMON ASSUMPTIONS

We consider a finite-horizon, periodic-review model for a perishable product with a fixed lifetime equal to two periods at a single manufacturer. At the beginning of each period, the manufacturer inspects his net inventory, x_t, which is one-period old, and places an order for quantity, q_t. For simplicity, we assume that the lead time is equal to zero. The assumption of zero lead times is a standard convention in both perishable inventory literature as well as in the literature on joint replenishment and pricing decisions.

At the same time, the manufacturer determines the price for that period. Similar to Li et al. [17], we assume that the manufacturer charges the same price for all the units. This assumption keeps the analysis simple. Let p_t be the price in period t. Let $d_t(p_t)$ be the expected demand corresponding to p_t. We assume that the function $d_t(\cdot)$ is strictly decreasing. As a result, there is a one-to-one correspondence between price and expected demand. This also means that we can use price and expected demand interchangeably in analysis. In fact, the exposition is considerably simplified when expected demand is used as a variable instead of price. Accordingly, throughout the article, we use expected demand as a variable to present results. In doing so, we omit the argument p_t, unless necessary, for simplicity and use only d_t to denote the expected demand.

We assume that customers pay the price prevailing in the period in which they arrive, even though the product may be out of stock. Hence, expected revenue is equal to $p_t \cdot d_t(p_t)$. We assume this expression to be strictly concave in p_t. One example of $d_t(p_t)$ for which the expected revenue is strictly concave is $d_t(p_t) = a_0 - b_0 p_t$, where $a_0, b_0 > 0$. Another example of such a function is $d_t(p_t) = a_0 \exp(-b_0 p_t)$, where $a_0, b_0 > 0$, for $p_t \in [0, 2/b_0]$.

The assumption of strict concavity of expected revenue is standard in the literature on joint inventory-price decisions. Although some papers make this assumption directly (e.g., Li and Zheng [16]), many others make assumptions that lead expected revenue to be a strictly concave function of p_t. For example, Federgruen and Heching [11] and Chen et al. [6] assume d_t to be concave and strictly decreasing function of p_t, which implies that $p_t \cdot d_t(p_t)$ is strictly concave in p_t.

Once the order is delivered, customer demand arrives through the rest of the period. We assume that given expected demand d_t, the realized demand in period t is equal to $D_t = d_t + \xi_t$, where ξ_t is a random variable with support $[-a, \infty)$, where $a > 0$, such that $E(\xi_t) = 0$. Bounding the support of ξ_t at $-a$ is necessary to ensure that demand D_t remains non-negative. We assume that ξ_t is independently and identically distributed over time. Let F and f be the cumulative distribution function (CDF) and probability density function (PDF) of ξ_t, respectively.

The above demand model is referred to as the additive demand model in the existing literature. The word "additive" arises from the additive nature of the randomness (ξ_t). A more general model is the multiplicative model, which has the following form: $D_t = d_t\zeta_t + \xi_t$. Thus, the randomness is present in both additive and multiplicative forms. Although we will use the multiplicative model in computational experiments, our methodology is not useful in developing insights for this demand model. The extension of our analytical results for this model thus remains a topic for future work.

We let $\mathcal{D}$ be the set of all feasible values of expected demand in a period. Two requirements for any d to be contained in $\mathcal{D}$ are that (a) the realized demand D_t remains non-negative for all $\xi_t \in [-a, \infty)$ and (b) the corresponding price be non-negative. Since $\xi_t \geq -a$, requirement (a) ensures that any d in $\mathcal{D}$ is greater than or equal to a. Further, we assume $\mathcal{D}$ to be convex, which implies that the set is an interval.

At the end of the period, once all the demand is realized, holding cost is charged on any remaining inventory at rate h per unit. On the other hand, if demand exceeds inventory, the excess demand is backlogged, and backlogging cost is charged at π per unit backlogged.

Given values of x_t, q_t, and d_t, the revenue and holding and shortage costs incurred in period t are equal to

$$L(x_t, q_t, d_t) = R(d_t) - hE[x_t + q_t - D_t]^+ \\ - \pi E[D_t - x_t - q_t]^+,$$

where $R(d_t)$ is the expected revenue, which is equal to the product of expected demand and the corresponding price, and $[\cdot]^+$ stands for $\max(\cdot, 0)$.

Also, at the end of the period, the inventory that is two-periods old is discarded. The amount of inventory discarded depends on the relative order of consumption. Under the FIFO scenario, the amount of inventory discarded is equal to $(x_t - D_t)^+$. On the other hand, the amount of inventory discarded under the LIFO scenario is $(x_t - (D_t - q_t)^+)^+$. We let θ (possibly negative) be the unit cost of discarding old inventory. The parameter θ can be both positive or negative depending on whether old inventory incurs a cost while being discarded or it is salvaged.

4. FIRST-IN, FIRST-OUT SCENARIO

In this section, we derive insights on the joint replenishment and price decisions when the order of inventory consumption is FIFO.

4.1. Analysis

We begin by formulating the model. For simplicity, we omit the subscript t from all the variables through the rest of this article unless necessary for exposition. The optimal profit from period t through the end of horizon v_t is equal to

$$v_t(x) = \max_{d\in\mathcal{D}, q\geq 0} L(x, q, d) - cq - \theta E(x - D)^+ \\ + \alpha E v_{t-1}(q - (D - x)^+), \qquad (1)$$

where $\alpha \in (0, 1)$ is the discount factor. We take the end of horizon profit $v_0(x)$ to be equal to $sx^+ - cx^-$, where s is the salvage value. Observe that the argument of v_{t-1}, $(q - (D - x)^+)^+$, is the amount of inventory that is one period old at the beginning of period $t - 1$. Also, define

$$G_t(x, q, d) = L(x, q, d) - cq - \theta E(x - D)^+ \\ + \alpha E v_{t-1}(q - (D - x)^+), \qquad (2)$$

so that $v_t(x) = \max_{d\in\mathcal{D}, q\geq 0} G_t(x, q, d)$.

We now state the structure of the optimal policy in the following theorem. We note that some (but not all) parts of the theorem are also derived by Li et al. [17]. In particular, they prove concavity of $G_t(x, \cdot, \cdot)$ and $v_t(\cdot)$; show the existence of a threshold $\bar{x}_t$ beyond which no order is placed; and show that $q^*(x) \in [-1, 0], d^*(x) \in [0, 1]$, where $q^*(x)$ and $d^*(x)$ are the optimal order quantity and expected demand when net inventory is equal to x. These results are proved under the assumption that the salvage value at the end of horizon is equal to the purchasing cost. (In comparison, we allow the salvage value to be lower.) In spite of the similarity, we state the result and proof in its entirety for two reasons. One, our proof approach is different. Two, some of the arguments and equations in the proof of the theorem are also used in establishing other results in the paper.

The theorem is stated below. Note that in the theorem statement and beyond, we have omitted to add subscript t to $d^*(x)$ and $q^*(x)$ to keep notation simple, even though both of these functions may vary with t.

THEOREM 1: Let $R''(d) \leq -h$ and $\pi > (1 - \alpha)c$. Also, let $\arg\max_d\{R(d) - cd\} \in \mathcal{D}$, and $0 < f(\xi) \leq 1$ for all $\xi \in [-a, \infty)$.

1. For each t, $G_t(x, q, d)$ is a jointly concave function of q and d.
2. For each t, there exists a unique $\bar{x}_t > 0$ such that $\bar{x}_T = \bar{x}_{T-1} = \ldots = \bar{x}_2 \geq \bar{x}_1$. For $x < \bar{x}_t$,
    a. $q^*(x) > 0$.
    b. $q^*(x), d^*(x)$ are unique; $q^*(x), d^*(x) \in \mathcal{C}^1$; $0 \leq d^{*\prime}(x) \leq 1$; and $-1 \leq q^{*\prime}(x) \leq 0$. Further, $1 + q^{*\prime}(x) - d^{*\prime}(x) \geq 0$.
    c. $d^*(x) = \max_{d\in\mathcal{D}} R(d) - cd$ for $x \leq 0$.
    d. $c(1 - \alpha) - \theta \leq v_t'(x) \leq c$ and $v_t'(x) = c$ for $x \leq 0$.
    e. $v_t(x)$ is a concave function of x.
3. On the other hand, when $x \geq \bar{x}_t$,
    a. $q^*(x) = 0$. Further, $d^*(x)$ is unique; $d^*(x) \in \mathcal{C}^1$; and $0 \leq d^{*\prime}(x) \leq 1$.

b. $-h - \theta \leq v_t'(x) \leq c$.
c. $v_t(x)$ is a concave function of x.

Because proof of the theorem is long, a brief sketch is as follows. We first establish the concavity of G_t in q and d by showing the Hessian matrix to be negative semidefinite. After that, we characterize the threshold value of net inventory ($\bar{x}_t$) beyond which optimal order quantity is zero. Because the characterizing equation is identical for all $t > 1$, the value of $\bar{x}_t$ is also identical for such t.

After proving the existence of $\bar{x}_t$, we use the implicit function theorem [23] to obtain bounds on $d^{*\prime}(x)$ and $q^{*\prime}(x)$ when x is less than $\bar{x}_t$. In particular, for $x \leq 0$, we find that the optimal d maximizes $R(d) - cd$. Subsequently, we use the bounds on $d^{*\prime}$ and $q^{*\prime}$ to bound v_t' and to prove concavity of v_t. The bounds on v_t' also have an economic interpretation, as we will see below. We use a similar approach to prove the results when $x \geq \bar{x}_t$.

A brief discussion on the four technical assumptions stated in the theorem is as follows. The first assumption requires that the second derivative of the expected revenue be less than $-h$. One consequence of this assumption is that the set of feasible values for expected demand $\mathcal{D}$ has bounded support. To see this, note that the left end-point of $\mathcal{D}$, a, is finite. As the slope of $R(d)$ strictly decreases, there must exist some finite d_0 such that $R(d) \leq 0$ for all $d \geq d_0$. A negative value of $R(d)$ is possible only if price is negative. Clearly, any such values of d cannot be feasible, so the feasible interval for expected demand will be $[a, d_0]$. The second assumption, which imposes a lower bound on π, simplifies analysis by ensuring that backlogs are always satisfied in the next period.

The next assumption that the maximizer of $R(d) - cd$ be feasible makes analysis convenient. This assumption is not used in most of the proofs. Finally, we assume that the magnitude of the density at each point be in $(0, 1]$. The assumption that the density is bounded by 1 holds for a wide range of parameter values for several common distributions. For instance, for the exponential distribution, the assumption is satisfied if rate $\lambda \leq 1$, which means that $E(\xi) \geq 1$. Similarly, for the uniform distribution, the assumption holds if the length of the support is greater than 1.

We next discuss insights from the theorem. The result shows that as net inventory x increases, the optimal order quantity decreases though the rate of reduction is less than 1. (The same relationship between order quantity and net inventory was shown by Nahmias [18] for a pure inventory system of a perishable product with general lifetime.) This means that the order-up-to level $x+q^*(x)$ increases with x. This is in contrast with nonperishable products, for which the order-up-to level is independent of x Federgruen and Heching [11]. This difference in the structure of the optimal order-up-to level is driven by the difference between new and old units for the two types of products. Although new and old nonperishable units are identical to each other for the inventory system, new and old perishable units are not identical for the inventory system because new perishable units last one period longer than old perishable units. As a consequence, although additional old units reduce requirement for new units, the reduction is less than the increase in older units, which causes the order-up-to level to rise with x. Once the level of old inventory becomes high enough, the order quantity becomes equal to 0. The threshold at which the order quantity becomes 0, interestingly, is the same for all $t \geq 2$, but has a lower value for period 1 due to the end-of-horizon effect.

We also note that the optimal expected demand for $x \leq 0$ is a constant that maximizes $R(d) - cd$. This is equal to the optimal demand for a nonperishable product if an order is placed [11]. But, as Part 2(b) shows, the optimal demand rises as the amount of old inventory increases, to reduce spoilage. Therefore, the optimal demand for a perishable product is greater than that of a nonperishable product whenever an order is placed.

The result that $v_t'(x) = c$ for $x \leq 0$ is intuitive. This result means that a unit reduction in backlogged quantity increases profit by c. Because lead-time is 0 and π is large enough, all the backlogged demand is satisfied in the following period. To satisfy each unit of backlog costs c. This cost will be saved if the amount of backlog decreases by one unit, resulting in a profit increase of c.

However, the marginal worth of a unit on-hand ($v_t'(x), x \geq 0$) is less than c even when an order is placed. This is in contrast to nonperishable products; for such products, the marginal worth of a unit is equal to c when an order is placed [11]. Because new and old units of a nonperishable product have identical remaining lifetimes (infinite), an additional unit on hand reduces order quantity by one and saves the manufacturer purchasing cost of one unit. But old and new units of a perishable product have different remaining lifetimes, so presence of an additional old unit reduces order quantity by less than one and the manufacturer saves less than c.

The slope of v_t is bounded from the lower side by $-\theta - h$. This bound is realized if a unit that is one-period old is certain not to be consumed. In that case, the unit does not contribute to the revenue, but incurs both holding and salvage costs, which add up to $-\theta - h$. The bounds on the slope of v_t are useful not only in establishing other parts of the theorem, but also in developing bounds on the optimal order quantity and expected demand, as we show in Subsection 4.2.

4.2. Bounds on Optimal Replenishment Quantity and Demand

In this subsection, we develop upper and lower bounds on the order quantity and expected demand for the model developed in the last subsection. We obtain these bounds by exploiting the structural properties of the profit function as well as the optimal order quantity and expected demand stated in Theorem 2. We briefly discuss the approach to obtain the bounds following the theorem statement, which is as follows.

THEOREM 2:

1. Let $y = F^{-1}[\frac{\pi-c+\alpha c}{\pi+h+\theta+\alpha c}]$. When $x \geq \bar{x}_t$, the expected demand is bounded from above by $\max\{0, x - y\}$ and when $x \leq \bar{x}_t$, it is bounded from above by $\max\{0, \bar{x}_t - y\}$.
2. Let d_c be the unique solution of $R'(d_c) = c$. For any value of x, the expected demand is bounded from below by d_c.
3. Let $\bar{r} = F^{-1}[\frac{\pi-c+\alpha c}{\pi+h}]$ and $\underline{r} = F^{-1}[\frac{\pi-c-\alpha(h+\theta)]}{\pi+h}]$. When $x \leq \bar{x}_t$, the order quantity is bounded from above by $\max\{0, \bar{r} + \bar{x}_t - y - x\}$ and bounded from below by $\max\{0, \underline{r} + d_c - x\}$.

Our approach for computing the bounds is as follows. The lower bound on the expected demand when $x > \bar{x}_t$ is obtained by using the upper bound on the marginal worth of an old unit, which is c (using Theorem 1), in the expression for $v_t'(x)$. Because the optimal expected demand is increasing, for $x \leq \bar{x}_t$ the optimal expected demand is bounded from above by $d^*(\bar{x}_t)$. This explains Part 1. The increasing nature of expected demand can also be used to obtain a lower bound on optimal demand. Thus, the optimal expected demand for any x is bounded from below by $d^*(0) = d_c$, which explains Part 2.

On the other hand, when $x \leq \bar{x}_t$, using the same approach as for Part 1, we obtain an upper bound on $q^*(x) - d^*(x)$, that is, we use the maximum marginal worth of an old unit, c, in the expression for $v_t'(x)$. Given the upper bound on $q^*(x) - d^*(x)$, we obtain an upper bound on $q^*(x)$ by replacing $d^*(x)$ by its upper bound. The approach to compute the lower bound on $q^*(x)$ is similar but exactly opposite, so that we now use the lower bound on the marginal worth of a unit, $-\theta - h$, to obtain a lower bound on $q^*(x) - d^*(x)$. Subsequently, we obtain a lower bound on $q^*(x)$ by using the lower bound on $d^*(x)$ computed in Part 2.

4.3. Relationship with One-period and Infinite Lifetime Systems

In this subsection, we compare the optimal policy for the model defined in Subsection 4.1 to that of a system in which the product has a single-period lifetime as well as another system in which the product has an infinite-period lifetime (or is nonperishable). When there is no demand management, it is understood that the order quantity for a two-period lifetime system in any period is greater than the order quantity for a one-period lifetime system [20]. The reason lies in the reduced risk of spoilage for a two-period lifetime system. The same reason also leads to the intuition of a usually larger order quantity for a nonperishable product as compared to a two-period lifetime system. Our objective is to explore if the conventional intuition modifies in the presence of demand management.

