謝 辭

研究所的生涯轉眼間已經要進入尾聲了，也就是說要畢業了。當初想讀研究所的原因是因爲一位曾經是大學部的學長有擔任大學部本系課程的助教，而進入研究所後，我也如願的當上大學部本系課程的助教。

到碩二的時候，就開始要準備資格考了，我很幸運第一次就考過了資格考，因此很快就開始和我的指導教授符聖珍老師討論論文的的部分。那我很感謝我的指導教授提供我講義，每一次當我覺得講義讀得差不多的時候，就會去找教授討論，但每次教授一問就發現原來我沒有讀得很透徹，所以花了很長的時間。討論完畢，就開始最後的論文的整理，過程中，符老師也會指導我寫論文的技巧。

除此之外，我還要感謝陳天進教授，經過一年的實變課程後，在我的論文部分用到實變的概念，並關心我門的狀況。另外，也要感謝其他的老師們也時時關心我們的狀況。當然也要謝謝系辦助教幫忙提醒我們該注意的地方，而且在政大也度過了難忘的時光。最後要再一次感謝我的指導教授符老師幫我修改我的論文。

鄭岱暘　謹誌于
國立政治大學應用數學系
中華民國103年1月

摘 要

在這篇論文中，我們探討以下格子動態系統之行進波解的存在性。

$$\begin{cases} \dfrac{du_j}{dt} = (u_{j+1}^m + u_{j-1}^m - 2u_j^m) + u_j(1 - u_j - kv_j), \\ \\ \dfrac{dv_j}{dt} = d(v_{j+1}^n + v_{j-1}^n - 2v_j^n) + rv_j(1 - v_j - hu_j), \end{cases}$$

其中 $t \in \mathbb{R}$, $j \in \mathbb{Z}$, $d > 0$, $h > 0$, $r > 0$, $k > 0$, 且$m, n \geq 1$. 此系統是洛特卡-沃特競爭模型的離散化系統。洛特卡-沃特競爭模型可用來描述兩個隨機分布且相互競爭的物種之數量變化的情況。然而當兩個競爭的物種呈群聚分布時，則格子動態系統模型會比洛特卡-沃特競爭模型更適合。我們證明當$0 < k < 1 < h$ 或 $0 < h < 1 < k$ ，則存在一個正的常數c_{min}使得格子動態系統中有行進波解若且唯若$c \geq c_{min}$。

Abstract

In this thesis, we investigate the existence of traveling wave solutions of the following lattice dynamical system (LDS):

$$\begin{cases} \dfrac{du_j}{dt} = (u_{j+1}^m + u_{j-1}^m - 2u_j^m) + u_j(1 - u_j - kv_j), \\[2ex] \dfrac{dv_j}{dt} = d(v_{j+1}^n + v_{j-1}^n - 2v_j^n) + rv_j(1 - v_j - hu_j), \end{cases}$$

where $t \in \mathbb{R}$, $j \in \mathbb{Z}$, $d > 0$, $h > 0$, $r > 0$, $k > 0$, and $m, n \geq 1$. This system is the spatial discretization of Lotka-Volterra competition system. It is known that Lotka-Volterra competition system can be used to describe the population dynamics of two competitive species whose individuals are randomly dispersed. However, if individuals of the species are clumped together in groups, then the LDS model is more suitable than the Lotka-Volterra competition model. We show that if $0 < k < 1 < h$ or $0 < h < 1 < k$ then there exists a positive constant c_{min} such that the LDS admits a traveling wave solution if and only if $c \geq c_{min}$.

Contents

Chapter 1

Introduction

We consider the lattice dynamical system (LDS):

$$\begin{cases} \dfrac{du_j}{dt} = (u_{j+1}^m + u_{j-1}^m - 2u_j^m) + u_j(1 - u_j - kv_j), \\ \\ \dfrac{dv_j}{dt} = d(v_{j+1}^n + v_{j-1}^n - 2v_j^n) + rv_j(1 - v_j - hu_j), \end{cases} \tag{1.1}$$

where $t \in \mathbb{R}$, $j \in \mathbb{Z}$, $d > 0$, $h > 0$, $r > 0$, $k > 0$, and $m, n \geq 1$. The system arises in the study of the competition between two species with migration when the habitat is of one-dimensional and divided into regions. Here $u_j(t)$ and $v_j(t)$ stand for the populations at time t and regions j of two species u, v, respectively. With a certain normalization, we assume that the birth rates of species u, v are given by 1, r, the carrying capacities are equal to 1, and the migration coefficients of species u, v are given by 1, d. Here all constants are positive. The constants h, k are inter- specific competition coefficients.

Note that the system (1.1) is a spatial discretization of the following PDE model:

$$\begin{cases} u_t = (u^m)_{xx} + u(1 - u - kv), \\ v_t = d(v^n)_{xx} + rv(1 - v - hu), \end{cases} \tag{1.2}$$

which is called Lotka-Volterra competition system. Here $u = u(x,t)$ and $v = v(x,t)$ represent the population densities of two species at position x and time t.

In the field of population biology, there are three distribution types of species: random, uniform and aggregated dispersion. The PDE system (1.2) can be used to describe the phenomenon of two competition species of random dispersion. For the aggregated dispersion, the LDS model (1.1) is more suitable than the PDE model (1.2).

From the biological point of view, the superior species shall invade the inferior one so that the inferior species will be extinct. To describe such an invading phenomenon, the traveling fronts play an important role. We remark that the existence of traveling waves of the PDE model (1.2) has been investigated by several researchers; see, for example, [1], [2],[3],[4], and [5]. For the LDS (1.1), the existence of traveling waves for the case $m = n = 1$ was studied by [6]. Hence, in this thesis, we turn to consider the more general case that $m, n \geq 1$.

Now we consider the corresponding kinetic system of system (1.2) which is the system (1.2) without diffusion and is given by the following ODE system

$$\begin{cases} \dfrac{du}{dt} = u(1 - u - kv), \\ \dfrac{dv}{dt} = rv(1 - v - hu). \end{cases} \tag{1.3}$$

Obviously, there are three trivial equilibrium points: $(0, 0)$, $(0, 1)$, and $(1, 0)$. When $0 < h, k < 1$ or $h, k > 1$, we have fourth equilibrium point

$$e_4 := (\frac{1-k}{1-hk}, \frac{1-h}{1-hk}).$$

Given initial condition, the long time behavior of the solution of (1.3) can be classified as follows:

(A) If $0 < k < 1 < h$, then $\lim_{t\to\infty}(u, v)(t) = (1, 0)$ (the species u will win).

(B) If $0 < h < 1 < k$, then $\lim_{t\to\infty}(u, v)(t) = (0, 1)$ (the species v will win).

(C) If $h, k > 1$, then $\lim_{t\to\infty}(u, v)(t) = (1, 0)$ or $(0, 1)$ (whether u or v will win depends on initial condition).

(D) If $0 < h, k < 1$, then $\lim_{t\to\infty}(u, v)(t) = e_4$ (u and v will coexist).