We begin by formulating the model for a system in which the lifetime is equal to one period. All the modeling assumptions remain the same as in Subsection 4.1; the sole difference is that any unsold units at the end of each period are discarded. The optimal profit from period t through the end of horizon is

$$v_t^1(x) = \max_{q^1 \geq 0, d^1 \in \mathcal{D}} L(x, q^1, d^1) - cq^1 - \theta E(x + q^1 - d^1 - \xi)^+ + \alpha E v_{t-1}^1(-(d^1 + \xi - x - q^1)^+), \quad (3)$$

where $v_0^1(x) = cx$. Observe that only backlogs are carried from one period to the next period. As a result, x can only take non-positive values.

Similarly, we can define a model for nonperishable products. The formulation is as follows:

$$v_t^\infty(x) = \max_{q^\infty \geq 0, d^\infty \in \mathcal{D}} L(x, q^\infty, d^\infty) - cq^\infty + \alpha E v_{t-1}^\infty(x + q^\infty - d^\infty - \xi), \quad (4)$$

where $v_0^\infty(x) = sx^+ - cx^-$. Because nothing ever perishes, there is no term involving θ in the above formulation. As noted before, the optimal demand when the product is nonperishable maximizes $R(d^\infty) - cd^\infty$ whenever an order is placed. The same can also be easily shown for a system in which the product lifetime is equal to one period; we omit the details. Coupling these observations with Theorem 2, in which we show that the optimal demand for a two-period lifetime system is bounded from below by the maximizer of $R(d) - cd$, we obtain the following corollary.

COROLLARY 1: Assuming an order is placed in a period, for given x the optimal demand for a two-period lifetime system is bounded from below by the optimal demands for one-period lifetime as well as infinite lifetime systems.

An intuitive explanation for the above result follows. For the two-period lifetime system, the optimal demand is selected not only to maximize the modified revenue ($R(d) - cd$), but also to lower spoilage of the old inventory. This results in the optimal demand being greater than the (modified) revenue maximizing demand. In the other two systems, the objective to lower spoilage does not exist. This is obvious in the case of infinite lifetime system. When lifetime is one period, fresh inventory is ordered every period, and there is no old inventory. As a consequence, the optimal demand is determined so as to maximize the (modified) revenue.

Next, we compare the optimal order quantities. Because net inventory at the beginning of a period in a one-period lifetime system is always non-positive, the comparison across the two systems can only be carried out for such values of x. It can be easily shown that for a one-period lifetime system the ordering policy is a basestock policy, so we omit the details. In the following proposition, we show that the optimal order quantity for a two-period lifetime system, $q^{2*}(x)$, is bounded from below by that for a one-period lifetime system, $q^{1*}(x)$, for all non-positive x. We also show that the threshold for order placement in a two-period lifetime system, $\bar{x}_t$, is bounded from below by the optimal basestock level, which is equal to $q^{1*}(0)$, of the one-period system.

PROPOSITION 1:

1. $q^{2*}(x) \geq q^{1*}(x)$ for all $x \leq 0$.
2. $\bar{x}_t \geq q_t^{1*}(0)$.

The above proposition (Part 1) shows that a two-period lifetime system continues to order more than a one-period lifetime system even when demand is managed. Thus, the relationship between the optimal order quantities of the two systems stays preserved when price can be varied.

The comparison of a two-period lifetime system with that of an infinite lifetime system, however, throws a different result. For instance, we find that the order quantity for the two-period lifetime system is larger than the infinite lifetime system for period 1 for any x. (When demand is not managed, the two orders are equal in the optimal solution.) Similarly, for any t, we find that the threshold for order placement for the two-period lifetime system, $\bar{x}_t$, is greater than the base-stock level, which is equal to $q^{\infty*}(0)$, for the infinite lifetime system. This means that for some values of net inventory x, the two-period lifetime system may place bigger orders than the infinite lifetime system. Even in the absence of demand management this result remains true, but demand management induces a bigger order in the two-period lifetime system. (See our arguments below.) However, there still exist many instances in which the infinite lifetime system places a bigger order. For instance, when $s = c$, so that the end-of-horizon effect vanishes, the infinite lifetime system orders more compared to a two-period lifetime system for $x \leq 0$.

The result is formally stated below.

PROPOSITION 2:

1. $q^{2*}(x) \geq q^{\infty*}(x)$ for all x when $t = 1$.
2. For any t and any $s \leq c$, $\bar{x}_t \geq q^{\infty*}(0)$.
3. If $s = c$, $q^{2*}(x) \leq q^{\infty*}(x)$ for $x \leq 0$.

Note that the end-of-horizon effect also vanishes when $t >> 1$ even though $s < c$. Therefore, $q^{2*}(x)$ will likely be less than $q^{\infty*}(x)$ for $x \leq 0$ for such t.

An intuitive explanation for the above result is as follows. Part 1, which shows this result for $t = 1$, is driven by two major factors. The first factor is that the optimal demand in the two-period lifetime system is greater than the infinite lifetime system, due to risk of spoilage. Greater expected demand induces the manufacturer to place a bigger order. The second factor is that the marginal worth of an additional new unit is the same in either system regardless of the amount of net inventory. (This is not true for other periods.) The difference in the marginal worths between the two systems occurs in the future profit term (or the recursive term). But since 1 is the last period, the perishability of a new unit does not influence the future profit term in the two-period lifetime system. As a consequence, in the optimal solution, the value of $q - d$ is equal in the two systems. Because the optimal demand in the two-period lifetime system is greater, the order quantity also is larger in this system.

Greater optimal demand in the two-period system is also a factor for Part 2, and once again it propels the manufacturer to place a bigger order. The other major factor that drives the result is the difference between the two systems in how they use the first few new units. In an infinite-lifetime system, the time of consumptions of new units (q) depends on the number of old units (x). (Without loss of generality, we can assume FIFO order of inventory consumption.) This means that larger the amount of older inventory, greater is the time lag before it is consumed. As a result, the marginal worth of the first few new units declines with x. On the other hand, in a two-period lifetime system, the worth of the first few new units is independent of the amount of old inventory. Because old units perish in the current period, the first few new units get used latest by the following period. This difference in the two systems potentially leads to a set of inventory values in which the marginal worth of a few new units is greater in the two-period lifetime system. For such inventory states, a bigger order is placed in the two-period lifetime system.

Observe that the second factor does not depend on the sensitivity of demand to price, so Part 2 also holds when demand is not managed. The modulation of demand, however, favors bigger orders in the two-period lifetime system compared to the infinite lifetime system due to the first factor.

Part 3 considers the case when there is nothing on-hand ($x \leq 0$). In this case, the optimal demand across the two systems is equal, so demand ceases to be a factor. The key driver of the result is greater future worth of an ordered unit in the infinite lifetime system due to nonperishability.

We end this subsection by noting that Proposition 1 can also be used to obtain a lower bound on the order quantity in a two-period lifetime system. (Proposition 2 can also be used to obtain an upper bound, but the bound turns out to be identical to Theorem 2, Part 3.) We state the lower bound in the following corollary.

COROLLARY 2: The optimal order quantity for $x \leq 0$ in a two-period lifetime system is bounded from below by $q^{1*}(x) = -x + F^{-1}\left(\frac{\pi - c + \alpha c}{\pi + h + \theta + \alpha c}\right)$.

4.4. Infinite-Horizon Model

In this subsection, we show that Theorem 1 extends to the infinite horizon case. To prove the result, we transform the dynamic program stated in Eqs. (1) and (2) so that the expectation of one-period profit is negative. We ensure this by subtracting $M := \max_{d\in\mathcal{D}} R(d)$ from the expectation of one-period profit.

The formulation of the transformed model is

$$\hat{v}_t(x) = \max_{q\geq 0, d\in\mathcal{D}} \hat{G}_t(x,q,d) \tag{5}$$

where

$$\hat{G}_t(x,q,d) = L(x,q,d) - M - cq - \theta E(x-D)^+ + \alpha E\hat{v}_{t-1}(q-(D-x)^+) \tag{6}$$

and

$$\hat{v}_0(x) = sx^+ - cx^-.$$

As M is a constant and is subtracted every period,

$$\hat{v}_t(x) = v_t(x) - \frac{M(1-\alpha^t)}{1-\alpha}$$

and

$$\hat{G}_t(x,q,d) = G_t(x,q,d) - \frac{M(1-\alpha^t)}{1-\alpha}.$$

The infinite-horizon equations of the transformed model is

$$\bar{v}(x) = \max_{q\geq 0, d\in\mathcal{D}} \bar{G}(x,q,d) \tag{7}$$

where

$$\bar{G}(x,q,d) = L(x,q,d) - M - cq - \theta E(x-D)^+ + \alpha E\bar{v}(q-(D-x)^+) \tag{8}$$

By removing—M from the above equation, we can define the infinite-horizon equations of the original model.

Because the single-period expected profit is negative for each period, it can be easily seen that $\hat{v}_t(x) \leq 0$ and $\hat{G}_t(x,q,d) \leq 0$. We state this result formally in the following corollary.

COROLLARY 3: For all $t \geq 1$, $\hat{v}_t(x) \leq 0$ and $\hat{G}_t(x,q,d) \leq 0$.

In the following proposition, we state the infinite-horizon counterpart of Theorem 1.

PROPOSITION 3: Assuming model parameters satisfy same relationships as in Theorem 1,

1. All of $v := \lim_{t\to\infty} v_t$, $\hat{v} := \lim_{t\to\infty} \hat{v}_t$, $G := \lim_{t\to\infty} G_t$ and $\hat{G} := \lim_{t\to\infty} \hat{G}_t$ exist. Further, $\hat{v} = v - \frac{M}{1-\alpha}$ and $\hat{G} = G - \frac{M}{1-\alpha}$.
2. The functions, $\hat{v}$ and $\hat{G}$, (v and G) satisfy the infinite-horizon optimality eq. (7) and (8), of the transformed model (original model).
3. $G(x,q,d)$ is jointly concave in q and d and so is $v(x)$ in x.
4. There exists a unique $\bar{x}$, which is equal to $\bar{x}_t, t > 1$, defined in Theorem 1, such that for $x < \bar{x}$,
   a. $q^*(x) > 0$.
   b. $q^*(x), d^*(x)$ are unique; $q^*(x), d^*(x) \in \mathcal{C}^1$; $0 \leq d^{*\prime}(x) \leq 1$; and $-1 \leq q^{*\prime}(x) \leq 0$. Further, $1 + q^{*\prime}(x) - d^{*\prime}(x) \geq 0$.
   c. $d^*(x) = \max_{d\in\mathcal{D}} R(d) - cd$ for $x \leq 0$.
   d. $c(1-\alpha) - \theta \leq v'(x) \leq c$ and $v'(x) = c$ for $x \leq 0$.
5. On the other hand, when $x \geq \bar{x}$,
   a. $q^*(x) = 0$. Further, $d^*(x)$ is unique; $d^*(x) \in \mathcal{C}^1$; and $0 \leq d^{*\prime}(x) \leq 1$.
   b. $-h - \theta \leq v'(x) \leq c$.

5. LAST-IN, FIRST-OUT SCENARIO

In this section, we discuss the case in which the order of inventory consumption is LIFO. We begin by developing a dynamic programming formulation. The optimal profit from period t to the end of horizon is equal to

$$v_t(x) = \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-(D-q)^+)^+ + \alpha E v_{t-1}(x+q-D-(x-(D-q)^+)^+), \tag{9}$$

such that $v_0(x) = sx^+ - cx^-$, where s is the salvage value of any inventory left at the end of horizon. The argument to v_{t-1} appears complex, but it can be derived in a simple manner. It is equal to net inventory at the end of period t $(x+q-D)$ less the amount of inventory spoiled $((x-(D-q)^+)^+)$. Similar to the model in Section 4, we consider an additive demand model, that is, $D = d + \xi$, where d takes values in $\mathcal{D}$.

Recall that for the FIFO scenario, the optimal demand depends on the value of net inventory. Fortunately, the LIFO scenario behaves differently, and the optimal expected demand can be obtained by maximizing $R(d) - cd$ whenever an order is placed. This also ensures that the objective function is separable in q and d. We state the result formally in the following proposition.

PROPOSITION 4: If the unconstrained optimal order quantity is non-negative in the optimal solution, the optimal expected demand is equal to $d^*(x) = \max_{d\in\mathcal{D}} R(d) - cd$.

It is well-known that the optimal profit function corresponding to the LIFO scenario lacks concavity (or any other type of simple structure) even when demand is not managed [20]. Naturally, the addition of another variable, expected demand, can only complicate the analysis, so the profit function continues to lack a simple structure. For instance, the optimal profit function defined in Eq. (9) is not necessarily concave in net inventory. We state this observation formally as follows.

OBSERVATION 1: The optimal profit function defined in Eq. (9) is not necessarily concave in x.

It is sufficient to consider the dynamic program for Period 1 to establish this observation. We provide an argument in Appendix.

The lack of concavity of the profit function makes it difficult to establish other properties of the optimal solution such as bounds on the slope of the optimal order quantity. In spite of this hardship, we are able to characterize a number of properties of the optimal profit function and optimal decisions by exploiting first-order conditions. The result is stated as follows.

THEOREM 3: Let $\pi > c(1-\alpha)$ and $\arg\max_d\{R(d) - cd\} \in \mathcal{D}$.

1. There exists a unique $\bar{x}$, which is independent of t, such that an order is placed if $x < \bar{x}$. For all $x \geq \bar{x}$, no order is placed.
2. $d^*(x) \in \mathcal{C}^1$. Further, $d^{*\prime}(x) = 0$ for $x < \bar{x}$ and $0 \leq d^{*\prime}(x) \leq 1$ for $x \geq \bar{x}$.
3. For $x \leq \bar{x}$ and $t > 1$, $-\theta - h \leq v_t'(x) \leq c + (\alpha h - (1-\alpha)\theta)^+$. For $x \leq \bar{x}$ and $t = 1$, $-\theta - h \leq v_t'(x) \leq c$. Further, for $x \leq 0$, $v_t'(x) = c$.
4. For $x \geq \bar{x}$, $v_t'(x) \geq -\theta - h$.
5. For $x \geq \bar{x}$, $v_t(x)$ is concave in x. Further, $G_t(x, 0, d)$ is strictly concave in d.

Some remarks on the above theorem are as follows. One, the above theorem can be extended to infinite horizon in a manner similar to Theorem 1. Two, similar to the FIFO scenario, there exists a unique threshold of net inventory beyond which the order quantity is zero. Unlike the FIFO scenario, however, this threshold is identical for all the periods, including period 1. Because of a lack of concavity of v_t, we are unable to say anything about how the order quantity changes with respect to net inventory. Extensive computational experiments, however, indicate that the optimal order quantity decreases with net inventory.

Three, similar to the FIFO scenario, $v_t'(x) = c$ for $x \leq 0$. The reason is also the same, and the details are omitted. Four, the optimal demand for a given x is unique. For $x \leq \bar{x}$, this follows from the strict concavity of $R(d)$. For $x > \bar{x}$, the uniqueness of $d^*(x)$ follows from the strict concavity of $G_t(x, 0, d)$ in d (Part 5 of the theorem). Five and final, the above structural results, if suitable, can also be extended to the case when demand is not managed, by taking $\mathcal{D}$ to be a singleton. To the best of our knowledge, such results do not exist in the literature. We state this observation more precisely in the following corollary.

COROLLARY 4: Consider an inventory system in which demand is exogenous, and let $\pi > c(1-\alpha)$.

1. There exists a unique $\bar{x}$, which is independent of t, such that an order is placed if $x < \bar{x}$. For all $x \geq \bar{x}$, no order is placed.
2. For $x \leq \bar{x}$ and $t > 1$, $-\theta - h \leq v_t'(x) \leq c + (\alpha h - (1-\alpha)\theta)^+$. For $x \leq \bar{x}$ and $t = 1$, $-\theta - h \leq v_t'(x) \leq c$. Further, for $x \leq 0$, $v_t'(x) = c$.
3. For $x \geq \bar{x}$, $v_t'(x) \geq -\theta - h$.
4. For $x \geq \bar{x}$, $v_t(x)$ is concave in x.

6. COMPUTATIONAL EXPERIMENTS

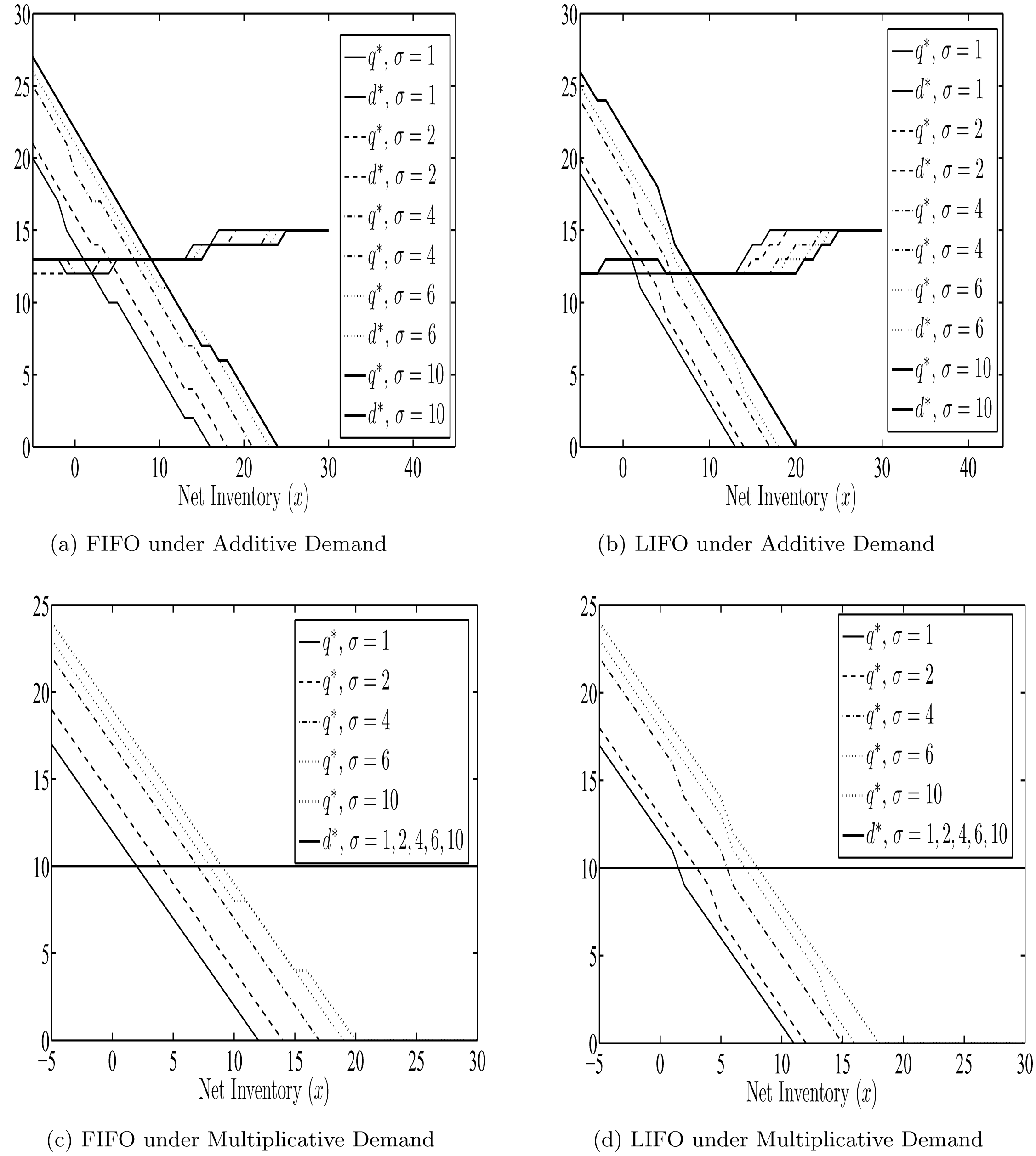
The purpose of the computational experiments is to obtain insights on how the optimal order quantity, demand, and expected profit vary for both FIFO and LIFO scenarios as a function of model parameters.

We consider both additive and multiplicative demand models. For the additive demand model, $D = a - bp + \xi$, and for the multiplicative demand model, $D = (a - bp)\zeta + \xi$. Although ζ is beta distributed with parameters γ and β, ξ is normally distributed with mean μ and standard deviation σ and truncated at 0 and 20. The values of other model parameters are as follows:

T	t	c	h	π	s	μ	a	β	γ	b	θ
4	4	5	1	40	1.5	10	20	1.2	1.2	1	−1

A summary of interesting observations is as follows:

1. For all the four scenario-demand model combinations, there appears to exist a threshold before which the optimal order quantity is positive and beyond which the optimal order quantity is 0. Further, on the left of the threshold, the optimal order quantity is decreasing in net inventory. (We have proved this result analytically for FIFO under the additive demand model.) On the other hand, the relationship between optimal demand and net inventory does not appear to be always monotone.



(a) FIFO under Additive Demand

(b) LIFO under Additive Demand

(c) FIFO under Multiplicative Demand

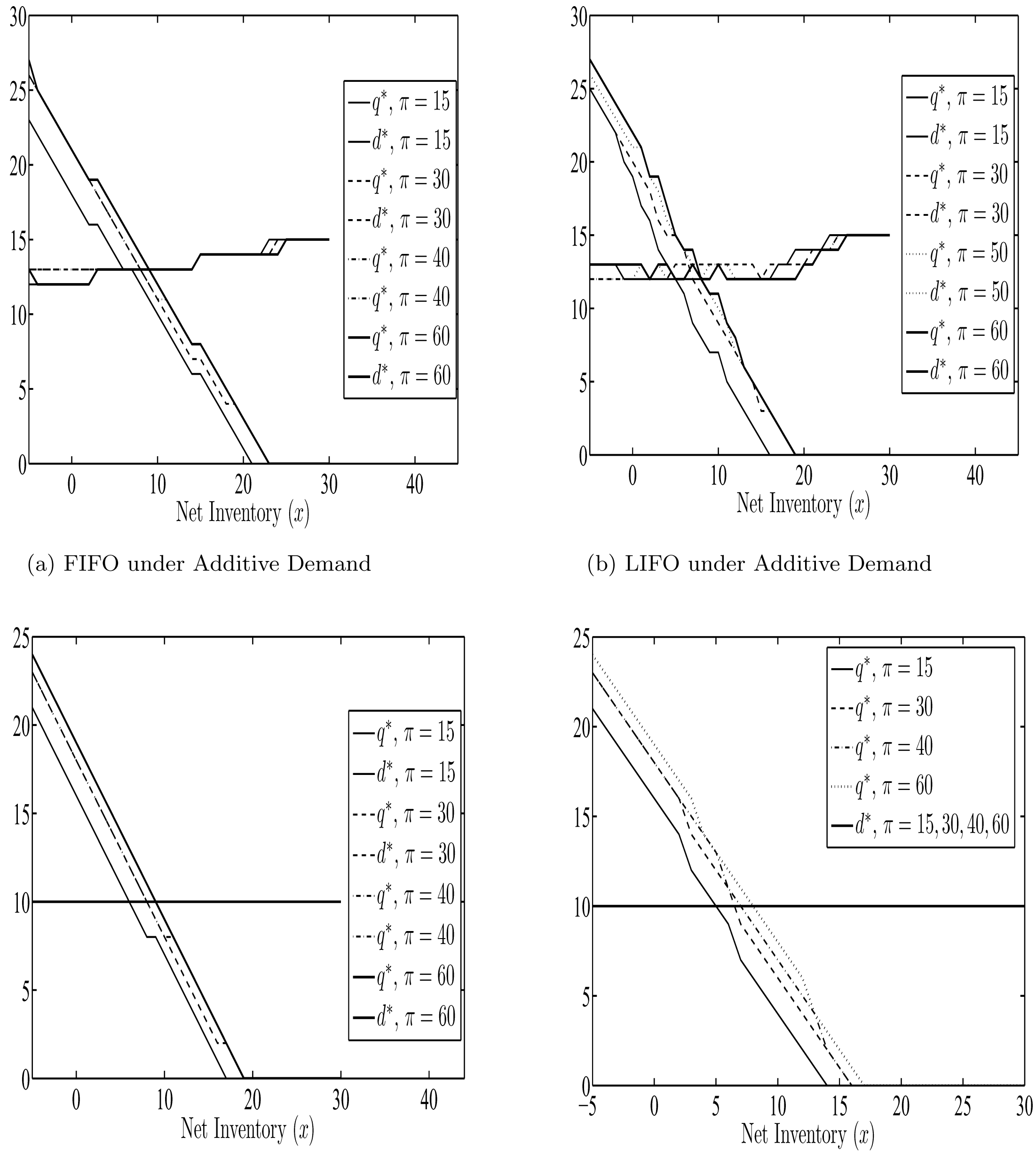
(d) LIFO under Multiplicative Demand

Figure 1. Optimal order quantity and demand as a function of σ.

2. For the multiplicative demand model, there appears to be little variation in optimal demand as a function of model parameters for both FIFO and LIFO scenarios. On the other hand, optimal order quantity appears to vary little with respect to b and θ, but increases with respect to π and σ, apparently to maintain higher safety stock.
3. For both additive and multiplicative demand models, the gap in expected profit between FIFO and LIFO widens as values of the model parameters (b, θ, σ, and π) increase. Further, the expected profits for both FIFO and LIFO scenarios decrease with model parameters.

The first observation is particularly encouraging as it indicates that it may be possible to extend Theorem 1, possibly in a limited sense, for the multiplicative demand model.

For sample plots, see Figs. 1–3. The figures show optimal order quantity, demand, and expected profit for FIFO and LIFO scenarios under additive and multiplicative demands



(a) FIFO under Additive Demand

(b) LIFO under Additive Demand

(c) FIFO under Multiplicative Demand

(d) LIFO under Multiplicative Demand

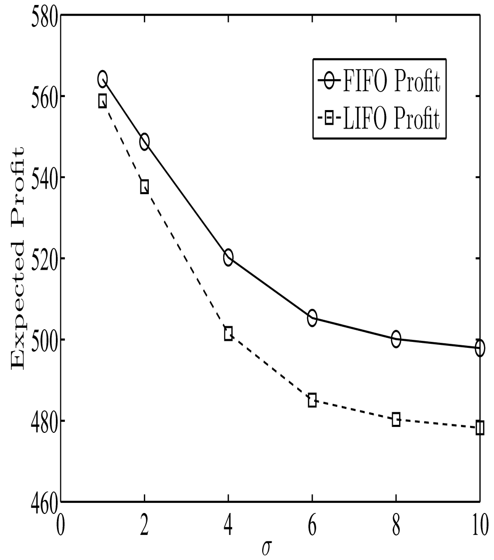
Figure 2. Optimal order quantity and demand as a function of backlogging cost (π).

for different values of σ and π. Although the order quantity and demand values correspond to $t = 4$, the expected profit corresponds to the whole planning horizon.

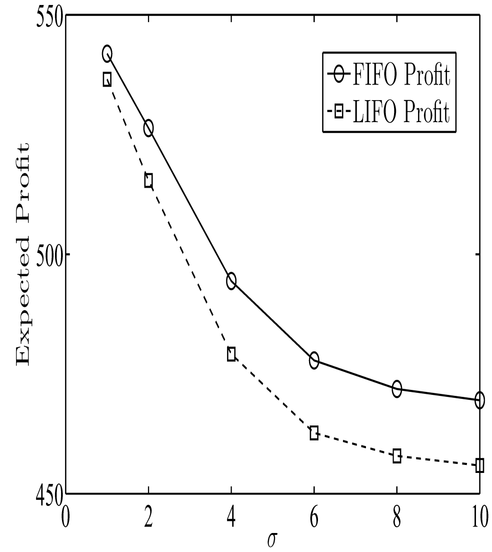
7. CONCLUSIONS AND FUTURE RESEARCH

We explore a periodic review model to develop insights on the optimal replenishment and pricing policies for a perishable product with a fixed lifetime of two periods. We consider two common scenarios: FIFO and LIFO. For the FIFO scenario, our analysis reveals that there exists a fixed threshold of inventory, which is the same for all but the last period, beyond which no order is placed. However, the order-up-to level and demand vary with the amount of inventory.

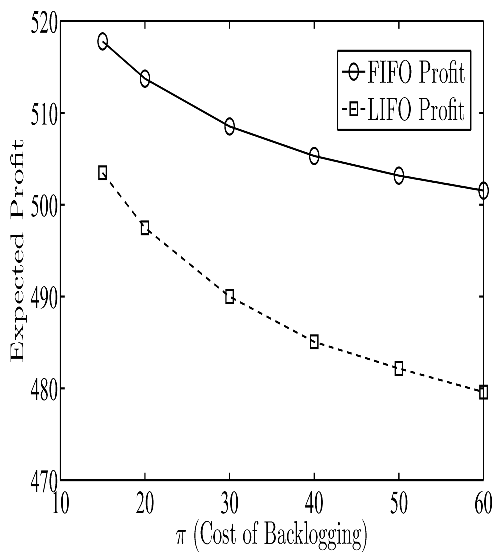
Although the profit function is concave under the FIFO scenario, this property is absent for the LIFO scenario. In spite of this difference, the two scenarios have similar structures for the optimal replenishment policy. In the optimal replenishment policy, both scenarios have thresholds beyond which no



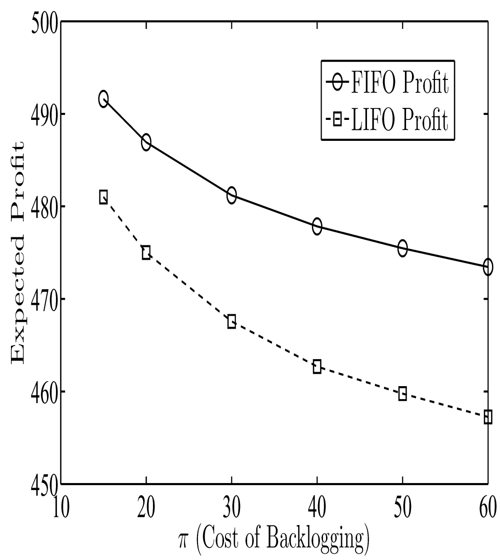
(a) Under Additive Demand



(b) Under Multiplicative Demand



(c) Under Additive Demand



(d) Under Multiplicative Demand

Figure 3. Optimal expected profit as a function of σ and π (backlogging cost).

order is placed. On the other hand, the optimal pricing policies for the two scenarios are notably different. Specifically, the pricing policy under the LIFO scenario is much simpler compared to the FIFO scenario: the optimal price is constant whenever an order is placed, unlike the FIFO scenario. These results hold for both finite and infinite horizon models.

Our analysis reveals that some well-understood relationships between inventory systems of perishable and nonperishable products may change in the presence of demand management. One such relationship is that the optimal order quantity for a two-period lifetime product is usually less than that of a nonperishable product due to the risk of spoilage. We find that demand management increases the likelihood of the optimal order quantity for the two-period lifetime product being greater than that of the nonperishable product.

One potential future research direction is the extension of our results to general lifetime. Given that the analysis of the problem in which demand is not managed is fairly complex, we believe that the extension will not be straightforward. Another interesting future direction could arise by relaxing the assumption that units of different ages are priced equally when customers choose units themselves, that is, the manager prices older units differently from newer units to maximize profit. We are currently working on this problem.

APPENDIX

Proof of Theorem 1

Before we begin the proof, three remarks are as follows. One, for brevity, we omit the dependence of $q^*(x)$ and $d^*(x)$ on x unless necessary for exposition. Two, as $v_0'(0)$ is not defined, we take it to be equal to its right derivative, which is equal to s. Three, any derivative at an end-point is actually a left or right derivative depending on the context.