In this thesis, we focus on the cases (A) and (B) for the LDS (1.1). In fact, by exchanging the roles of u and v, we know that (A) is equivalent to (B). Without loss of generality, we only consider (A). Hence in the remaining of this thesis we always assume that $0 < k < 1 < h$.

By a traveling wave solution of equation (1.1), we mean a solution of (1.1) of the form

$$(u_j(t), v_j(t)) = (U(\xi), V(\xi)), \ \ \xi = j + ct,$$

such that $(U, V)(-\infty) = (0, 1)$ and $(U, V)(\infty) = (1, 0)$. Here the wave speed c is a constant to be determined and the wave profile $(U, V) \in C^1(\mathbb{R}) \times C^1(\mathbb{R})$ is a pair of nonnegative functions. Upon substituting the ansatz on (U, V) into (1.1), we are led to the governing system for (U, V) as follows:

$$\begin{cases} cU' = D_2^m[U] + U(1 - U - kV), \\ cV' = dD_2^n[V] + rV(1 - V - hU), \\ (U, V)(-\infty) = (0, 1), \ \ (U, V)(\infty) = (1, 0), \\ 0 \leq U, V \leq 1, \end{cases} \text{ on } \mathbb{R}. \tag{1.4}$$

Here the prime indicates differentiation with respect to ξ and $D_2^\alpha[f](\xi) := f^\alpha(\xi + 1) + f^\alpha(\xi - 1) - 2f^\alpha(\xi)$. Hence to investigate the existence of traveling waves of system (1.1) is equivalent to find (c, U, V) satisfying (1.4).

Our main result is stated in the following theorem.

Theorem 1.1. *If $0 < k < 1 < h$, then there exists a positive constant c_{min} such that the system (1.4) admits a solution (c, U, V) satisfying $U' > 0$ and $V' < 0$ on $\mathbb{R}$ if and only if $c \geq c_{min}$.*

We remark that the proof of Theorem 1.1 is obtained by following the arguments of [6].

Chapter 2

Basic Properties and The Monotone Operators

For simplicity of mathematical analysis, we let $W = 1 - V$. Then the system (1.4) can be written in the following form

$$cU' = D_2^m[U] + U[1 - U - k(1 - W)] \text{ on } \mathbb{R}, \tag{2.1a}$$
$$cW' = d\widetilde{D}_2^n[W] + r(1 - W)(hU - W) \text{ on } \mathbb{R}, \tag{2.1b}$$
$$(U, W)(-\infty) = (0, 0), (U, W)(\infty) = (1, 1), \tag{2.1c}$$
$$0 \le U, W \le 1, \tag{2.1d}$$

where $m, n \ge 1$, $0 < k < 1 < h$, and $r, d > 0$ are given, and

$$\widetilde{D}_2^n[W](\cdot) := -D_2^n[1 - W](\cdot) = 2[1 - W(\cdot)]^n - [1 - W(\cdot + 1)]^n - [1 - W(\cdot - 1)]^n.$$

In addition, Theorem 1.1 can be restated in the following theorem.

Theorem 2.1. *If $0 < k < 1 < h$, then there exists a positive constant c_{min} such that the equation (2.1) admits a solution (c, U, W) satisfying $U' > 0$ and $W' > 0$ on $\mathbb{R}$ if and only if $c \ge c_{min}$.*

2.1 The Property of Traveling Wave Solution

To show Theorem 2.1, we first establish some basic properties for solutions of (2.1).

Lemma 2.2. *If (c, U, W) is a solution of (2.1), then $0 < U, W < 1$ in $\mathbb{R}$ and $c > 0$.*

Proof. First, we show that $U > 0$ and $W < 1$ in $\mathbb{R}$. For contradiction, we assume that there exists ξ_0 such that $U(\xi_0) = 0$. Since $U \ge 0$, it following that

$U'(\xi_0) = 0$. Together with (2.1a), we get $U^m(\xi_0+1) = U^m(\xi_0-1) = 0$. Hence, $U(\xi_0+1) = U(\xi_0-1) = 0$. Arguing as above, we can further have $U(\xi_0+n) = 0$ for all $n \in \mathbb{N}$, which contradicts the fact that $U(\infty) = 1$. Hence $U > 0$ in $\mathbb{R}$. By a similar argument, we also find that $W < 1$ in $\mathbb{R}$.

Next, we show that $U < 1$ and $W > 0$ in $\mathbb{R}$. For contradiction, we assume that there exists ξ_1 such that $U(\xi_1) = 1$. Since $U \leq 1$, it implies that $U'(\xi_1) = 0$. Together with (2.1a), we obtain $U^m(\xi_1+1) + U^m(\xi_1-1) = 2 + k(1-W) > 2$, which contradicts the fact $U \leq 1$. Hence $U < 1$ in $\mathbb{R}$. By a similar argument, we also find that $W > 0$ in $\mathbb{R}$.

Finally, we show that $c > 0$. To this end, we need to claim that the functions $R(\xi) := \int_{-\infty}^{\xi} U(s)ds$ and $R_m(\xi) := \int_{-\infty}^{\xi} U^m(s)ds$ are well-defined. Recall that $(U,W)(-\infty) = (0,0)$ and $0 < k < 1$. It follows that there exists $N \gg 0$ such that

$$1 - U - k(1-W) > \frac{1-k}{2} > 0 \tag{2.2}$$

in $(-\infty, -N)$. Integrating the equation (2.1a) from $-\infty$ to ξ and using the fact that $U(-\infty) = 0$, we get that

$$\begin{aligned} cU(\xi) &= \int_{\xi}^{\xi+1} U^m(s)ds - \int_{\xi-1}^{\xi} U^m(s)ds + \int_{-\infty}^{\xi} U[1-U-k(1-W)](s)ds \\ &> \int_{\xi}^{\xi+1} U^m(s)ds - \int_{\xi-1}^{\xi} U^m(s)ds + \frac{1-k}{2}\int_{-\infty}^{\xi} U(s)ds, \end{aligned} \tag{2.3}$$

for all $\xi < -N$. Here we have used (2.2). Since $0 < U < 1$, we can easily deduce from (2.3) that

$$|c| + 1 > cU(\xi) - \int_{\xi}^{\xi+1} U^m(s)ds + \int_{\xi-1}^{\xi} U^m(s)ds > \frac{1-k}{2}\int_{-\infty}^{\xi} U(s)ds.$$

This implies the function $R(\xi)$ is well-defined. Together with the fact that $0 < U^m(\xi) \leq U(\xi)$, we infer that $R_m(\xi)$ is also well-defined. Obviously, it is positive and increasing. Integrating (2.3) from $-\infty$ to x, we deduce that

$$\begin{aligned} cR(x) &= \int_{x}^{x+1} R_m(\xi)d\xi - \int_{x-1}^{x} R_m(\xi)d\xi + \int_{-\infty}^{x}\int_{-\infty}^{\xi} U[1-U-k(1-W)](s)dsd\xi \\ &\geq \int_{x}^{x+1} R_m(\xi)d\xi - \int_{x-1}^{x} R_m(\xi)d\xi + \frac{1-k}{2}\int_{-\infty}^{x} R(\xi)d\xi \quad \text{(by (2.2))} \\ &\geq \frac{1-k}{2}\int_{-\infty}^{x} R(\xi)d\xi \quad \text{(since } R_m(\xi) \text{ is increasing)} \\ &> 0, \quad \text{(since } R > 0 \text{ and } k < 1) \end{aligned} \tag{2.4}$$

for all $x < -N$. Since $R > 0$, (2.4) implies that $c > 0$. This completes the proof of this lemma. $\square$