PROOF OF PART 1

We prove all the results by induction. We first establish the result for period 1. Note that

$$\begin{aligned}G_1(x,q,d) &= R(d) - cq - h\int_{-a}^{x+q-d}(x+q-d-\xi)f(\xi)d\xi\\ &- \pi\int_{x+q-d}^{\infty}(\xi+d-x-q)f(\xi)d\xi\\ &- \theta\int_{-a}^{x-d}(x-d-\xi)f(\xi)d\xi + \alpha s\int_{-a}^{x-d}qf(\xi)d\xi\\ &+ \alpha s\int_{x-d}^{x+q-d}(x+q-d-\xi)f(\xi)d\xi\\ &- \alpha c\int_{x+q-d}^{\infty}(d+\xi-x-q)f(\xi)d\xi,\end{aligned}$$

where we use the definition of $v_0(x) = sx^+ - cx^-$. We have represented partial expectations in the integral form for ease of computation, and we will pursue this approach throughout the proof. Now,

$$\begin{aligned}\frac{\partial G_1(x,q,d)}{\partial q} &= -c(1-\alpha)\\ &+ \pi - (h+\pi-\alpha s+\alpha c)F(x+q-d), \qquad (10)\\ \frac{\partial G_1(x,q,d)}{\partial d} &= R'(d) - \alpha c + (h+\pi-\alpha s+\alpha c)F(x+q-d)\\ &- \pi + (\theta+\alpha s)F(x-d), \qquad (11)\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial^2 G_1(x,q,d)}{\partial q^2} &= -(h+\pi-\alpha s+\alpha c)f(x+q-d) \le 0.\\ \frac{\partial^2 G_1(x,q,d)}{\partial d^2} &= R''(d) - (h+\pi-\alpha s+\alpha c)f(x+q-d)\\ &- (\theta+\alpha s)f(x-d) \le 0.\\ \frac{\partial^2 G_1(x,q,d)}{\partial d\partial q} &= (h+\pi-\alpha s+\alpha c)f(x+q-d).\end{aligned}$$

It can be easily shown that the determinant of the Hessian matrix of $G_1(x,q,d)$ is non-negative; the details are omitted. Further, since $\frac{\partial^2 G_1(x,q,d)}{\partial d^2} \le 0$ and $\frac{\partial^2 G_1(x,q,d)}{\partial q^2} \le 0$, $G_1(x,q,d)$ is jointly concave in q and d for $q \ge 0$ and $d \in \mathcal{D}$.

Suppose now that the result is true for periods $t-1, \cdots, 1$, and consider period t. Then

$$\begin{aligned}\frac{\partial G_t(x,q,d)}{\partial q} &= -c - h[F(x+q-d)] + \pi[1-F(x+q-d)]\\ &+ \alpha\int_{x-d}^{\infty}v_{t-1}'(x+q-d-\xi)f(\xi)d\xi\\ &+ \alpha\int_{-a}^{x-d}v_{t-1}'(q)f(\xi)d\xi \qquad (12)\\ \frac{\partial^2 G_t(x,q,d)}{\partial q^2} &= -(h+\pi)f(x+q-d)\\ &+ \alpha\int_{x-d}^{\infty}v_{t-1}''(x+q-d-\xi)f(\xi)d\xi\\ &+ \alpha v_{t-1}''(q)F(x-d),\end{aligned}$$

which is non-positive as $v_{t-1}'' \le 0$ by induction hypothesis. Also,

$$\begin{aligned}\frac{\partial G_t(x,q,d)}{\partial d} &= R'(d) + (h+\pi)F(x+q-d)\\ &- \pi + \theta F(x-d)\\ &- \alpha\int_{x-d}^{\infty}v_{t-1}'(x+q-d-\xi)f(\xi)d\xi \qquad (13)\\ \frac{\partial^2 G_t(x,q,d)}{\partial d^2} &= R''(d) - (h+\pi)f(x+q-d) - \theta f(x-d)\\ &+ \alpha\int_{x-d}^{\infty}v_{t-1}''(x+q-d-\xi)f(\xi)d\xi\\ &- \alpha v_{t-1}'(q)f(x-d)\\ &\le R''(d) - \theta f(x-d) + \alpha(\theta+h)f(x-d), \qquad (14)\end{aligned}$$

where the inequality follows as by induction hypothesis $v_{t-1}''(\cdot) \le 0$ and $v_{t-1}'(\cdot) \ge -\theta - h$. The expression in (14) is clearly non-positive if $\theta(1-\alpha) \ge \alpha h$. When $\theta(1-\alpha) < h\alpha$, the non-positivity of the expression in (14) can be easily established using $f(\cdot) \le 1$ and $R''(d) \le -h$. Now,

$$\begin{aligned}\frac{\partial^2 G_t(x,q,d)}{\partial d\partial q} &= (h+\pi)f(x+q-d)\\ &- \alpha\int_{x-d}^{\infty}v_{t-1}''(x+q-d-\xi)f(\xi)d\xi.\end{aligned}$$

Finally, we compute the determinant of the Hessian matrix as follows:

$$\begin{aligned}&= [-(h+\pi)f(x+q-d)\\ &+ \alpha\int_{x-d}^{\infty}v_{t-1}''(x+q-d-\xi)f(\xi)d\xi]\\ &[R''(d) - (\theta+\alpha v_{t-1}'(q))f(x-d)]\\ &+ [R''(d) - (h+\pi)f(x+q-d)\\ &+ \alpha\int_{x-d^*}^{\infty}v_{t-1}''(x+q-d-\xi)f(\xi)d\xi - (\theta+\alpha v_{t-1}'(q))f(x-d)]\\ &[\alpha v_{t-1}''(q)F(x-d)].\end{aligned}$$

Since v_{t-1} is concave, to show that the above expression is non-negative it is sufficient to prove that $R''(d) - [\theta + \alpha v_{t-1}'(q)]f(x-d) \le 0$. Since $v_{t-1}'(q) \ge -\theta - h$,

$$R''(d) - [\theta + \alpha v_{t-1}'(q)]f(x-d) \le R''(d) - [\theta + \alpha(-h-\theta)]f(x-d).$$

The RHS can be show to be non-positive using the same argument that was used for proving that the RHS of (14) is non-positive. Therefore, $G_t(x,q,d)$ is a jointly concave function of d and q for $q \ge 0$ and $d \in \mathcal{D}$.

PROOF OF PART 2

We first consider periods $t \in \{T, T-1, \dots, 2\}$. Given any x, suppose that the order quantity is 0. To determine the optimal expected demand in this scenario, note that

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0} = R'(d) + hF(x-d) - \pi[1-F(x-d)] + \theta F(x-d) - \alpha c[1-F(x-d)], \quad (15)$$

where we use $v'_{t-1}(x) = c$ for $x \le 0$ by induction hypothesis. If there exists a $d_0 \in \mathcal{D}$ for which the RHS is equal to 0, then d_0 is the optimal demand. By setting Eq. (15) equal to 0, we fine that d_0 satisfies the following relationship: $F(x-d) = \frac{\pi+\alpha c - R'(d)}{\pi+h+\theta+\alpha c}$. However, if the RHS of Eq. (15) is negative (positive) for all $d \in \mathcal{D}$, then the optimal demand is equal to $\min \mathcal{D}$ ($\max \mathcal{D}$).

When the order quantity is 0, the values of net inventory x can be segmented into three mutually disjoint (one or more of which could be possibly empty) sets depending on the value of corresponding optimal expected demand. Let the three sets be denoted by A, B, and C. The sets A and C consist, of all the inventory values for which $\frac{\partial G_t}{\partial d}|_{q=0} < 0$ for all $d \in \mathcal{D}$ (so $d^* = \min \mathcal{D}$) and $\frac{\partial G_t}{\partial d}|_{q=0} > 0$ for all $d \in \mathcal{D}$ (so $d^* = \max \mathcal{D}$), respectively. On the other hand, the set B consists of all x such that the unconstrained optimal value of d lies in $\mathcal{D}$.

For each $x \in B$, $d^{*\prime}(x) \in (0,1)$. To see this, we use implicit function theorem [23] to obtain $d^{*\prime}(x) = -\frac{\partial^2 G_t/\partial x \partial d}{\partial^2 G_t/\partial d^2} = \frac{(\pi+h+\theta+\alpha c)f(x-d^*)}{(\pi+h+\theta+\alpha c)f(x-d^*)-R''(d^*)} \in (0,1)$. As a result, $x - d^*$ (and hence $F(x-d^*)$) is strictly increasing in x. On the other hand, for $x \in A, C$, $d^{*\prime} = 0$. Once again, $x - d^*$ (and hence $F(x-d^*)$) is strictly increasing in x for all $x \in A, C$.

Now, observe that any value of x for which the unconstrained optimal order quantity is 0 must satisfy the following equation for given optimal demand d^*:

$$\begin{aligned}\frac{\partial G_t(x,q,d^*)}{\partial q}|_{q=0} = 0 = -c - h[F(x-d^*)] &+ \pi[1-F(x-d^*)] \\ &+ \alpha \int_{-a}^{x-d^*} v'_{t-1}(0) f(\xi) d\xi \\ &+ \alpha \int_{x-d^*}^{\infty} v'_{t-1}(x-d^*-\xi) f(\xi) d\xi \\ &= \pi - c(1-\alpha) - (h+\pi)[F(x-d^*)], \quad (16)\end{aligned}$$

where we use $v'_{t-1}(x) = c$ for $x \le 0$ by induction hypothesis. As $F(x-d^*)$ increases with x (as we proved above), there exists a unique value of x for which the RHS is equal to 0. This value is denoted by $\bar{x}_t$. Further, since $d^*(\bar{x}_t)$ is independent of t, it is obvious that $\bar{x}_t$ is also independent of t.

For period 1, the analysis is same as above except that

$$\frac{\partial G_1(x,q,d^*)}{\partial q}|_{q=0} = \pi - c(1-\alpha) - (h+\pi-\alpha s+\alpha c)F(x-d^*). \quad (17)$$

As a consequence, $\frac{\partial G_1(x,q,d)}{\partial q}|_{q=0} \le \frac{\partial G_t(x,q,d)}{\partial q}|_{q=0}$, $t > 1$. Further, $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=0}$ is identical to $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0}, t > 1$. As a consequence, the optimal expected demand when $q = 0$ is the same for all t, including period 1. Therefore, $\bar{x}_1 \le \bar{x}_t$.

PROOF OF PART 2(A)

The uniqueness of $q^*(x)$ and $d^*(x)$ follows from the strict concavity of G_t in q and d.

Consider some $x < \bar{x}_t$ and by way of contradiction, let $q^*(x) = 0$. We argued in the proof of Part 2 that $F(x - d^*(x))$ is increasing in x if the order quantity is zero. As a consequence, $F(x-d^*(x)) < F(\bar{x}_t - d^*(\bar{x}_t))$. Using Eq. (16), this means that $\frac{\partial G_t(x,q,d^*)}{\partial q}|_{q=0} > 0$ implying that the optimal profit will increase if q is increased, which contradicts the assumption that $q^*(x) = 0$.

PROOF OF PART 2(B)

First consider period 1. Suppose that q^* and d^* satisfy $\frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = \frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$. As a consequence, we can add Eqs. (10) and (11) to obtain

$$R'(d^*) - c + (\theta + \alpha s)F(x-d^*) = 0.$$

Using the implicit function theorem [23], $d^*(x)$ is unique and continuously differentiable. Further, using the same theorem,

$$d^{*\prime}(x) = \frac{-(\theta+\alpha s)f(x-d^*)}{R''(d^*) - (\theta+\alpha s)f(x-d^*)} \in (0,1).$$

Similarly, $q^*(x)$ is unique and continuously differentiable. Further,

$$q^{*\prime}(x) = \frac{-R''(d^*)}{R''(d^*) - (\theta+\alpha s)f(x-d^*)} \in (-1,0). \quad (18)$$

Consider now the case in which $\frac{\partial G_1}{\partial d}|_{d=d^*,q=q^*} \neq 0$. Since $q^*(x) > 0$, we still have $\frac{\partial G_1}{\partial q}|_{d=d^*,q=q^*} = 0$. If we substitute Eq. (10), which is equal to 0, in Eq. (11) for any given d, we get

$$\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*} = R'(d) - c + (\theta+\alpha s)F(x-d).$$

Given the concavity of G_1 in d for any x, if no d satisfies $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*} = 0$, the optimal value of d must lie at one of the boundary points of $\mathcal{D}$. Therefore, $d^{*\prime}(x) = 0$. Further, a straightforward application of the implicit function theorem implies that $q^*(x)$ is unique and continuously differentiable and that $q^{*\prime}(x) = -1$.

Next, we consider a generic period $t > 1$. Suppose that q^* and d^* are such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = \frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$. Let $H_1(x,q,d) := \frac{\partial G_t(x,q,d)}{\partial q}$ and $H_2(x,q,d) := \frac{\partial G_t(x,q,d)}{\partial q} + \frac{\partial G_t(x,q,d)}{\partial d}$. That is,

$$H_2(x,q,d) = R'(d) - c + [\theta + \alpha v'_{t-1}(q)]F(x-d).$$

Note that $H_1(x,q^*,d^*) = H_2(x,q^*,d^*) = 0$. Also, $H_1, H_2 \in \mathcal{C}^1$. Let

$$A = \begin{bmatrix} \frac{\partial H_1(x,q,d)}{\partial q} & \frac{\partial H_1(x,q,d)}{\partial d} \\ \frac{\partial H_2(x,q,d)}{\partial q} & \frac{\partial H_2(x,q,d)}{\partial d} \end{bmatrix}|_{q=q^*,d=d^*}.$$

Using the implicit function theorem [23], $q^*(x), d^*(x) \in \mathcal{C}^1$ provided A^{-1} exists. Further, $q^*(x)$ and $d^*(x)$ are unique and

$$\begin{bmatrix} q^{*\prime}(x) \\ d^{*\prime}(x) \end{bmatrix} = A^{-1} \times \begin{bmatrix} \frac{\partial H_1(x,q,d)}{\partial x} \\ \frac{\partial H_2(x,q,d)}{\partial x} \end{bmatrix}|_{q=q^*,d=d^*},$$

Now,

$$\frac{\partial H_1(x,q,d)}{\partial q} = -(h+\pi)f(x+q-d) + \alpha\int_{x-d}^{\infty} v''_{t-1}(x+q-d-\xi)f(\xi)d\xi + \alpha v''_{t-1}(q)F(x-d)$$

$$\frac{\partial H_2(x,q,d)}{\partial d} = R''(d) - (\theta+\alpha v'_{t-1}(q))f(x-d)$$

$$\frac{\partial H_1(x,q,d)}{\partial d} = (h+\pi)f(x+q-d) - \alpha\int_{x-d}^{\infty} v''_{t-1}(x+q-d-\xi)f(\xi)d\xi$$

$$\frac{\partial H_2(x,q,d)}{\partial q} = \alpha v''_{t-1}(q)F(x-d).$$

Similarly,

$$\frac{\partial H_1(x,q,d)}{\partial x} = -(h+\pi)f(x+q-d) + \alpha\int_{x-d}^{\infty} v''_{t-1}(x+q-d-\xi)f(\xi)d\xi$$

$$\frac{\partial H_2(x,q,d)}{\partial x} = (\theta+\alpha v'_{t-1}(q))f(x-d).$$

Using the above expressions,

$$\begin{aligned}Det(A) = {} & [R''(d) - (\theta+\alpha v'_{t-1}(q))f(x-d)] \\ & \cdot \Big[-(h+\pi)f(x+q-d) + \alpha\int_{x-d}^{\infty} v''_{t-1}(x+q-d-\xi) \\ & + \alpha v''_{t-1}(q)F(x-d)\Big] - \Big[(h+\pi)f(x+q-d) \\ & - \alpha\int_{x-d}^{\infty} v''_{t-1}(x+q-d-\xi)f(\xi)d\xi\Big] \\ & \cdot [\alpha v''_{t-1}(q)F(x-d)],\end{aligned}$$

which is strictly positive for any q and d due to concavity of v_{t-1} and $R(d)$; since $f > 0$; and since $R''(d) - (\theta+\alpha v'_{t-1}(q))f(x-d) < 0$. To see this inequality, recall that $v'_{t-1} \geq -\theta - h$, $f \leq 1$, $\alpha < 1$, and $R'' \leq -h$. As a result,

$$\begin{aligned}& R''(d) - (\theta+\alpha v'_{t-1}(q))f(x-d) \\ & \leq R''(d) - (\theta - \alpha(\theta+h))f(x-d) \\ & \leq R''(d) + \alpha h < 0.\end{aligned}$$

Now,

$$q^{*\prime}(x) = \frac{R''(d^*)((h+\pi)f(x+q^*-d^*) - \alpha\int_{x-d^*}^{\infty} v''_{t-1}(x+q^*-d^*-\xi)f(\xi)d\xi)}{Det(A)} < 0. \quad (19)$$

We next show that $q^{*\prime}(x) \geq -1$. It is sufficient to show that the denominator ($Det(A)$) is greater than the numerator in (19). Rearranging terms in the denominator, it can be written as

$$\begin{aligned}& - R''(d^*)[(h+\pi)f(x+q^*-d^*) \\ & - \alpha\int_{x-d^*}^{\infty} v''_{t-1}(x+q^*-d^*-\xi)f(\xi)d\xi \\ & + R''(d^*)\cdot[\alpha v''_{t-1}(q^*)F(x-d^*)] \\ & - (\theta+\alpha v'_{t-1}(q^*))f(x-d^*)\cdot[-(h+\pi)f(x+q^*-d^*) \\ & + \alpha\int_{x-d^*}^{\infty} v''_{t-1}(x+q^*-d^*-\xi)f(\xi)d\xi + \alpha v''_{t-1}(q^*)F(x-d^*)] \\ & - [(h+\pi)f(x+q^*-d^*) \\ & - \alpha\int_{x-d^*}^{\infty} v''_{t-1}(x+q^*-d^*-\xi)f(\xi)d\xi]\cdot[\alpha v''_{t-1}(q^*)F(x-d^*)],\end{aligned}$$

The first line is identical to the numerator in (19). The terms in the second and fifth lines are non-negative. Thus, it is sufficient to show that the terms in third and fourth lines are non-negative. In particular, it is enough to show that $(\theta+\alpha v'_{t-1}(q^*)) \geq 0$. The following lemma, in fact, shows the same, but it is only valid for $q \leq \bar{x}_{t-1}$. We will prove shortly that this restriction would not matter as $q^*(x) \leq \bar{x}_{t-1}$ for all $x \leq \bar{x}_t$ anyway.

LEMMA 1: If $z \leq \bar{x}_{t-1}$, then $\theta + \alpha v'_{t-1}(z) \geq 0$.

PROOF: Note that using the induction hypothesis, $v'_k(\cdot) \leq c, k \leq t-1$. Let $t-1 \geq 2$. Then,

$$\begin{aligned}\theta + \alpha v'_{t-1}(z) &= \theta + \alpha c - \alpha[\theta + \alpha v'_{t+2}(q^*(z))]F(z-d^*(z)) \\ &\geq \theta + \alpha c - \alpha(\theta+\alpha c)F(z-d^*(z)) \\ &= (\theta+\alpha c)[1-\alpha F(z-d^*(z))] \quad \geq 0.\end{aligned}$$

where we have used the following expression for $v'_{t-1}(z)$:

$$v'_{t-1}(z) = c - [\theta + \alpha v'_{t-2}(q^*(z))F(z-d^*(z))], \quad z \leq \bar{x}_{t-1},$$

which we formally derive later in the proof of Part 2(d). [See Eq. (24).] A similar argument can be developed when $t-1=1$ using Eq. (23); the details are omitted. □

For $t > 2$, since $q^{*\prime}(x) \leq 0$ for all $x \leq \bar{x}_t$ and $\bar{x}_t = \bar{x}_{t-1}$, $q^*(x) \leq \bar{x}_{t-1}$ if $q^*(0) \leq \bar{x}_t$. Accordingly, we next show that $q^*(0) \leq \bar{x}_t$.

If $q^*(0) = 0$, then we have nothing to prove since $\bar{x}_t \geq 0$ by Part 2(a). So assume that $q^*(0) > 0$. Then, it must satisfy the following equation obtained by setting $\frac{\partial G_t(0,q,d^*)}{\partial q}|_{q=q^*(0)} = 0$:

$$F(q^*(0)-d^*(0)) = \frac{\pi - c + \alpha\int_{-a}^{\infty} v'_{t-1}(q^*(0)-d^*(0)-\xi)f(\xi)d\xi}{\pi+h}. \quad (20)$$

Since $v'_{t-1}(\cdot) \leq c$, $F(q^*(0)-d^*(0)) \leq \frac{\pi-c+\alpha c}{\pi+h}$. On the other hand, from the proof of Part 2, we know that $F(\bar{x}_t - d^*(\bar{x}_t)) = \frac{\pi-c+\alpha c}{\pi+h}$. Clearly, $q^*(0)-d^*(0) \leq \bar{x}_t - d^*(\bar{x}_t)$. Thus, it suffices to show that $d^*(0) \leq d^*(\bar{x}_t)$ to show that $q^*(0) \leq \bar{x}_t$.

By substituting $\frac{\partial G_t(0,q,d^*(0))}{\partial q}|_{q=q^*(0)}$, which is equal to 0, into $\frac{\partial G_t(0,q^*(0),d)}{\partial d}$, we get $\frac{\partial G_t(0,q^*(0),d)}{\partial d} = R'(d) - c$. By assumption, the value of d at which $R'(d) - c = 0$ is feasible. Hence $d^*(0)$ is equal to the solution of $R'(d) = c$.

Now, at $x = \bar{x}_t$, if we substitute $\frac{\partial G_t}{\partial q}$, which is equal to 0 by definition of $\bar{x}_t$, into $\frac{\partial G_t}{\partial d}$, we get

$$\frac{\partial G_t}{\partial d} = R'(d) - c + [\theta+\alpha c]F(\bar{x}_t - d).$$

If there exists a $d_0 \in \mathcal{D}$ at which the above equation is equal to 0, then $R'(d_0) \leq c$, implying that $d_0 \geq d^*(0)$. If however d_0 is not feasible, then the optimal solution should be $\max \mathcal{D}$. (Note that the optimal solution cannot be at $\min \mathcal{D}$ since that would imply $R'(d) < c$ for all $d \in \mathcal{D}$, which violates the assumption that the value of that d satisfies $R'(d) = c$ is feasible.) Since $\arg\max_d (R(d) - cd) \in \mathcal{D}$, $R'(\max \mathcal{D}) \leq c$, implying that $\max \mathcal{D} \geq d^*(0)$. Consequently, $d^*(0) \leq d^*(\bar{x}_t)$. Hence, $q^*(0) \leq \bar{x}_t$.

The above argument requires that $\bar{x}_t$ be equal to $\bar{x}_{t-1}$. As a result, it does not hold for $t = 2$ since $\bar{x}_1$ may be less than $\bar{x}_2$. Fortunately, it is sufficient to show that $\theta + \alpha v_1'(z) \geq 0$ for $z \leq \bar{x}_2$.

To show that $\theta + \alpha v_1'(z) \geq 0$ for $z \leq \bar{x}_2$, observe that since $v_1'(\cdot)$ decreases due to the hypothesized concavity of v_1, it is enough to establish that $\theta + \alpha v_1'(z) \geq 0$ for $z = \bar{x}_2$. Using Eq. (16), we get

$$F(\bar{x}_2 - d^*(\bar{x}_2)) = \frac{\pi - c + \alpha c}{\pi + h}.$$

However, the proof of Part 2 argues that whenever $q = 0$, the optimal value of d is the same for all periods for any given x. Therefore, the above equation continues to hold for Period 1 as well. Now, since $\bar{x}_2 \geq \bar{x}_1$,

$$v_1'(\bar{x}_2) = -(h + \pi + \theta + \alpha c)F(\bar{x}_2 - d^*(\bar{x}_2)) + \pi + \alpha c,$$

where we use Eq. (25), which is derived in the proof of Part 3(b). Substituting for $F(\bar{x}_2 - d^*(\bar{x}_2))$, we get $v_1'(\bar{x}_2) = -(h+\pi+\theta+\alpha c)\left(\frac{\pi-c+\alpha c}{\pi+h}\right)+\pi+\alpha c$. On simplification, we find that $\theta + \alpha v_1'(\bar{x}_2) \geq (\theta + \alpha c)h > 0$.

We now consider $d^{*\prime}(x)$, which is equal to

$$\begin{aligned}&\big([\alpha v_{t-1}''(q^*)F(x - d^*)] \cdot [-(h + \pi) f(x + q^* - d^*)\\
&+ \alpha \int_{x-d^*}^{\infty} v_{t-1}''(x + q^* - d^* - \xi) f(\xi) d\xi]\\
&+ [(h + \pi) f(x + q^* - d^*)\\
&- \alpha \int_{x-d^*}^{\infty} v_{t-1}''(x + q^* - d^* - \xi) f(\xi) d\xi\\
&- \alpha v_{t-1}''(q^*)F(x - d^*)]\\
&\times [\theta + \alpha v_{t-1}'(q^*) f(x - d^*)]\big) / (Det(A))\end{aligned}$$

Since $Det(A) > 0$, $d^{*\prime}(x) \geq 0$. Further, the numerator in the above expression is less than $Det(A)$, so $d^{*\prime}(x) < 1$.

Finally, we consider the case in which $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*,d=d^*}$ is not necessarily equal to 0 for all $x \leq \bar{x}_t$. Recall from our argument above that $d^*(0)$ is in the interior of $\mathcal{D}$. Thus, $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*,d=d^*}$ is equal to 0 at $x = 0$. As x increases, $d^*(x)$ also increases because its slope is non-negative. Suppose now that there exists a $\underline{x}_t < \bar{x}_t$ such that the unconstrained optimal value of d at $x = \underline{x}_t$ is equal to $\max(\mathcal{D})$. We claim that for all $x \in (\underline{x}_t, \bar{x}_t]$, $d^*(x) = \max(\mathcal{D})$. To see this, observe that for $x > \underline{x}_t$,

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} < \frac{\partial G_t(\underline{x}_t,q,d)}{\partial q}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0,$$

and

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} > \frac{\partial G_t(\underline{x}_t,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0.$$

Both of these inequalities can be proved by showing that the derivatives of $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}}$, and $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}}$ with respect to x increase and decrease, respectively. As a consequence of above inequalities, profit at x can be improved in only one way: decrease q. (Increasing d will also increase profit but that is not possible.)

Suppose now that we gradually decrease the value of q to q_0 such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q_0,d=\max\mathcal{D}}$ becomes equal to 0. If for $q = q_0$, $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}}$ is still non-negative, then that would imply that $(q_0, \max\mathcal{D})$ is optimal at x since profit can only be further increased by increasing d, but that is not possible. Now, adding Eq. (12), which is equal to 0, to Eq. (13), we get

$$\begin{aligned}&\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}}\\
&\qquad = R'(\max\mathcal{D}) - c + [\theta + \alpha v_{t-1}'(q_0)]F(x - \max\mathcal{D}).\end{aligned}$$

Recall that $\theta + \alpha v_{t-1}'(q^*(\underline{x}_t)) \geq 0$ from above, since $q^*(\underline{x}_t) \leq x_{t-1}$. This combined with the concavity of v_{t-1} and $q_0 \leq q^*(\underline{x}_t)$ implies that $\theta + \alpha v_{t-1}'(q_0) \geq \theta + \alpha v_{t-1}'(q^*(\underline{x}_t))$. Finally, $F(x - \max\mathcal{D}) > F(\underline{x}_t - \max\mathcal{D})$. As a result,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}} > \frac{\partial G_t(\underline{x}_t,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0.$$

As a consequence, $d^*(x) = \max\mathcal{D}$ and so $d^{*\prime}(x) = 0$. Since Eq. 12 is equal to 0 for $x \leq \bar{x}_t$, we can apply the implicit function theorem to obtain

$$q^{*\prime}(x) = \frac{\begin{array}{c}(h+\pi)f(x+q^*-d^*)\\ -\alpha\int_{x-d^*}^{\infty} v_{t-1}''(x+q^*-d^*-\xi)f(\xi)d\xi\end{array}}{\begin{array}{c}-(h+\pi)f(x+q^*-d^*)+\alpha\int_{x-d^*}^{\infty} v_{t-1}''(x+q^*-d^*-\xi)\\ \times f(\xi)d\xi + \alpha v_{t-1}''(q^*)F(x-d^*)\end{array}} \in (-1, 0).$$

PROOF OF PART 2(C)

For any t, when $x \leq 0$,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*} = R'(d) - c + \frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*}.$$

Since $\bar{x}_t \geq 0$, $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*} = 0$ for $x \leq 0$. Therefore,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*} = R'(d) - c,$$

which when set to zero yields $R'(d^*) = c$. Hence, the optimal expected demand for $x \leq 0$ maximizes $R(d) - cd$.

PROOF OF PART 2(D)

Using the implicit differentiation rule,

$$\begin{aligned}\frac{\partial v_1(x)}{\partial x} &= \frac{\partial G_1(x,q,d)}{\partial x}|_{q=q^*,d=d^*} + \frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \cdot q^{*\prime}(x)\\
&\quad + \frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}(x), \qquad (21)\end{aligned}$$

where the second term is equal to zero at $q = q^*$ since $\frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$ for $x \leq \bar{x}_t$. The third term is also 0 as either d^* satisfies $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$ or d^* is equal to $\max\mathcal{D}$. Recall that we showed in the proof of Part 2(b) that if $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \neq 0$ for all $x \leq \bar{x}_t$, then there exists $\underline{x}_t$ such that $d^*(x) = \max\mathcal{D}$ for $x \in (\underline{x}_t, \bar{x}_t]$ and

$d^*(x)$ satisfies $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$ for $x \le \underline{x}_t$. Clearly, $d^{*\prime}(x) = 0$ for $x > \underline{x}_t$. Therefore,

$$v_1'(x) = \frac{\partial G_1(x,q,d)}{\partial x}|_{q=q^*,d=d^*} \tag{22}$$

$$= \alpha c - (h + \pi - \alpha s + \alpha c)F(x + q^* - d^*) + \pi - (\theta + \alpha s)F(x - d^*)$$

$$= c - (\theta + \alpha s)F(x - d^*) + \frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \tag{23}$$

where $\frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$. In a similar manner,

$$v_t'(x) = -(h+\pi)F(x + q^* - d^*) + \pi - \theta F(x - d^*) + \alpha \int_{x-d^*(x)}^{\infty} v_{t-1}'(x + q^* - d^* - \xi) f(\xi) d\xi$$

$$= c - \alpha \int_{-a}^{x-d^*} v_{t-1}'(q^*) f(\xi) d\xi - \theta F(x - d^*) + \frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*}$$

$$= c - [\theta + \alpha v_{t-1}'(q^*)]F(x - d^*). \tag{24}$$

Now, from Eq. (23),

$$c \ge v_1'(x) \ge c - (\theta + \alpha s).$$

On the other hand, using Eq. (24),

$$v_t'(x) \ge c - (\theta + \alpha c)F(x - d^*) \ge c(1 - \alpha) - \theta,$$

where the inequality holds by using the induction hypothesis, $v_{t-1}'(\cdot) \le c$. Further, since $\theta + \alpha v_{t-1}(q^*(x)) \ge 0$ for $x \le \bar{x}_t$, as we proved in the proof of Part 2(b), $v_t'(x) \le c$.

Now, when $x \le 0$ $v_1'(x) = c$ since $F(x - d^*) = 0$ for $x \le 0$. Using the same argument, $v_t'(x) = c$. for $x \le 0$.

PROOF OF PART 2(E)

Using Eq. (24),

$$v_t''(x) = -\alpha v_{t-1}''(q^*) q^{*\prime}(x) F(x - d^*) - (\theta + \alpha v_{t-1}'(q^*)) f(x - d^*)(1 - d^{*\prime}(x)) \le 0,$$

where we use the concavity of v_{t-1}, the non-positivity of $q^{*\prime}(x)$, the non-negativity of $(\theta + \alpha v_{t-1}'(q^*))$, as proved in the Proof of Part 2(b), and $d^{*\prime}(x) \in [0, 1]$. The proof for period 1 is similar, and the details are omitted.