2.2 The Monotone Operators

Next, we will introduce some monotone operators. For this, we choose a parameter μ large enough such that

$$\mu \geq \mu_0 := \frac{1}{c} \cdot \max\{2m+1+k, 2nd + r(1+h)\}$$

and we define the operators H_1 and H_2 as follows:

$$H_1(U,W) := \mu U + \frac{1}{c} D_2^m[U] + \frac{1}{c} U[1-U-k(1-W)],$$

$$H_2(U,W) := \mu W + \frac{d}{c} \widetilde{D}_2^n[W] + \frac{r}{c}(1-W)(hU-W).$$

Using $H_1(U,W)$ and $H_2(U,W)$, we can rewrite (2.1a)-(2.1b) as

$$U' = H_1(U,W) - \mu U \text{ and } W' = H_2(U,W) - \mu W.$$

Then, multiplying the above two equations by integrating factor $e^{\mu\xi}$, and then integrating the resulting equations from $-\infty$ to ξ, we find that

$$U(\xi) = T_1(U,W)(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H_1(U,W)(s) ds,$$

$$W(\xi) = T_2(U,W)(\xi) := e^{-\mu\xi} \int_{-\infty}^{\xi} e^{\mu s} H_2(U,W)(s) ds.$$

Lemma 2.3. *For $i = 1, 2$, the operators H_i and T_i satisfy the monotonic property in the following sense:*

$$H_i(U_1,W_1)(\cdot) \leq H_i(U_2,W_2)(\cdot) \text{ and } T_i(U_1,W_1)(\cdot) \leq T_i(U_2,W_2)(\cdot) \text{ in } \mathbb{R},$$

provided that

$$0 \leq U_1(\cdot) \leq U_2(\cdot) \leq 1 \text{ and } 0 \leq W_1(\cdot) \leq W_2(\cdot) \leq 1 \text{ in } \mathbb{R}.$$

Proof. Suppose that $0 \leq U_1(\cdot) \leq U_2(\cdot) \leq 1$ and $0 \leq W_1(\cdot) \leq W_2(\cdot) \leq 1$ in $\mathbb{R}$. Then, by simple computations, we get that

$$\begin{aligned}
& c[H_1(U_2,W_2)(\xi) - H_1(U_1,W_1)(\xi)] \\
= \; & [c\mu + (1-k) - (U_2+U_1)(\xi)](U_2-U_1)(\xi) \\
& -2(U_2^m(\xi) - U_1^m(\xi)) + (U_2^m - U_1^m)(\xi+1) \\
& +(U_2^m - U_1^m)(\xi-1) + k(U_2W_2 - U_1W_1)(\xi) \quad (\text{ by definition of } H_1) \\
\geq \; & [c\mu - 2m - 2 + (1-k)](U_2-U_1)(\xi) \\
& \quad (\text{ by } 0 \leq U_1 \leq U_2 \leq 1 \text{ and } 0 \leq W_1 \leq W_2 \leq 1) \\
\geq \; & 0 \quad (\text{ by } \mu \geq (2m+1+k)/c \text{ and } U_1 \leq U_2)
\end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
& c[H_2(U_2,W_2)(\xi) - H_2(U_1,W_1)(\xi)] \\
= \ & [c\mu - r + r(W_2+W_1)(\xi)](W_2-W_1)(\xi) \\
& +2d[(1-W_2(\xi))^n - (1-W_1(\xi))^n] \\
& -d[(1-W_2(\xi+1))^n - (1-W_1(\xi+1))^n] \\
& -d[(1-W_2(\xi-1))^n - (1-W_1(\xi-1))^n] \\
& +rh(U_2-U_1)(\xi) - rh(U_2W_2 - U_1W_1)(\xi) \quad (\text{ by definition of } H_2) \\
\geq \ & [c\mu - 2nd - r(1+h)](W_2-W_1)(\xi) \\
& \quad (\text{ by } 0 \leq U_1 \leq U_2 \leq 1 \text{ and } 0 \leq W_1 \leq W_2 \leq 1) \\
\geq \ & 0. \quad (\text{ by } \mu \geq [(2nd + r(1+h))]/c \text{ and } W_1 \leq W_2)
\end{aligned}
$$

Hence,

$$H_1(U_1,W_1)(\cdot) \leq H_1(U_2,W_2)(\cdot) \text{ and } H_2(U_1,W_1)(\cdot) \leq H_2(U_2,W_2)(\cdot) \text{ in } \mathbb{R}.$$

This, together with the definitions of T_1 and T_2, implies that

$$T_1(U_1,W_1)(\cdot) \leq T_1(U_2,W_2)(\cdot) \text{ and } T_2(U_1,W_1)(\cdot) \leq T_2(U_2,W_2)(\cdot) \text{ in } \mathbb{R}.$$

Hence we complete the proof of this lemma. $\square$

Chapter 3

A Truncation Problem

In this chapter, we consider the following truncation problem:

$$cU' = D_2^m[U] + U[1 - U - k(1 - W)], \forall \xi \in [-\gamma, 0], \tag{3.1a}$$

$$cW' = d\widetilde{D}_2^n[W] + r(1 - W)(hU - W), \forall \xi \in [-\gamma, 0], \tag{3.1b}$$

together with the boundary condition

$$(U, W)(\xi) = (\varepsilon, \varepsilon), \forall \xi \in (-\infty, -\gamma], \tag{3.2}$$

$$(U, W)(\xi) = (1, 1), \forall \xi \in (0, \infty), \tag{3.3}$$

where $\varepsilon \in [0, 1)$. Here

$$U'(-\gamma) := \lim_{h \searrow 0}(U(-\gamma + h) - U(-\gamma))/h, \quad U'(0) := \lim_{h \searrow 0}(U(0) - U(-h))/h,$$

$$W'(-\gamma) := \lim_{h \searrow 0}(W(-\gamma + h) - W(-\gamma))/h, \quad W'(0) := \lim_{h \searrow 0}(W(0) - W(-h))/h.$$

We will use the monotone iteration method to investigate the existence of solutions of (3.1)-(3.3). To begin with, we rewrite (3.1) as follows:

$$U' = H_1(U, W) - \mu U \text{ and } W' = H_2(U, W) - \mu W.$$

Then, multiplying the above the two equations by integrating factor $e^{\mu\xi}$, and then integrating the resulting equations from $-\gamma$ to ξ and using $(U, W)(-\gamma) = (\varepsilon, \varepsilon)$, we find that

$$U(\xi) = T_1^n(U, W)(\xi) \text{ and } W(\xi) = T_2^n(U, W)(\xi), \ \forall \xi \in [-\gamma, 0], \tag{3.4}$$

where

$$T_1^n(U, W)(\xi) = e^{-\mu\xi}\left(\int_{-\infty}^{-\gamma} \varepsilon\mu e^{\mu s} ds + \int_{-\gamma}^{\xi} e^{\mu s} H_1(U, W)(s) ds\right),$$

$$T_2^n(U, W)(\xi) = e^{-\mu\xi}\left(\int_{-\infty}^{-\gamma} \varepsilon\mu e^{\mu s} ds + \int_{-\gamma}^{\xi} e^{\mu s} H_2(U, W)(s) ds\right).$$