PROOF OF PART 3(A)

Consider the following for $x > \bar{x}_t$:

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} = -c - (h+\pi)F(x - d^*(\bar{x}_t)) + \pi + \alpha c,$$

and

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} = R'(d^*(\bar{x}_t)) + (h + \pi + \theta + \alpha c)F(x - d^*(\bar{x}_t)) - \pi - \alpha c.$$

Clearly, $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} < \frac{\partial G_t(\bar{x}_t,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} = 0$ and $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} < \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)}$. At x, we can increase profit in the following three ways: (i) Increase q by $\delta > 0$, (ii) Increase or decrease d from $d^*(\bar{x}_t)$ by $\delta > 0$, and (iii) A combination of (i) and (ii). Implementing (i) will reduce profit as $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} < 0$, so we ignore it. To consider (ii), assume first that $d^*(\bar{x}_t) < \max \mathcal{D}$. Thus, $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} = 0$. In this case, increasing d by δ increases profit at x. On the other hand, if $d^*(\bar{x}_t) = \max \mathcal{D}$, then $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} > 0$, implying that $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} > 0$. Although the profit will improve if we increase d, but that is not possible as d is already at the upper bound. Decreasing the value of d will definitely not increase the profit. Finally, using a similar argument as above, we can show that the profit improvement by (iii) cannot be more than that by (ii). Hence, we can ignore it.

Suppose now that we are able to increase the value of d by δ. Although this will increase the value of $\frac{\partial G_t(x,q,d)}{\partial q}$ and decrease the value of $\frac{\partial G_t(x,q,d)}{\partial d}$, their signs, however, remain unchanged. As a result, once again, the profit can only be improved by increasing the value of d from $d^*(\bar{x}_t) + \delta$ to $d^*(\bar{x}_t) + 2\delta$, provided the new value of d remains feasible. We can keep repeating this procedure until one of $\frac{\partial G_t(x,q,d)}{\partial q}$ and $\frac{\partial G_t(x,q,d)}{\partial d}$ hits zero or d reaches the upper bound. If d reaches the upper bound before either of the two partial derivatives hits zero, the optimal solution will be $q^* = 0$ and $d^* = \max D$.

Suppose now that the either of the two derivatives hits zero before d reaches the upper bound. We claim that the partial derivative of G_t with respect to d will hit zero before the partial derivative of G_t with respect to q does. To see this, suppose on the contrary that as we are increasing the value of d, there exists a d_0 such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d_0} = 0 < \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0}$. This implies that $x - d_0 = \bar{x}_t - d^*(\bar{x}_t)$. Therefore,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0} = R'(d_0) + (h + \pi + \theta + \alpha c)F(\bar{x}_t - d^*(\bar{x}_t)) - \pi - \alpha c.$$

Since $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} = 0$, $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0} = R'(d_0) - R'(d^*(\bar{x}_t)) < 0$ as $d_0 < d^*(\bar{x}_t)$. This is a contradiction.

Thus, $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*} < 0$, and it is optimal to not order. Further, in this case, using the Implicit Function Theorem [23], $d^*(x) \in \mathcal{C}^1$ (since $\frac{\partial G_t(x,q,d)}{\partial d} \in \mathcal{C}^1$) and

$$d^{*\prime}(x) = \frac{(h + \pi + \theta + \alpha c) f(x - d^*)}{R''(d^*) + (h + \pi + \theta + \alpha c) f(x - d^*)} \in (0, 1),$$

if $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(x)} = 0$. Otherwise, $d^{*\prime}(x) = 0$.

PROOF OF PART 3(B)

First, we consider Period 1.

$$v_1(x) = G_1(x, 0, d^*) = R(d^*) - (h + \theta) \int_{-a}^{x-d^*} (x - d^* - \xi) f(\xi) d\xi - (\pi + \alpha c) \int_{x-d^*}^{\infty} (d^* + \xi - x) f(\xi) d\xi$$

$$v_1'(x) = R'(d^*) d^{*\prime}(x) - (h + \theta)(1 - d^{*\prime}(x))F(x - d^*) + (\pi + \alpha c)(1 - d^{*\prime}(x))[1 - F(x - d^*)]$$

$$= d^{*\prime}(x)\{R'(d^*) + (h + \theta)F(x - d^*) - (\pi + \alpha c)[1 - F(x - d^*)]\} - (h + \theta)F(x - d^*) + (\pi + \alpha c)[1 - F(x - d^*)]$$

$$= d^{*\prime}(x)[\frac{\partial G_1(x,q,d)}{\partial d}|_{d=d^*(x),q=0}] - (h+\theta)F(x-d^*)$$
$$+ (\pi + \alpha c)[1 - F(x - d^*(x))]$$
$$= -(h+\theta+\pi+\alpha c)F(x-d^*) + \pi + \alpha c \quad (25)$$

where either $\frac{\partial G_1(x,q,d)}{\partial d}|_{d=d^*(x),q=0}$ or $d^{*\prime} = 0$; the reason is similar to the one given in the proof of Part 2(c) following Eq. (21), and the details are omitted. Next, we consider a period $t > 1$.

$$v_t(x) = G_t(x, 0, d^*)$$
$$= R(d^*) - (h+\theta)\int_{-a}^{x-d^*}(x - d^* - \xi)f(\xi)d\xi$$
$$- \pi \int_{x-d^*}^{\infty}(d^* + \xi - x)f(\xi)d\xi$$
$$+ \alpha \int_{x-d^*}^{\infty} v_{t-1}(x - d^* - \xi)f(\xi)d\xi + \alpha v_{t-1}(0)F(x-d^*)$$
$$v_t'(x) = R'(d^*)d^{*\prime}(x) - (h+\theta)(1 - d^{*\prime}(x))F(x-d^*)$$
$$+ \pi(1 - d^{*\prime}(x))[1 - F(x-d^*)]$$
$$+ \alpha \int_{x-d^*}^{\infty} v_{t-1}'(x - d^* - \xi)f(\xi)d\xi(1 - d^{*\prime}(x))$$
$$= d^{*\prime}(x)[\frac{\partial G_t(x,q,d)}{\partial d}|_{d=d^*,q=0}] - (h+\theta)F(x-d^*)$$
$$+ (\pi+\alpha c)[1 - F(x-d^*)]$$

where we use $v_{t-1}'(x) = c$ for $x \le 0$. Because either $d^{*\prime}(x) = 0$ or $\frac{\partial G_t(x,q,d)}{\partial d}|_{d=d^*,q=0} = 0$, the rationale for which can be explained in the same manner as in the proof of Part 2(c) following Eq. (21), we get

$$v_t'(x) = -(h+\theta+\pi+\alpha c)F(x-d^*) + \pi + \alpha c. \quad (26)$$

From Eqs. (25) and (26) for any given period t,

$$v_t'(x) \ge -(h+\theta+\pi+\alpha c) + \pi + \alpha c \ge -(h+\theta)$$

which provides a lower bound. Also,

$$\frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*} = -c(1-\alpha) + \pi$$
$$- (h + \pi - \alpha s + \alpha c)[F(x-d^*)] \le 0, \quad (27)$$
$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*} = -c - h[F(x-d^*)] + \pi[1 - F(x-d^*)$$
$$+ \alpha c \le 0, \quad t > 1, \quad (28)$$

where the inequalities follow from the proof of Part 3(a). Using (27) and (28), $F(x-d^*) \ge \frac{\pi - c + \alpha c}{\pi + h - \alpha s + \alpha c}$ for $t = 1$, and $F(x-d^*) \ge \frac{\pi - c + \alpha c}{\pi + h}$ for $t > 1$. Substituting these lower bound on $F(x-d^*)$ in Eqs. (25) and (26), we get

$$v_1'(x) - c \le \frac{(\pi - c + \alpha c)(-\theta - \alpha s)}{\pi + h - \alpha s + \alpha c} \le 0,$$
$$\le \frac{(\pi - c + \alpha c)(-\theta - \alpha c)}{\pi + h} \le 0, \quad t > 1,$$

implying that $v_t'(x) \le c$.

PROOF OF PART 3(C)

Using Eqs. (25) and (26),

$$v_t''(x) = -(h+\theta+\pi+\alpha c)f(x - d^*(x))[1 - d^{*\prime}(x)]$$

in any given period t. Since $d^{*\prime}(x) \in [0,1]$, $v_t''(x) \le 0$.

Proof of Theorem 2

PROOF OF PART 1

We first derive the upper bound on the optimal expected demand. When $x > \bar{x}$ and $t \ge 1$, using Eqs. (25) and (26),

$$v_t'(x) = -(h+\theta+\pi+\alpha c)F(x - d^*(x)) + \pi + \alpha c \le c,$$

where the inequality follows from Theorem 1 Part 3(b). Therefore,

$$d^*(x) \le x - F^{-1}[\frac{\pi - c + \alpha c}{h + \theta + \pi + \alpha c}] = x - y,$$

where we define $y = F^{-1}[\frac{\pi - c + \alpha c}{h+\theta+\pi+\alpha c}]$. Thus, the expected demand corresponding to net inventory x is bounded from above by $x - y$.

On the other hand, when $x \le \bar{x}_t$, $d^*(x) \le d^*(\bar{x}_t) \le \bar{x}_t - y$, since $d^*(x)$ is increasing in x.

PROOF OF PART 2

Let d_c be defined such that $\frac{\partial[R(d)]}{\partial d}|_{d=d_c} = c$. Recall from the proof of Theorem 1 Part 2(a), $d^*(x) = d_c$, $x \le 0$. Further, Theorem 1 shows that $d^{*\prime}(x) \ge 0$ for all x. As a consequence, $d^*(x) \ge d_c$ for all x.

PROOF OF PART 3

When $x \le \bar{x}_t$,

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$$
$$= -c - (h+\pi)F(x + q^* - d^*) + \pi + \alpha[v_{t-1}'(q^*)F(x-d^*)$$
$$+ \alpha \int_{x-d^*}^{\infty} v_{t-1}'(x + q^* - d^* - \xi)f(\xi)d\xi]$$
$$\le (-c + \pi + \alpha c) - (h+\pi)F(x + q^* - d^*),$$

where the inequality follows since $v_{t-1}'(x) \le c$ using Theorem 1. Therefore,

$$x + q^* - d^* \le F^{-1}\left[\frac{-c + \pi + \alpha c}{h + \pi}\right] =: \bar{r}.$$

Consequently,

$$x + q^* \le F^{-1}\left[\frac{-c + \pi + \alpha c}{h + \pi}\right] + d^* \le \bar{r} + \bar{x}_t - y,$$

where the second inequality follows since $d^*(x) \le \bar{x}_t - y$ from Part 1 above. Thus, the order quantity is bounded from above by $\bar{r} + \bar{x}_t - y - x$ when $x \le \bar{x}_t$.

The lower bound on order quantity is obtained by using $v_{t-1}'(x) \ge -h-\theta$. As a consequence,

$$x + q^* - d^* \ge F^{-1}\left[\frac{\pi - c - \alpha(h+\theta)}{h + \pi}\right] = \underline{r},$$

which implies $q^* \ge \underline{r} + (d^* - x) \ge \underline{r} + d_c - x$, where d_c is the maximizer of $(R(d) - cd)$.

PROOF OF PROPOSITION 1

Recall from Eq. (20) that

$$F(q^2(0) - d^2(0)) = \frac{\pi - c + \alpha \int_{-a}^{\infty} v'_{t-1}(q^2(0) - d^2(0) - \xi) f(\xi) d\xi}{h + \pi}.$$

Further, since $\bar{x}_t \geq 0$, using Lemma 1, $\theta + \alpha v'_{t-1}(q^2(0) - d^2(0) - \xi) \geq 0$. By substituting $v'_{t-1}(q^2(0) - d^2(0) - \xi) \geq -\frac{\theta}{\alpha}$ and $v'_{t-1}(z) = c$ for $z \leq 0$ in the above equation, we get

$$F(q^2(0) - d^2(0)) \geq \frac{\pi + \alpha c - c}{h + \theta + \pi + \alpha c}. \tag{29}$$

On the other hand, let $G_t^1(x, q, d)$ be the maximand in Eq. (3). Now,

$$\begin{aligned}\frac{\partial G_t^1(x, q^1, d^1)}{\partial q^1} &= -c - (h + \theta + \pi) F(x + q^1 - d^1) + \pi] \\ &\quad + \alpha \int_{x+q^1-d^1}^{\infty} v'_{t-1}(x + q^1 - d^1 - \xi) f(\xi) d\xi \\ &= c(-1 + \alpha) + \pi - (h + \theta + \pi + \alpha c) F(x + q^1 - d^1)\end{aligned} \tag{30}$$

where we use $v'_{t-1}(z) = c$ (without proof) for $z \leq 0$. Thus, the optimal value of q^1 satisfies

$$F(x + q^{1*} - d^{1*}) = \frac{\pi - c + \alpha c}{\pi + h + \theta + \alpha c}. \tag{31}$$

Comparing Eqs. (31) and (29), $q^{1*}(0) \leq q^{2*}(0)$ where we use $d^{1*}(0) = d^{2*}(0)$.

Next, using Eq. (15),

$$F(\bar{x}_t^2 - d^2(\bar{x}_t^2)) = \frac{\pi + \alpha c - c}{h + \pi}.$$

Comparing the above equation with Eq. (31),

$$q_t^1(0) - d_t^1(0) \leq \bar{x}_t^2 - d^2(\bar{x}_t^2).$$

Because the optimal value of $d^2(x)$ is increasing in x, $d^{2*}(\bar{x}_t^2) \geq d^{2*}(0) = d^{1*}(0)$. Therefore, $q_t^1(0) \leq \bar{x}_t^2$.

Proof of Proposition 2

PROOF OF PART 1

Let $G^{\infty}(x, q^{\infty}, d^{\infty})$ be used to denote the maximand in Eq. (4). For $t = 1$,

$$\begin{aligned}&\frac{\partial G_1^{\infty}(x, q^{\infty}, d^{\infty})}{\partial q^{\infty}} \\ &\quad = -c(1 - \alpha) + \pi - (h + \pi - \alpha s + \alpha c) F(x + q^{\infty} - d^{\infty}),\end{aligned}$$

which is identical to Eq. (10). Hence for any x, $q^{\infty *}(x) - d^{\infty *}(x) = q^{2*}(x) - d^{2*}(x)$. Since $d^{2*}(x) \geq d^{2*}(0) = d^{\infty *}(x)$, $q^{\infty *}(x) \leq q^{2*}(x)$.

PROOF OF PART 2

Recall from the proof of Proposition 1 that $\bar{x}_t$ satisfies

$$F(\bar{x}_t - d^2(\bar{x}_t)) = \frac{\pi + \alpha c - c}{h + \pi}.$$

On the other hand,

$$\begin{aligned}\frac{\partial G_t^{\infty}(x, q^{\infty}, d^{\infty})}{\partial q^{\infty}} &= -c - h[F(x + q^{\infty} - d^{\infty})] \\ &\quad + \pi[1 - F(x + q^{\infty} - d^{\infty})] \\ &\quad + \alpha E v'_{t-1}(x + q^{\infty} - d^{\infty} - \xi) f(\xi) d\xi \\ &\leq -c - h[F(x + q^{\infty} - d^{\infty})] \\ &\quad + \pi[1 - F(x + q^{\infty} - d)] + \alpha c,\end{aligned}$$

where we use $v'_{t-1}(\cdot) \leq c$. Since $\frac{\partial G_t^{\infty}(x, q^{\infty}, d^{\infty})}{\partial q^{\infty}}|_{q=q^{\infty *}, d=d^{\infty *}} = 0$,

$$F(x + q^{\infty *} - d^{\infty *}) \leq \frac{\pi - c + \alpha c}{\pi + h}.$$

Setting $x = 0$,

$$F(q^{\infty *}(0) - d^{\infty *}(0)) \leq \frac{\pi - c + \alpha c}{\pi + h}.$$

Therefore,

$$q^{\infty *}(0) - d^{\infty *}(0) \leq \bar{x}_t - d^2(\bar{x}_t).$$

Since $d^2(\bar{x}_t)) \geq d^2(0) = d^{\infty *}(0)$, $q^{\infty *}(0) \leq \bar{x}_t$.

PROOF OF PART 3

When $s = c$, $v'_t(x) = v'_{t-1}(x)$ for $t \geq 2$. (This result can be easily established, but we omit the details.) Essentially, the system becomes stationary. As a result, the optimal base-stock level, $q^{\infty *}(0)$, is independent of t. Another fact that we omit to prove but can be easily established is that $v'_t(x) = c$ for $x \leq q^{*\infty}(0)$. Using this fact, for $x \leq 0$,

$$\begin{aligned}\frac{\partial G_t^{\infty}(x, q^{\infty}, d^{\infty})}{\partial q^{\infty}} &= -c - h[F(x + q^{\infty} - d^{\infty})] \\ &\quad + \pi[1 - F(x + q^{\infty} - d^{\infty})] \\ &\quad + \alpha E v'_{t-1}(x + q^{\infty} - d^{\infty} - \xi) f(\xi) d\xi \\ &= -c - h[F(x + q^{\infty} - d^{\infty})] \\ &\quad + \pi[1 - F(x + q^{\infty} - d)] + \alpha c.\end{aligned}$$

By setting the above equation to 0, we find that the optimal value of $q^{\infty}(x)$ must satisfy the following equation:

$$F(x + q^{\infty *} - d^{\infty *}) = \frac{\pi - c + \alpha c}{\pi + h}. \tag{32}$$

On the other hand, using Eq. (20), which is also valid for $x \leq 0$,

$$F(x + q^{2*}(x) - d^{2*}(x)) \leq \frac{\pi - c + \alpha c}{\pi + h},$$

where we use the upper bound on $v'_{t-1}(\cdot)$, which is equal to c, to obtain the upper bound. Comparing the above equation with Eq. (32),

$$q^{2*}(x) - d^{2*}(x) \leq q^{\infty *}(x) - d^{\infty *}(x).$$

Since $d^{2*}(x) = d^{\infty *}(x)$, $q^{2*}(x) \leq q^{\infty *}(x)$.

Proof of Proposition 3

PROOF OF PART 1

Because one-period expected profits are negative, it is enough to show that for all λ, the sets

$$U_t(x,\lambda) = \{(q,d) : q \geq 0, d \in \mathcal{D} \text{ and } \hat{G}_t(x,q,d) \geq -\lambda\}, \quad t = 1,2,\ldots$$

are compact subsets of $\Re^2$ (Proposition 9.17 in [1]).

Consider any sequence $\{q_n, d_n\} \in U_t(x,\lambda)$. For all n, $\hat{G}_t(x,q_n,d_n) \geq -\lambda$. Let $\{q_n, d_n\} \to \{\bar{q}, \bar{d}\}$. As $\hat{G}_t(x,q,d)$ is continuous in (q,d), $\hat{G}_t(x,\bar{q},\bar{d}) \geq -\lambda$. Hence, $\{\bar{q},\bar{d}\} \in U_t$, which proves that U_t is a closed set.

To show that U_t is bounded, observe that

$$\hat{G}_t(x,q,d) \leq -cq - hE(x+q-d-\xi)^+ - \pi E(d+\xi-x-q)^+ := UB.$$

Let $z = q - d$. Then,

$$UB = -cq - hE(x+z-\xi)^+ - \pi E(\xi - z - x)^+.$$

Note that UB is separable in q and z. As $-hE(x+z-\xi)^+ - \pi E(\xi-z-x)^+$ is concave in z, there exist z_1, z_2 such that $UB \geq -\lambda$ for $z \in [z_1, z_2]$ and for any $q \geq 0$. Similarly, for any z, $UB \geq -\lambda$ for $q \in [0, \frac{\lambda}{c}]$.

The values of d for which $z \in [z_1, z_2]$ such that $q \in [0, \frac{\lambda}{c}]$ are $[-z_2, \frac{c}{\lambda} - z_1]$. Hence, $U_t(x,\lambda) \subset [0, \frac{\lambda}{c}] \times [-z_2, \frac{c}{\lambda} - z_1]$, so it is bounded.

PROOF OF PART 2

Using Proposition 9.8 in Bertsekas and Shreve [1], $\hat{v}$ and $\hat{G}$ satisfy the infinite-horizon optimality Eqs. (7–8) for the transformed model. That is,

$$\hat{v}(x) = \max_{q\geq 0, d\in\mathcal{D}} \hat{G}(x,q,d)$$

where

$$\hat{G}(x,q,d) = -L(x,q,d) - M - cq - \theta E(x-D)^+ + \alpha E\hat{v}(q-(D-x)^+)$$

Substituting $v = \hat{v} + \frac{M}{1-\alpha}$ and $G = \hat{G} + \frac{M}{1-\alpha}$ in the above equations, we get

$$v(x) - \frac{M}{1-\alpha} = \max_{q\geq 0, d\in\mathcal{D}} G(x,q,d) - \frac{M}{1-\alpha}$$

and

$$G(x,q,d) - \frac{M}{1-\alpha} = -L(x,q,d) - M - cq - \theta E(x-D)^+ + \alpha Ev(q-(D-x)^+) - \frac{\alpha M}{1-\alpha}.$$

All the terms involving M cancel out, and the resulting equations are the infinite horizon equations for the original model.

PROOF OF PART 3

The concavity of G in q and d and v in x follows from Part 1 and from concavity of G_t and v_t.

PROOF OF PARTS 4 AND 5

We first prove $v'(x) = c$ for $x \leq 0$. We will also show that for such x the optimal demand is the maximizer of $R(d) - cd$.

The infinite-horizon equation of the original model is

$$v(x) = \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-D)^+ + \alpha Ev(q-(D-x)^+).$$

For $x \leq 0$, the equation becomes

$$v(x) = \max_{q\geq 0, d\in\mathcal{D}} R(d) - hE(x+q-D)^+ - \pi E(D-x-q)^+ - cq + \alpha Ev(x+q-D).$$

Let $y = x + q$ and replace D by $d + \xi$.

$$v(x) = cx + \max_{y\geq x, d\in\mathcal{D}} R(d) - hE(y-d-\xi)^+ - \pi E(d+\xi-y)^+ - cy + \alpha Ev(y-d-\xi).$$

Define $z = y - d$. The above formulation may be written as

$$v(x) = cx + \max_{d\in\mathcal{D}}(R(d) - cd + \max_{z\geq x-d}(-hE(z-\xi)^+ - \pi E(\xi-z)^+ - cz + \alpha Ev(z-\xi))).$$

For a given d, consider the optimization problem for z:

$$\max_{z\geq x-d} (-\pi E(\xi - z) - cz + \alpha Ev(z-\xi)).$$

Let z^* be the unconstrained optimal value of z. Using Karush's lemma, the above function can be written as $A + F(x-d)$, where A is a constant and F is a decreasing, concave function, which is zero if $z^* \geq x - d$ or $d \geq x - z^*$. Using A and F, the optimization problem for d may be stated as

$$\max_{d\in\mathcal{D}} R(d) - cd + A + F(x-d).$$

The maximand is a concave function of d. Further, while F is a decreasing function, as a function of d, it is an increasing function.

Define $d_0 := \arg\max R(d) - cd$. If $x - z^* \leq d_0$, the optimal value of d is d_0. Otherwise, it is greater than d_0. If $x - z^* \leq d_0$, $v(x) = cx +$ a constant independent of x and $v'(x) = c$.

To prove the result, it is sufficient to show that $x - z^* > d_0$ (or $z^* < x - d_0$) is not possible for any $x \leq 0$. Suppose, on the contrary, this is not so and the optimal value of $z^* = x - d_0$ for some $x = \hat{x} < 0$. As a result, $z^* < x - d_0$ for all $x > \hat{x}$. Now, by definition of z^*,

$$\frac{\partial}{\partial z}(-\pi E(\xi - z) - cz + \alpha Ev(z-\xi))|_{z=\hat{x}-d_0} = \pi - c + \alpha Ev'(\hat{x} - d_0 - \xi) = 0.$$

For any given ξ, since $\hat{x} - d_0 - \xi \leq \hat{x}$, $v'(\hat{x} - d_0 - \xi) = c$. As a result,

$$\pi - c + \alpha Ev'(\hat{x} - d_0 - \xi) = \pi - c + \alpha c,$$

which is strictly positive. But this is a contradiction. Hence, z^* cannot be less than $x - d_0$ for any $x \leq 0$. This also means that the optimal value of d is d_0. Finally, as v is concave, $v'(x) \leq c$ for all $x \geq 0$.

The remaining results can be proved in the same manner as Theorem 1 by using the concavity of v and G and $v'(x) = c, x \leq 0$. The details are omitted.

Proof of Proposition 4

$v_t(x) = \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-(D-q)^+)^+ + \alpha E v_{t-1}(x + q - D - (x-(D-q)^+)^+)$.

Let $G_t(x,q,d)$ be the maximand on the RHS of Eq. (9). That is,

$$G_t(x,q,d) = R(d) - cq - hE[x+q-d-\xi]^+ - \pi E[d+\xi-x-q]^+ - \theta E[x-[d+\xi-q]^+]^+ + \alpha E v_{t-1}(x+q-d-\xi-(x-(d+\xi-q)^+)^+).$$

Let $z = q - d$. Then, the optimization problem in any period t becomes

$$G_t(x,q,d) = \max_{d\in\mathcal{D}, z\geq -d} R(d) - c(z+d) - hE[x+z-\xi]^+ - \pi E[\xi-x-z]^+ - \theta E[x-[\xi-z]^+]^+ + \alpha E v_{t-1}(x+z-\xi-(x-(\xi-z)^+)^+).$$

Consider some x such that $q^*(x) > 0$. In that case, the constraint $z \geq -d$ becomes redundant. Thus, the above optimization problem becomes

$$G_t(x,q,d) = \max_{d\in\mathcal{D}}\{R(d) - cd\} + \max_{z}\{-cz - hE[x+z-\xi]^+ - \pi E[\xi-x-z]^+ - \theta E[x-[\xi-z]^+]^+ + \alpha E v_{t-1}(x+z-\xi-(x-(\xi-z)^+)^+)\}.$$

Clearly, the optimal value of d is the maximizer of $R(d) - cd$.

Justification Behind Observation 1

As noted in the main body of the paper, it is sufficient to demonstrate this observation for period

1. Define

$$v_1(x) = \max_{d\in\mathcal{D}, q\geq 0} R(d) - cq - hE[x+q-d-\xi]^+ - \pi E[d+\xi-x-q]^+ - \theta E[x-(d+\xi-q)^+]^+ + \alpha s E[q-d-\xi]^+ - \alpha c[d+\xi-x-q]^+.$$

Consider some x such that $q^* > 0$ so that $\frac{\partial G_1(x,q,d)}{\partial q} = 0 = \frac{\partial G_1(x,q,d)}{\partial d}$. (From Proposition 4, d^* maximizes $R(d) - cd$ and thus lies in the interior of $\mathcal{D}$.) Now,

$$\frac{\partial v_1(x)}{\partial x} = \frac{\partial G_1(x,q^*,d^*)}{\partial x} + \underbrace{\frac{\partial G_1(x,q,d)}{\partial q}\Big|_{q=q^*,d=d^*}}_{=0} \cdot \frac{\partial q^*(x)}{\partial x} + \underbrace{\frac{\partial G_1(x,q,d)}{\partial d}\Big|_{q=q^*,d=d^*}}_{=0} \cdot \frac{\partial d^*(x)}{\partial x} = -(\pi+h+\theta+\alpha c)F(x+q^*-d^*) + \pi + \alpha c.$$

Therefore,

$$\frac{\partial^2 v_1(x)}{\partial x^2} = -(\pi+h+\theta+\alpha c)f(x+q^*-d^*)(1+q^{*\prime}-d^{*\prime}), \quad (33)$$

where $d^{*\prime} = 0$ since d^* is a constant. To show that v_1 is not necessarily concave in x, it suffices to show that $1+q^{*\prime}$ is not necessarily non-negative. Using the implicit function theorem,

$$q^{*\prime}(x) = \frac{(\pi+\alpha c+h+\theta)f(x+q^*-d^*)}{(\alpha s+\theta)f(q^*-d^*) - (\pi+\alpha c+h+\theta)f(x+q^*-d^*)},$$

which is not necessarily greater than -1 if s or θ are strictly positive.

Proof of Theorem 3

PROOF OF PART 1

In Period 1,

$$\frac{\partial G_1(x,q,d)}{\partial q} = -c + \pi + \alpha c - (h+\pi+\alpha c+\theta) F(x+q-d) + (\theta+\alpha s)F(q-d), \quad (34)$$

$$\frac{\partial G_1(x,q,d)}{\partial d} = R'(d) - \pi - \alpha c + (h+\pi+\alpha c+\theta) F(x+q-d) - (\theta+\alpha s)F(q-d). \quad (35)$$

Define $x = \bar{x}$ such that the constrained optimal value of q, $q^*(\bar{x}) = 0$. If there exist multiple such values of x, then we take the maximum of those values. It can be easily shown, however, that there exists at least one such value. Therefore,

$$\frac{\partial G_1(x,q,d)}{\partial q}\Big|_{q=0} = -c + \pi + \alpha c - (h+\pi+\alpha c+\theta)F(x-d),$$

$$\frac{\partial G_1(x,q,d)}{\partial d}\Big|_{q=0} = R'(d) - \pi - \alpha c + (h+\pi+\alpha c+\theta)F(x-d),$$

where we take $F(-d) = 0$ since $d + \xi \geq 0$. Since $\frac{\partial G_1(\bar{x},q,d)}{\partial q}|_{q=0} = 0$, $\frac{\partial G_1(\bar{x},q,d)}{\partial d}|_{q=0} = R'(d) - c$. Thus, $d^*(\bar{x})$ satisfies the equation $R'(d) = c$.

Now, for $x < \bar{x}$,

$$\frac{\partial G_1(x,q,d)}{\partial q}\Big|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_1(\bar{x},q,d)}{\partial q}\Big|_{q=0,d=d^*(\bar{x})} = 0, \text{ and}$$

$$\frac{\partial G_1(x,q,d)}{\partial d}\Big|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_1(\bar{x},q,d)}{\partial d}\Big|_{q=0,d=d^*(\bar{x})} = 0.$$

Given the signs of the slopes of G_1, there are three ways to improve the profit at x: (i) increase q by δ or (ii) decrease d by δ or (iii) increase q by $\frac{\delta}{2}$ and decrease d by $\frac{\delta}{2}$ where $\delta > 0$ and sufficiently small. In Case (i), the profit increases by

$$G_1(x,\delta,d^*(\bar{x})) - G_1(x,0,d^*(\bar{x})) = -c\delta + A,$$

where

$$A = -hE[x+\delta-d^*(\bar{x})-\xi]^+ + hE[x-d^*(\bar{x})-\xi]^+ - \pi E[d^*(\bar{x})+\xi-x-\delta]^+ + \pi E[d^*(\bar{x})+\xi-x]^+ - \theta E[x-(d^*(\bar{x})+\xi-\delta)^+]^+ + \theta E[x-(d^*(\bar{x})+\xi)^+]^+ + \alpha s E[\delta-d^*(\bar{x})-\xi]^+ - \alpha s E[-d^*(\bar{x})-\xi]^+.$$

In Case (ii), the profit improves by

$$G_1(x,0,d^*(\bar{x})-\delta) - G_1(x,0,d^*(\bar{x})) = [R(d^*(\bar{x})-\delta) - R(d^*(\bar{x}))] + A < -c\delta + A,$$

where the inequality follows as $R'(d^*(\bar{x})) = c$ and $R(d)$ is a strictly concave function of d. Consequently, $[R(d^*(\bar{x})-\delta) - R(d^*(\bar{x}))] < -c\delta$. Finally, in case (iii), the profit improves by

$$G_1\left(x, \frac{\delta}{2}, d^*(\bar{x}\right) - \frac{\delta}{2}) - G_1(x,0,d^*(\bar{x})) = \left[R\left(d^*(\bar{x}) - \frac{\delta}{2}\right) - R(d^*(\bar{x}))\right] - c\left[\frac{\delta}{2}\right] + A < -c\frac{\delta}{2} - c\frac{\delta}{2} + A.$$

The reason for the inequality remains the same as for Case (ii). From the above analysis, Case (i) produces the best profit improvement.