Hence one can easily verify that a pair of functions (U, W) with $(U, W)(-\gamma) = (\varepsilon, \varepsilon)$ satisfies (3.1) iff it satisfies the integral equations (3.4). Next, using the monotonic property, we have the fact that

$$0 \leq U_1(\cdot) \leq U_2(\cdot) \leq 1 \text{ and } 0 \leq W_1(\cdot) \leq W_2(\cdot) \leq 1 \text{ in } \mathbb{R}$$

imply that

$$T_1^n(U_1, W_1)(\cdot) \leq T_1^n(U_2, W_2)(\cdot) \text{ and } T_2^n(U_1, W_1)(\cdot) \leq T_2^n(U_2, W_2)(\cdot) \text{ on } [-\gamma, 0].$$

With this property, we have the following lemma.

Lemma 3.1. *If*

$$\varepsilon \leq U(\cdot) \leq 1, \text{ and } \varepsilon \leq W(\cdot) \leq 1 \text{ in } \mathbb{R},$$

then

$$\varepsilon \leq T_1^n(U, W)(\cdot) \leq 1 \text{ and } \varepsilon \leq T_2^n(U, W)(\cdot) \leq 1 \text{ in } [-\gamma, 0].$$

Proof. By the monotonic property, we get

$$\mu\varepsilon + \frac{\varepsilon(1-k)(1-\varepsilon)}{c} = H_1(\varepsilon, \varepsilon) \leq H_1(U, W) \leq H_1(1, 1) = \mu \tag{3.5}$$

and

$$\mu\varepsilon + \frac{r\varepsilon(h-1)(1-\varepsilon)}{c} = H_2(\varepsilon, \varepsilon) \leq H_2(U, W) \leq H_2(1, 1) = \mu \tag{3.6}$$

in $\mathbb{R}$. Together with definitions of $T_1^n(U, W)$ and $T_2^n(U, W)$, we deduce that for $\xi \in [-\gamma, 0]$,

$$T_1^n(U, W)(\xi) \leq e^{-\mu\xi} \left(\int_{-\infty}^{-\gamma} \varepsilon\mu e^{\mu s} + \int_{-\gamma}^{\xi} \mu e^{\mu s} ds \right) = 1 - (1-\varepsilon)e^{-\mu(\gamma+\xi)} \leq 1$$

and

$$T_1^n(U, W)(\xi) \quad \geq \quad \varepsilon + \frac{\varepsilon(1-k)(1-\varepsilon)(1-e^{-\mu(\gamma+\xi)})}{c\mu} \geq \varepsilon.$$

Hence we have $\varepsilon \leq T_1^n(U, W)(\xi) \leq 1$ for all $\xi \in [-\gamma, 0]$. By a similar argument, we also have $\varepsilon \leq T_2^n(U, W)(\xi) \leq 1$ for all $\xi \in [-\gamma, 0]$. □

Now we are in a position to establish the existence of solutions of (3.1)-(3.3).

Lemma 3.2. *For all $n \in \mathbb{N}$ and $\varepsilon \in [0, 1)$, there exists a unique pair of functions $(U^{n,\varepsilon}, W^{n,\varepsilon})$: $\mathbb{R} \mapsto [\varepsilon, 1] \times [\varepsilon, 1]$ satisfying (3.1)-(3.3). Moreover, $(U^{n,\varepsilon}, W^{n,\varepsilon})$ possesses the following properties:*

(1) $U^{n,\varepsilon}, W^{n,\varepsilon} \in C^1((-\gamma, 0)) \cap C((-\infty, 0])$;
(2) $(U^{n,\varepsilon})' > 0$ and $(W^{n,\varepsilon})' > 0$ on $[-\gamma, 0]$;
(3) $\frac{d}{d\varepsilon} U^{n,\varepsilon}(\xi) \geq e^{-\mu(\xi+\gamma)}$ and $\frac{d}{d\varepsilon} W^{n,\varepsilon}(\xi) \geq e^{-\mu(\xi+\gamma)}$, $\forall \xi \in [-\gamma, 0]$.

Proof. Let $n \in \mathbb{N}$ and $\varepsilon \in [0,1)$ be given. We divide the proof into several steps.

Step 1: We construct a pair of functions (U_*, W_*): $\mathbb{R} \mapsto [\varepsilon,1]\times[\varepsilon,1]$ satisfying (3.1)-(3.3) and show that it satisfies property (1).

Define $(U_0^{n,\varepsilon}, W_0^{n,\varepsilon})$ by

$$\begin{cases} (U_0^{n,\varepsilon}, W_0^{n,\varepsilon})(\xi) = (1,1) \text{ if } \xi \in [-\gamma,\infty), \\ (U_0^{n,\varepsilon}, W_0^{n,\varepsilon})(\xi) = (\varepsilon,\varepsilon) \text{ if } \xi \in (-\infty,-\gamma). \end{cases}$$

For each $j \in \mathbb{N}$, we also define $(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})$ as follows:

$$\begin{aligned} &(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})(\xi) = \left(T_1^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon}), T_2^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon})\right)(\xi) \text{ if } \xi \in [-\gamma,0], \\ &(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})(\xi) = (1,1) \text{ if } \xi \in (0,\infty), \\ &(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})(\xi) = (\varepsilon,\varepsilon) \text{ if } \xi \in (-\infty,-\gamma]. \end{aligned}$$

Note that

$$U_1^{n,\varepsilon} = T_1^n(1,1) \le 1 = U_0^{n,\varepsilon} \text{ on } [-\gamma,0],$$
$$W_1^{n,\varepsilon} = T_2^n(1,1) \le 1 = W_0^{n,\varepsilon} \text{ on } [-\gamma,0].$$

Together with the monotonic property and Lemma 3.1, one can easily show that for all $j \in \mathbb{N}$,

$$\varepsilon \le T_1^n(U_j^{n,\varepsilon}, W_j^{n,\varepsilon}) \le T_1^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon}) \le 1,$$
$$\varepsilon \le T_2^n(U_j^{n,\varepsilon}, W_j^{n,\varepsilon}) \le T_2^n(U_{j-1}^{n,\varepsilon}, W_{j-1}^{n,\varepsilon}) \le 1$$

on $[-\gamma,0]$. This, together with definition of $(U_j^{n,\varepsilon}, W_j^{n,\varepsilon})$, implies that the functions $U_j^{n,\varepsilon}$ and $W_j^{n,\varepsilon}$ are non-increasing in j and $\varepsilon \le U_j^{n,\varepsilon}, W_j^{n,\varepsilon} \le 1$ for all $j \in \mathbb{N}$. Therefore,

$$(U_*, W_*)(\xi) := (\lim_{j\to\infty} U_j^{n,\varepsilon}, \lim_{j\to\infty} W_j^{n,\varepsilon})(\xi), \forall \xi \in \mathbb{R}$$

exists. Clearly, $(U_*, W_*)(\xi) = (1,1)$ for all $\xi > 0$ and $(U_*, W_*)(\xi) = (\varepsilon,\varepsilon)$ for all $\xi \le -\gamma$. Moreover, applying the Lebesgue Dominated Convergence Theorem, we get

$$U_*(\xi) = T_1^n(U_*, W_*)(\xi) \text{ and } W_*(\xi) = T_2^n(U_*, W_*)(\xi), \ \forall \xi \in \ [-\gamma,0].$$

Then it is easy to see that $U_*, W_* \in C^1((-\gamma,0)) \cap C((-\infty,0])$.

Step 2: We prove the uniqueness.

For this, we let $(U^*, W^*) : \mathbb{R} \mapsto [\varepsilon,1]\times[\varepsilon,1]$ be another solution of (3.1)-(3.3), then $U^*, W^* \in C^1((-\gamma.0)) \cap C((-\infty,0])$. It suffices to claim that $U^* \equiv U_*$ and $W^* \equiv W_*$ on $[-\gamma,0]$. Since $0 < \varepsilon \le U^* \le 1 = U_0$ and $0 < \varepsilon \le W^* \le 1 = W_0$ on $[-\gamma,0]$, the monotonic property gives that

$$U^* = T_1^n(U^*, W^*) \le T_1^n(U_0, W_0) = U_1 \text{ on } [-\gamma,0],$$

$$W^* = T_2^n(U^*, W^*) \le T_2^n(U_0, W_0) = W_1 \text{ on } [-\gamma, 0].$$

Then, using the iteration, we find that $U^* \le U_*$ and $W^* \le W_*$.

To prove the reverse inequality, we define

$$\bar{\eta} := \inf\{\eta > 0 | U^*(\xi) \ge U_*(\xi - y), W^*(\xi) \ge W_*(\xi - y), \forall \xi \in [-\gamma + y, 0], y \ge \eta\}.$$

Then it is easy to see that $\bar{\eta}$ is well-defined and $0 \le \bar{\eta} \le \gamma$ since $U^*(0) \ge \varepsilon = U_*(-\gamma)$ and $W^*(0) \ge \varepsilon = W_*(-\gamma)$. Indeed, $\bar{\eta} = 0$. To see this, we first claim that

$$H_1(U^*, W^*)(\xi) \ge H_1(U_*, W_*)(\xi - \bar{\eta}) \tag{3.7}$$

and

$$H_2(U^*, W^*)(\xi) \ge H_2(U_*, W_*)(\xi - \bar{\eta}) \tag{3.8}$$

for all $\xi \in \mathbb{R}$. For this, we note that the continuity implies that

$$U^*(\xi) \ge U_*(\xi - \bar{\eta}) \text{ and } W^*(\xi) \ge W_*(\xi - \bar{\eta}), \forall \xi \in [-\gamma + \bar{\eta}, 0].$$

Together with the boundary condition, we have

$$U^*(\xi) \ge U_*(\xi - \bar{\eta}) \text{ and } W^*(\xi) \ge W_*(\xi - \bar{\eta}), \forall \xi \in \mathbb{R}.$$

Hence, using the monotonic property, we get (3.7) and (3.8). Next, using (3.7), we deduce that

$$\begin{aligned}
& U^*(\xi) - U_*(\xi - \bar{\eta}) \\
= \; & T_1^n(U^*, W^*)(\xi) - T_1^n(U_*, W_*)(\xi - \bar{\eta}) \\
= \; & \int_{-\gamma-\xi}^{0} e^{\mu s} H_1(U^*, W^*)(s + \xi) ds - \int_{-\gamma-\xi+\bar{\eta}}^{0} e^{\mu s} H_1(U_*, W_*)(s + \xi) ds \\
& + \int_{-\infty}^{-\gamma-\xi} \varepsilon \mu e^{\mu s} ds - \int_{-\infty}^{-\gamma-\xi+\bar{\eta}} \varepsilon \mu e^{\mu s} ds \\
\ge \; & \int_{-\gamma-\xi}^{-\gamma-\xi+\bar{\eta}} e^{\mu s} [H_1(U^*, W^*)(s + \xi) - \mu \varepsilon] ds \\
\ge \; & \frac{\varepsilon(1-k)(1-\varepsilon)}{c} \int_{-\gamma-\xi}^{-\gamma-\xi+\bar{\eta}} e^{\mu s} ds, \forall \xi \in [-\gamma + \bar{\eta}, 0].
\end{aligned} \tag{3.9}$$

Similarly, using (3.8), we also get

$$W^*(\xi) - W_*(\xi - \bar{\eta}) \ge \frac{r\varepsilon(h-1)(1-\varepsilon)}{c} \int_{-\gamma-\xi}^{-\gamma-\xi+\bar{\eta}} e^{\mu s} ds, \forall \xi \in [-\gamma + \bar{\eta}, 0]. \tag{3.10}$$

Now we are ready to claim that $\bar{\eta} = 0$. For contradiction, we assume that $\bar{\eta} > 0$. If $\varepsilon > 0$, then it follows from (3.9) and (3.10) that $U^*(\xi) - U_*(\xi - \bar{\eta}) > 0$ and $W^*(\xi) - W_*(\xi - \bar{\eta}) > 0$ for all $\xi \in [-\gamma + \bar{\eta}, 0]$. By continuity, there exists $0 < \delta \ll 1$

such that $U^*(\xi) - U_*(\xi - (\bar{\eta} - \delta)) > 0$ and $W^*(\xi) - W_*(\xi - (\bar{\eta} - \delta)) > 0$ for all $\xi \in [-\gamma + \bar{\eta} - \delta, 0]$. This contradicts definition of $\bar{\eta}$. If $\varepsilon = 0$, then (3.9) and (3.10) imply that

$$U^*(\xi) - U_*(\xi - \bar{\eta}) \geq \int_{-\gamma-\xi}^{-\gamma-\xi+\bar{\eta}} e^{\mu s} H_1(U^*, W^*)(s + \xi) ds > 0, \forall \xi \in [-\gamma + \bar{\eta}, 0],$$

where we have used the property that $H_1(U^*, W^*) \geq H_1(\varepsilon, \varepsilon) > 0$. Similarly, we also have $W^*(\xi) - W_*(\xi - \bar{\eta}) > 0$ for all $\xi \in [-\gamma + \bar{\eta}, 0]$. This contradicts definition of $\bar{\eta}$ again. Therefore, $\bar{\eta} = 0$, which implies that $U^* \geq U_*$ and $W^* \geq W_*$. Hence the uniqueness has been proven.

Step 3: We prove (U_*, W_*) satisfies property (2).