Suppose now that $q = \delta$, $d = d^*(\bar{x})$, $\frac{\partial G_1(x,q,d)}{\partial q}|_{q=\delta,d=d^*(\bar{x})} > 0$, and $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=\delta,d=d^*(\bar{x})} < 0$. Once again, we repeat the same argument as above, and find that increasing q by δ will improve profit the most. We keep increasing q by δ until $\frac{\partial G_1(x,q,d)}{\partial q}|_{q,d=d^*(\bar{x})} = 0$. Observe that the value of q that results in $\frac{\partial G_1(x,q,d)}{\partial q}|_{q,d=d^*(\bar{x})} = 0$ will also result in $\frac{\partial G_1(x,q,d)}{\partial d}|_{q,d=d^*(\bar{x})} = 0$. Therefore, $q^*(x) > 0$ and $d^*(x) = d^*(\bar{x})$ for $x < \bar{x}$.

On the other hand, for $x > \bar{x}$,

$$\frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_1(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = 0 = \frac{\partial G_1(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}.$$

Given the signs of the slopes of G_1, to increase profit, we must either decrease q or increase d (or do both). Given that q must remain non-negative, we cannot decrease q. Therefore, our only option is to increase d. Suppose we increase d by δ where $\delta > 0$ and sufficiently small. Then,

$$\frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = B,$$

where $B = -(h + \pi + \theta + \alpha c)(F(x - d^*(\bar{x}) - \delta) - F(x - d^*(\bar{x})))$. Furthermore,

$$\frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} = [R'(d^*(\bar{x}) + \delta) - R'(d^*(\bar{x}))] - B.$$

As $R(\cdot)$ is strictly concave, $R'(d^*(\bar{x}) + \delta) - R'(d^*(\bar{x})) < 0$ and so

$$\left|\frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}\right| > \left|\frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_1(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})}\right|.$$

This means that the slope of G_1 with respect to q increases less in magnitude than the amount by which the slope of G_1 with respect to d decreases. Now, if we repeat the same procedure of increasing expected demand by a small amount $\delta > 0$ over and over again, the above inequality continues to hold. As a result, either we will reach the upper bound of $\mathcal{D}$ or the slope of G_1 with respect to d will become 0 before the slope of G_1 with respect to q does. Thus, the optimal value of q will remain equal to 0.

Now, consider period $t > 1$.

$$\frac{\partial G_t(x,q,d)}{\partial q} = -c + \pi - (h + \pi + \theta)F(x + q - d) + \theta F(q - d) + \alpha \int_{-a}^{q-d} v'_{t-1}(q - d - \xi) f(\xi) d\xi + \alpha \int_{x+q-d}^{\infty} v'_{t-1}(x + q - d - \xi) f(\xi) d\xi \quad (36)$$

$$\frac{\partial G_t(x,q,d)}{\partial d} = R'(d) - \pi + (h + \pi + \theta)F(x + q - d) - \theta F(q - d) - \alpha \int_{-a}^{q-d} v'_{t-1}(q - d - \xi) f(\xi) d\xi - \alpha \int_{x+q-d}^{\infty} v'_{t-1}(x + q - d - \xi) f(\xi) d\xi \quad (37)$$

Once again, set $x = \bar{x}$ such that the unconstrained optimal value of $q^*(\bar{x}) = 0$. In case of multiple such values, choose the highest one. Using the induction hypothesis, $v'_{t-1}(x) = c$ for $x \leq 0$ and for any given d,

$$\frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0} = -c + \pi + \alpha c - (h + \pi + \theta + \alpha c)F(\bar{x} - d) \quad (38)$$

$$\frac{\partial G_t(\bar{x},y,d)}{\partial d}|_{q=0} = R'(d) - \pi - \alpha c + (h + \pi + \theta + \alpha c)F(\bar{x} - d) \quad (39)$$

Following the same argument as for Period 1, we can show that $R'(d^*(\bar{x})) = c$. Now, for $x < \bar{x}$,

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = 0 = \frac{\partial G_t(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}.$$

Using the same argument as for Period 1, we can show that q^* is strictly positive and the optimal demand satisfies $R'(d^*(\bar{x})) = c$. For $x > \bar{x}$,

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = 0 = \frac{\partial G_t(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}.$$

Given the signs of the slopes of G_t, we must either decrease q or increase d to improve profit. However, as q is constrained to be non-negative, we cannot decrease q. Our only option is to increase d. Suppose we increase d by δ where $\delta > 0$ and is sufficiently small. Then

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = B,$$

where $B = -(h+\pi+\theta+\alpha c)(F(x-d^*(\bar{x})-\delta) - F(x-d^*(\bar{x})))$. Similarly,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} = [R'(d^*(\bar{x}) + \delta) - R'(d^*(\bar{x}))] - B.$$

Observe that the value of B is the same as for $t = 1$. Hence, we can easily replicate the argument for $t = 1$; the details are omitted.

PROOF OF PART 2

When $x \geq \bar{x}$, $q^*(x) = 0$. As a result,

$$\frac{\partial G_1(x,0,d)}{\partial d} = R'(d) - \pi - \alpha c + (h + \pi + \alpha c + \theta)F(x - d). \quad (40)$$

Now, when there exists a feasible d at which the above equation is 0, then we can use the implicit function theorem [23] to obtain

$$d^{*\prime}(x) = -\frac{\partial^2 G_t(x,0,d)/\partial x \partial d}{\partial^2 G_t(x,0,d)/\partial d^2}|_{d=d^*} = \frac{(h + \pi + \theta + \alpha c) f(x - d^*)}{-R''(d^*) + (h + \pi + \theta + \alpha c) f(x - d^*)} \in (0,1).$$

Otherwise, if no d feasible exists at which $\frac{\partial G_1(x,0,d)}{\partial d} = 0$, then as we argued in the proof of Part 1, $d^* = \max \mathcal{D}$ and so $d^{*\prime}(x) = 0$.

Using the same theorem, $d^*(x) \in \mathcal{C}^1$ as $G_1(x,0,d)$ is continuously differentiable in (x,d).

For $x \leq \bar{x}$, $d^*(x)$ satisfies $R'(d) - c = 0$. Clearly, $d^*(x) \in \mathcal{C}^1$.

The argument for any other period $t > 1$ is identical as the expression for $\frac{\partial G_t}{\partial d}$ is same as for $\frac{\partial G_1}{\partial d}$, and the details are omitted. For $x < \bar{x}$, $d^{*\prime}(x) = 0$ as the optimal order quantity is a constant.

PROOF OF PART 3

First, consider $x \le 0$. For such x,

$$v_t(x) = \max_{d\in\mathcal{D},q\ge 0} R(d) - cq - hE[x+q-d-\xi]^+ - \pi E[d+\xi-x-q]^+ + \alpha E v_{t-1}(x+q-d-\xi).$$

Let $y = x+q$. Then, the above formulation becomes

$$v_t(x) = cx + \max_{d\in\mathcal{D},y\ge x} R(d) - cy - hE[y-d-\xi]^+ - \pi E[d+\xi-y]^+ + \alpha E v_{t-1}(y-d-\xi). \quad (41)$$

Similar to Proposition 4, it can be easily shown that the optimal value of d satisfies $R'(d) = c$ for all such x. Now, we claim that there exists a value of $y > x$ that produces strictly greater profit than $y = x$. To see this, we compute the derivative of the maximand with respect to y at $y = x$ as follows:

$$-c - (h+\pi)F(x-d^*) + \pi + \alpha c,$$

where we use $v'_{t-1}(x) = c$ for $x \le 0$ by induction hypothesis. Since $x - d^* \le -a$, $F(x-d^*) = 0$. Thus, the derivative becomes $\pi - c + \alpha c > 0$. Hence, increasing y from x to $x+\delta$ will increase profit, so $y^* > x$. With this observation, we can now drop the constraint $y \ge x$ from the RHS in Eq. (41). It can now be easily seen that $v'_t(x) = c$.

Suppose now that $0 < x < \bar{x}$. Using the implicit differentiation rule,

$$v'_1(x) = \frac{\partial G_1(x,q,d)}{\partial x}|_{q=q^*,d=d^*} + \frac{\partial G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \cdot q^{*\prime} + \frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}$$

Because we know that d^* is a constant when $x \le \bar{x}$, the third term is equal to zero. Also, $\frac{G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$. Therefore,

$$\begin{aligned} v'_1(x) &= \frac{\partial G_1(x,q,d)}{\partial x}|_{q=q^*,d=d^*} \\ &= \pi + \alpha c - (h+\pi+\alpha c+\theta)F(x+q^*-d^*) \\ &= c - (\theta+\alpha s)F(q^*-d^*) + \underbrace{\frac{G_1(x,q,d)}{\partial q}|_{q=q^*,d=d^*}}_{=0}. \end{aligned}$$

Although the first equation shows that $v'_1(x) \ge -h-\theta$, the second equation establishes that $v'_1(x)$ is bounded from above by c. Similarly, for period t,

$$\begin{aligned} v'_t(x) &= \frac{\partial G_t(x,q,d)}{\partial x}|_{q=q^*,d=d^*} \\ &= \pi - (h+\pi+\theta)F(x+q^*-d^*) \\ &\quad + \alpha\int_{x+q^*-d^*}^{\infty} v'_{t-1}(x+q^*-d^*-\xi)f(\xi)d\xi \quad (42) \\ &= \pi + \alpha c - (h+\pi+\theta+\alpha c)F(x+q^*-d^*), \end{aligned}$$

where we use $v'_{t-1}(x) = c$ for $x \le 0$. It can be easily seen that $v'_t(x) \ge -\theta - h$. Now, using Eqs. (36) and (42) can also be written as

$$v'_t(x) = c - \theta F(q^*-d^*) - \alpha\int_{-a}^{q^*-d^*} v'_{t-1}(q^*-d^*-\xi)f(\xi)d\xi + \underbrace{\frac{G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*}}_{=0}.$$

Using the induction hypothesis, $v'_{t-1}(x) \ge -\theta - h$. Therefore, $v'_t(x) \le c + (\alpha h - (1-\alpha)\theta)F(q^*-d^*) \le c + (\alpha h - (1-\alpha)\theta)^+$.

PROOF OF PARTS 4 AND 5

For $x \ge \bar{x}$, $q^*(x) = 0$.

$$\begin{aligned} \frac{\partial v_1(x)}{\partial x} &= \frac{\partial G_1(x,q^*,d^*)}{\partial x} + \underbrace{\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}(x)}_{=0} \\ &= -(\pi+h+\theta+\alpha c)F(x-d^*) + \pi + \alpha c. \end{aligned}$$

where $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}(x) = 0$ as either $d^{*\prime}(x)$ or $\frac{\partial G_1(x,q,d)}{\partial d}|_{q=0,d=d^*(x)} = 0$. Clearly, $v'_1(x) \ge -h-\theta$, and

$$v''_1(x) = -(h+\pi+\theta+\alpha c)f(x-d^*(x))[1-d^{*\prime}(x)] \le 0$$

as $d^{*\prime}(x) < 1$, as shown in Part 4 above. Similarly, in period t,

$$\begin{aligned} v'_t(x) &= \frac{\partial G_t(x,q,d)}{\partial x}|_{q=0,d=d^*} \\ &= \pi - (h+\pi+\theta)F(x-d^*) \\ &\quad + \alpha\int_{x-d^*}^{\infty} v'_{t-1}(x-d^*-\xi)f(\xi)d\xi \\ &= \pi + \alpha c - (h+\pi+\theta+\alpha c)F(x-d^*), \end{aligned}$$

where we use $v'_{t-1}(x) = c$ for $x \le 0$. It can be easily seen that $v'_t(x) \ge -h-\theta$. Further,

$$v''_t(x) = -(h+\pi+\theta+\alpha c)F(x-d^*(x))(1-d^{*\prime}(x)) \le 0$$

as $d^{*\prime}(x) < 1$.

The strict concavity of $G_1(x,0,d)$ in d is clear from Eq. (40), as $R(d)$ is strictly concave in d.

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REFERENCES

[1] D.P. Bertsekas and S.E. Shreve, Stochastic optimal control: The discrete-time case, Athena Scientific, Belmont, Massachusetts, 1978.

[2] A. Chande, S. Dhekane, N. Hemachandra, and N. Rangaraj, "Fixed life perishable inventory problem and approximation under price promotion," Technical Report, Industrial Engineering and Operations Research, IIT Bomaby, Mumbai, India, 2004.

[3] A. Chande, S. Dhekane, N. Hemachandra, and N. Rangaraj, Perishable inventory management and dynamic pricing using RFID technology, Sadhana 30 (2005), 445–462.

[4] K. Chandrashekar, N. Dave, N. Hemachandra, and N. Rangaraj, "Timing of discount offers for perishable inventories," in Proceedings of Sixth Asia Pacific Operations Research Society, Allied Publishers, New Delhi, 2003.

[5] X. Chen and D. Simchi-Levi, Coordinating inventory control and pricing strategies with random demand and fixed order cost: The finite horizon case, Oper Res 52 (2004), 887–896.

[6] Y.F. Chen, S. Ray, and Y. Song, Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales, Nav Res Logis 53 (2006), 117–136.

[7] Z.-L. Chen, Integrated production and outbound distribution scheduling in a supply chain: Review and extensions, Oper Res 58 (2010), 130–148.
[8] M.A. Cohen and D. Pekelman, LIFO inventory systems, Manage Sci 24 (1978), 1150–1162.
[9] U. Dave, Survey of literature on continuously deteriorating inventory models: A rejoinder, J Oper Res Soc 42 (1991), 725.
[10] M.J. Fontaine, Y.T. Chung, W.M. Rogers, H.D. Sussmann, P. Quach, S.A. Galel, L.T. Goodnough, and F. Erhun, Improving platelet supply chains through collaborations between blood centers and transfusion services, Transfusion 49 (2009), 2040–2047.
[11] A. Federgruen and A. Heching, Combined pricing and inventory control under uncertainty, Oper Res 47 (1999), 454–475.
[12] B.E. Fries, Optimal ordering policy for a perishable commodity with fixed lifetime, Oper Res 23 (1975), 46–61.
[13] J. Gallagher, Kroger food donations reduce waste and disposal costs, Supermarket News (January 18, 2008). http://supermarketnews.com/center-store/kroger-food-donations-reduce-waste-and-disposal-costs. Accessed on 29 March, 2013.
[14] S.K. Goyal and B.C. Giri, Recent trends in modeling of deteriorating inventory, Eur J Oper Res 134 (2001), 1–16.
[15] I.Z. Karaesmen, A. Scheller-Wolf, and B. Deniz, "Managing perishable and aging inventories: Review and future directions," in: Planning production and inventories in the extended enterprise Vol. 151 of International Series in Operations Research and Management Science, K. Kempf, P. Keskinocak, and P. Uzsoy (Editors), Springer, US, 2011, pp. 393–436.
[16] Q. Li and S. Zheng, Joint inventory replenishment and pricing control with uncertain yield and demand, Oper Res 54 (2006), 696–705.
[17] Y. Li, A. Lim, and B. Rodrigues, Note: Price and Inventory Control for a Perishable Product, Manuf Serv Oper Manag 11 (2009), 538–542.
[18] S. Nahmias, Optimal ordering policies for perishable inventory—II, Oper Res 23 (1975), 735–749.
[19] S. Nahmias, Myopic approximations for the perishable inventory problem, Manag Sci 22 (1976), 1002–1008.
[20] S. Nahmias, Perishable inventory theory: A review, Oper Res 30 (1982), 680–708.
[21] M. Parlar, D. Perry, and W. Stadje, FIFO versus LIFO issuing policies for stochastic perishable inventory systems, Method Comput Appl Probab 12 (2010), 1–13.
[22] F. Raafat, Survey of literature on continuously deteriorating inventory models, J Oper Res Soc 42 (1991), 27–37.
[23] W. Rudin, Principles of mathematical analysis, 3rd ed., McGraw-Hill, Singapore, 1976.
[24] A. Webb, Ripeness sticker takes the guesswork out of picking and eating, The Albuquerque Journal (2006 May 25).
[25] C.A. Yano and S.M. Gilbert, "Coordinating pricing and production/procurement decisions: A review," in: Managing Business Interfaces Vol. 16 of International Series in Quantitative Marketing, J. Eliashberg and A.K. Chakravarty (Editors), Springer, US, 2006, pp. 65–103.