By the uniqueness and the fact that $\bar{\eta} = 0$, we deduce that $U_*(\xi) \geq U_*(\xi - s)$ and $W_*(\xi) \geq W_*(\xi - s)$ for all $s \geq 0$ and $\xi \in \mathbb{R}$. This implies that $(U_*)' \geq 0$ and $(W_*)' \geq 0$ for all $\xi \in [-\gamma, 0]$. By the monotonic property, we find

$$H_i(U_*, W_*)(s) \leq H_i(U_*, W_*)(\xi), \ \forall s \leq \xi, \ i = 1, 2. \tag{3.11}$$

For $i = 1, 2$ and $\xi \in [-\gamma, 0]$, differentiating $T_i^n(U_*, W_*)(\xi)$, we get

$$\begin{aligned}
& T_i^n(U_*, W_*)'(\xi) \\
= \; & -\mu\varepsilon e^{-\mu(\gamma+\xi)} + H_i(U_*, W_*)(\xi) - \mu \int_{-\gamma}^{\xi} e^{\mu(s-\xi)} H_i(U_*, W_*)(s) ds \\
= \; & -\mu\varepsilon e^{-\mu(\gamma+\xi)} + H_i(U_*, W_*)(\xi) e^{-\mu(\gamma+\xi)} \\
& + \mu \int_{-\gamma}^{\xi} e^{\mu(s-\xi)} H_i(U_*, W_*)(\xi) ds - \mu \int_{-\gamma}^{\xi} e^{\mu(s-\xi)} H_i(U_*, W_*)(s) ds \\
\geq \; & e^{-\mu(\gamma+\xi)} [H_i(U_*, W_*)(\xi) - \mu\varepsilon] \quad \text{(by (3.11))} \\
> \; & 0. \quad \text{(by (3.5) and (3.6))}
\end{aligned}$$

Since $U_*'(\xi) = T_1^n(U_*, W_*)'(\xi)$ and $W_*'(\xi) = T_2^n(U_*, W_*)'(\xi)$, the above inequality implies that (U_*, W_*) satisfies property (2).

Step 4: We prove that property (3) is satisfied.

Let $0 \leq \varepsilon_1 < \varepsilon_2 \leq 1$. Then, by the monotonic property, we know that $U^{n,\varepsilon_2} \geq U^{n,\varepsilon_1}$ for all $\xi \in [-\gamma.0]$. From this, we get that for all $\xi \in [-\gamma.0]$,

$$\begin{aligned}
& U^{n,\varepsilon_2}(\xi) - U^{n,\varepsilon_1}(\xi) \\
= \; & \int_{-\gamma}^{\xi} e^{\mu(s-\xi)} [H_1(U^{n,\varepsilon_2}, W^{n,\varepsilon_2})(s) - H_1(U^{n,\varepsilon_1}, W^{n,\varepsilon_1})(s)] ds + (\varepsilon_2 - \varepsilon_1) e^{-\mu(\gamma+\xi)} \\
\geq \; & (\varepsilon_2 - \varepsilon_1) e^{-\mu(\gamma+\xi)},
\end{aligned}$$

which follows that

$$\frac{d}{d\varepsilon} U^{n,\varepsilon}(\xi) \geq e^{-\mu(\gamma+\xi)}.$$

By a similar way, we also get

$$\frac{d}{d\varepsilon}W^{n,\varepsilon}(\xi) \geq e^{-\mu(\gamma+\xi)}, \forall \xi \in [-\gamma.0].$$

Hence $(U^{n,\varepsilon}, W^{n,\varepsilon})$ satisfies property (3). This completes the proof of this lemma. $\square$

Chapter 4

Proof of Theorem 2.1

To prove the existence of the solution of (2.1), we need to use the super-solution.

4.1 Super-solution and Its Role

First, we give definition of super-solutions.

Definition 4.1. *Given a constant $c > 0$. A continuous function $(U^+, W^+) : \mathbb{R} \mapsto (0,1] \times (0,1]$ is called a super-solution of (2.1), if W^+ is a non-constant function, $U^+(\infty) = W^+(\infty) = 1$, and both U^+ and W^+ are differentiable a.e in $\mathbb{R}$ such that*

$$\begin{cases} c(U^+)' \geq D_2^m[U^+] + U^+[1 - U^+ - k(1 - W^+)] \\ c(W^+)' \geq d\widetilde{D}_2^n[W^+] + r(1 - W^+)(hU^+ - W^+) \end{cases} \quad a.e \ in \ \mathbb{R}. \tag{4.1}$$

The following lemma gives us an information on the role of super-solution in the existence of solutions of (2.1).

Lemma 4.2. *If there exists a super-solution (U^+, W^+) satisfying $U^+ = W^+ = 1$ on $[0, \infty)$ for a given $c > 0$, then (2.1) admits a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$.*

To show Lemma 4.2, we need Helly's Lemma. For readers' convenience, we state it in the following.

Lemma 4.3. *(Helly's Lemma) Let $\{U_n\}_{n \in \mathbb{N}}$ be a sequence of uniformly bounded and non-decreasing functions in $\mathbb{R}$. Then there exists a subsequence $\{U_{n_i}\}$ of $\{U_n\}$ and a non-decreasing function U such that $U_{n_i} \to U$ pointwise in $\mathbb{R}$ as $i \to \infty$.*

In the sequel, we say that a vector-valued function (U, W) is non-decreasing in $\mathbb{R}$ if both U and W are non-decreasing in $\mathbb{R}$.

proof of Lemma 4.2: Since W^+ is a non-constant function with $0 < W^+ \leq 1$ on $\mathbb{R}$ and $W^+ = 1$ on $[0, \infty)$, there exists $\gamma_0 > 0$ such that $W^+(-\gamma_0) = \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$. Then, for each $n > 2\gamma_0$, we claim that there exists only one $\varepsilon = \varepsilon_n \in (0,1)$ such that $W^{n,\varepsilon_n}(-\frac{n}{2}) = \varepsilon_0$.

To this end, we need to show that $W^{n,0}(-\frac{n}{2}) < \varepsilon_0$ for any given $n > 2\gamma_0$. For this, we define

$$\bar{\eta} := \inf\{\eta > 0 | U^+(\xi) \geq U^{n,0}(\xi - \eta), W^+(\xi) \geq W^{n,0}(\xi - \eta), \forall \xi \in (-\infty, 0]\}.$$

Then $\bar{\eta}$ is well-defined and $0 \leq \bar{\eta} \leq \gamma$ since $U^+(\xi) \geq 0 = U^{n,0}(\xi - \gamma)$ and $W^+(\xi) \geq 0 = W^{n,0}(\xi - \gamma)$ for all $\xi \in (-\infty, 0]$. By continuity, $U^+(\xi) \geq U^{n,0}(\xi - \bar{\eta})$ and $W^+(\xi) \geq W^{n,0}(\xi - \bar{\eta})$ for all $\xi \in (-\infty, 0]$, which, together with the monotonic property of H_1 and H_2, implies that

$$H_1(U^+, W^+)(\xi) \geq H_1(U^{n,0}, W^{n,0})(\xi - \bar{\eta}) \tag{4.2}$$

and

$$H_2(U^+, W^+)(\xi) \geq H_2(U^{n,0}, W^{n,0})(\xi - \bar{\eta}) \tag{4.3}$$

for all $\xi \in (-\infty, 0]$. Then, using (4.4), we deduce that

$$\begin{aligned}
& W^+(\xi) - W^{n,0}(\xi - \bar{\eta}) \\
\geq\ & T_2(U^+, W^+)(\xi) - T_2^n(U^{n,0}, W^{n,0})(\xi - \bar{\eta}) \\
=\ & \int_{-\infty}^{0} e^{\mu s} H_2(U^+, W^+)(s + \xi) ds - \int_{-\gamma-\xi+\bar{\eta}}^{0} e^{\mu s} H_2(U^{n,0}, W^{n,0})(s + \xi - \bar{\eta}) ds \\
\geq\ & \int_{-\infty}^{-\gamma-\xi+\bar{\eta}} e^{\mu s} H_2(U^+, W^+)(s + \xi) ds > 0,
\end{aligned}$$

for all $\xi \in [-\gamma + \bar{\eta}, 0]$. Similarly, using using (4.3), we also have $U^+(\xi) - U^{n,0}(\xi - \bar{\eta}) > 0$ for all $\xi \in [-\gamma + \bar{\eta}, 0]$. By a similar proof of lemma 3.2, we obtain $\bar{\eta} = 0$, which implies that $W^+(\xi) \geq W^{n,0}(\xi)$ on $(-\infty, 0]$. Together with property (2) of Lemma 3.2, we get

$$W^{n,0}\left(-\frac{n}{2}\right) < W^{n,0}(-\gamma_0) \leq W^+(-\gamma_0) = \varepsilon_0.$$

On the other hand, from definition of $W^{n,\varepsilon}$ and its increasing property, we see that $W^{n,\varepsilon}(-\frac{n}{2}) \to 1$ as $\varepsilon \to 1$. Since $\frac{d}{d\varepsilon} W^{n,\varepsilon}(\xi) > 0$ for all $\xi \in [-\gamma, 0]$, we conclude that there exists an unique $\varepsilon = \varepsilon_n \in (0, \varepsilon_0] \subset (0,1)$ such that $W^{n,\varepsilon_n}(-\frac{n}{2}) = \varepsilon_0$.

Now, we are in a position to show the existence of a solution (c, U, W). To begin with, we consider the sequence of functions $\{U^{n,\varepsilon_n}(\cdot - \frac{n}{2}), W^{n,\varepsilon_n}(\cdot - \frac{n}{2})\}_{n > 2\gamma_0}$ in $\mathbb{R}$. By Helly's Lemma, there exists a subsequence $\{U^{n_i,\varepsilon_{n_i}}(\cdot - n_i/2), W^{n_i,\varepsilon_{n_i}}(\cdot - n_i/2)\}$ and a non-decreasing function $(U, W) : \mathbb{R} \mapsto [0,1] \times [0,1]$ such that

$$(U^{n_i,\varepsilon_{n_i}}(\xi - n_i/2), W^{n_i,\varepsilon_{n_i}}(\xi - n_i/2)) \to (U, W)(\xi), \forall \xi \in \mathbb{R},$$

as $i \to \infty$. By LDCT, $U(\xi) = T_1(U, W)(\xi)$ and $W(\xi) = T_2(U, W)(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, it is easy to verify that $W(0) = \lim_{i \to \infty} W^{n_i,\varepsilon_{n_i}}(-n_i/2) = \varepsilon_0$, $0 \leq U, W \leq 1$ in $\mathbb{R}$, and $U, W \in C^1(\mathbb{R})$.

Next, we prove that (U, W) satisfies the boundary condition. Since U and W are non-decreasing in $\mathbb{R}$ and $0 \leq U, W \leq 1$ in $\mathbb{R}$, both $U(\pm\infty)$ and $W(\pm\infty)$ exist. By definition of U and W and L'Hospital's rule, we have

$$\begin{aligned}\lim_{\xi\to\pm\infty} U(\xi) &= \lim_{\xi\to\pm\infty} T_1(U,W)(\xi) \\ &= \lim_{\xi\to\pm\infty} \left\{ U(\xi) + \frac{1}{c\mu}[D_2^m[U](\xi) + U(\xi)(1 - U(\xi) - k(1 - W(\xi)))] \right\}\end{aligned}$$

and

$$\begin{aligned}\lim_{\xi\to\pm\infty} W(\xi) &= \lim_{\xi\to\pm\infty} T_2(U,W)(\xi) \\ &= \lim_{\xi\to\pm\infty} \left\{ W(\xi) + \frac{1}{c\mu}[d\widetilde{D}_2^n[W](\xi) + r(1 - W(\xi))(hU(\xi) - W(\xi))] \right\}.\end{aligned}$$

This implies that

$$\begin{cases} U(\pm\infty)(1 - U(\pm\infty) - k(1 - W(\pm\infty))) = 0, \\ (1 - W(\pm\infty))(hU(\pm\infty) - W(\pm\infty)) = 0. \end{cases} \tag{4.4}$$

Hence $U(\pm\infty), W(\pm\infty) \in \{0, 1\}$. Since W is non-decreasing and $W(0) = \varepsilon_0 \in (0, 1)$, we have $W(-\infty) = 0$ and $W(\infty) = 1$. Moreover, using $W(-\infty) = 0$, we deduce from (4.4) that $U(-\infty) = 0$. Note that $U \not\equiv 0$ in $\mathbb{R}$, since otherwise, by integrating both sides of (2.1b) over $\mathbb{R}$, we have $0 < c = -r\int_{\mathbb{R}} W(\xi)(1 - W(\xi))d\xi < 0$, a contradiction. Using the fact that $U(-\infty) = 0$, U is nondecreasing, and $U \not\equiv 0$ in $\mathbb{R}$, we obtain that $U(\infty) = 1$.

Finally, differentiating as in lemma 3.2, we know that $U' > 0$, and $W' > 0$ in $\mathbb{R}$. The proof of lemma 4.2 is thus accomplished. □

Indeed, the condition $U^+ = W^+ = 1$ on $[0, \infty)$ in Lemma 4.2 can be replaced by the monotonic property. Specifically, we state it in the following lemma.

Lemma 4.4. *If there exists a super-solution (U^+, W^+) of (2.1) with $(U^+)' > 0$, $(W^+)' > 0$ for a given $c > 0$, then (2.1) admits a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$.*

Proof. Let $c > 0$ be given. Suppose (U^+, W^+) is a super-solution of (2.1) with $(U^+)' > 0$ and $(W^+)' > 0$. In the sequel, we will use Lemma 4.2 to get a solution of (2.1). For each $0 < \delta \ll 1$, we define

$$(U_\delta^+(\xi), W_\delta^+(\xi)) := (\min\{1, (1+\delta)U^+(\xi)\}, \min\{1, (1+\delta)W^+(\xi)\}), \forall \xi \in \mathbb{R},$$

then there exist $M_1 = M_1(\delta) \gg 1$ and $M_2 = M_2(\delta) \gg 1$ such that

$$U_\delta^+ \equiv 1 \text{ on } [M_1, \infty), U_\delta^+ < 1 \text{ on } (-\infty, M_1),$$

$$W_\delta^+ \equiv 1 \text{ on } [M_2, \infty), W_\delta^+ < 1 \text{ on } (-\infty, M_2).$$

Then we will claim (U_δ^+, W_δ^+) is a super-solution of the problem P_δ:

$$\begin{cases} cU' = (D_2^m[U])_\delta + f_\delta(U,W), \\ cW' = d(\widetilde{D}_2^n[W])_\delta + g_\delta(U,W), \\ (U,W)(-\infty) = (0,0), (U,W)(\infty) = (1,1), \\ 0 \le U, W \le 1, \end{cases}$$

where

$$(D_2^m[U])_\delta = \frac{D_2^m[U]}{(1+\delta)^{m-1}},$$

$$(\widetilde{D}_2^n[W])_\delta = -\frac{D_2^n[(1+\delta)^n - W]}{(1+\delta)^{n-1}},$$

$$f_\delta(U,W) = \min\left\{ U\left(1 - k - \frac{1}{1+\delta}(U - kW)\right), U(1 - U - k(1-W))\right\},$$

$$g_\delta(U,W) = \min\left\{ r(hU - W)\left(1 - \frac{1}{1+\delta}W\right), r(hU - W)(1-W)\right\}.$$

Without loss of generality, we assume that $M_1 \le M_2$. If $M_1 = M_2$, then it is trivial. Now we turn to consider the case $M_1 < M_2$. It is clear that P_δ holds on $(-\infty, M_1) \cup (M_2, \infty)$. Suppose that $M_1 < \xi < M_2$, then

$$U_\delta^+(\xi) = 1,\ (U_\delta^+)'(\xi) = 0, \tag{4.5}$$

and

$$W_\delta^+(\xi) = (1+\delta)W^+(\xi),\ (W_\delta^+)'(\xi) = (1+\delta)(W^+)'(\xi). \tag{4.6}$$

Note that $W_\delta^+(\xi+1) \le (1+\delta)W^+(\xi+1)$ and $U_\delta^+(\xi) \le (1+\delta)U^+(\xi)$. Using (4.5) and the inequalities

$$(D_2^m[U_\delta^+])_\delta(\xi) = U_\delta^+(\xi-1)^m - 1 \le 0$$

and

$$f_\delta(U_\delta^+, W_\delta^+)(\xi) \le U_\delta^+(\xi)[1 - U_\delta^+(\xi) - k(1 - W_\delta^+(\xi))] = -k(1 - W^+(\xi)) \le 0,$$

we deduce that

$$c(U_\delta^+)'(\xi) \ge (D_2^m[U_\delta^+])_\delta(\xi) + f_\delta(U_\delta^+, W_\delta^+)(\xi).$$

Using (4.6), we get

$$\begin{aligned} c(W_\delta^+)'(\xi) &= (1+\delta)(W^+)'(\xi) \\ &\ge (1+\delta)[d\widetilde{D}_2^n[W^+](\xi) + r(hU^+ - W^+)(1 - W^+)(\xi)] \\ &\ge d(\widetilde{D}_2^n[W_\delta^+])_\delta(\xi) + g_\delta(U_\delta^+, W_\delta^+)(\xi). \end{aligned}$$

These imply that P_δ holds for $M_1 < \xi < M_2$. Therefore, (U_δ^+, W_δ^+) is a super-solution of the problem P_δ.

Next, we set $\widehat{U}_\delta^+(\xi) = U_\delta^+(\xi + M_2)$ and $\widehat{W}_\delta^+(\xi) = W_\delta^+(\xi + M_2)$. Then $\widehat{U}_\delta^+ = \widehat{W}_\delta^+ = 1$ on $[0, \infty)$ and $(\widehat{U}_\delta^+, \widehat{W}_\delta^+)$ is a super-solution of P_δ. By lemma 4.2, we obtain a solution (U_δ, W_δ) of P_δ with $U'_\delta > 0$ and $W'_\delta > 0$ in $\mathbb{R}$.

Now, let $\{(c, U_{\delta_i}, W_{\delta_i})\}$ be a sequence of monotone increasing of P_{δ_i} such that $W_{\delta_i}(0) = \frac{1}{2}$ for all i and $\delta_i \to 0$ as $i \to \infty$. By Lemma 4.3 (Helly's Lemma), there exists a subsequence $\{(c, U_{\delta_{i_j}}, W_{\delta_{i_j}})\}$ and a monotone non-decreasing function (U_0, W_0) such that $(c, U_{\delta_{i_j}}, W_{\delta_{i_j}}) \to (c, U_0, W_0)$ pointwise in $\mathbb{R}$ as $j \to \infty$. In addition $0 \le U_0, W_0 \le 1$ and $W_0(0) = \frac{1}{2}$. Therefore, (c, U_0, W_0) satisfies (2.1) such that $U'_0 > 0$ and $W'_0 > 0$ in $\mathbb{R}$ (By Lemma 2.2). The proof of lemma 4.4 is thus completed. □

Finally, we construct a super-solution of (2.1) for $c \gg 1$.

Lemma 4.5. *For sufficiently large $c > 0$, (U^+, W^+), where $U^+(\xi) = W^+(\xi) = \min\{1, e^\xi\}$, is a super-solution of (2.1).*

Proof. By choosing

$$c \ge c_1 := \max\{(e^m + e^{-m} - 2) + (1 - k), d[(e - 1)^n - (1 - e^{-1})^n] + r(h - 1)\},$$

we will find that (U^+, W^+) is a super-solution of (2.1). □

4.2 Proof of Theorem 2.1

Now we are in a position to prove Theorem 2.1.

Proof. By Lemma 4.2 and 4.5, (2.1) admits a solution (c, U, W) with $U' > 0$ and $W' > 0$ for all $c \ge c_1$. It follows that the constant

$$c_{min} := \inf\{c > 0 | (2.1) \text{ has a solution } (c, U, W) \text{ with } U' > 0 \text{ and } W' > 0 \text{ in } \mathbb{R}\}$$

is well-defined. Since a monotone front with speed c_0 gives a super-solution of (2.5) for all $c > c_0$, Lemma 4.4 implies that (2.1) has a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$ for all $c > c_{min}$.

Now, we need to claim that (2.1) has a solution (c, U, W) with $U' > 0$ and $W' > 0$ in $\mathbb{R}$ for $c = c_{min}$. To this end, we let $\{c_i, U_i, W_i\}$ be a sequence of solutions of (2.1) with $c = c_i$ such that $W_i(0) = \frac{1}{2}$, $U'_i, W'_i > 0$ in $\mathbb{R}$ for all $i \in \mathbb{N}$, and $c_i \downarrow c_{min}$ as $i \to \infty$. By the same proof of lemma 4.4, (2.1) has a solution (c, U_*, W_*) with $U'_* > 0$ and $W'_* > 0$ in $\mathbb{R}$ when $c = c_{min}$.

Finally, the constant $c_{min} > 0$ by Lemma 2.2. The proof is accomplished. □

Bibliography

[1] Y. Hosono, The minimal speed of traveling fronts for a diffusive Lokta-Volterra competition model, Bulletin of Math. Biology 60 (1998), 435-448.

[2] Y. Kan-on, Instability of stationary solutions for Lokta-Volterra competition model with diffusion, J. Math. Anal. Appl. 208 (1997), 158-170.

[3] C. Conley, R. Gardner, An application of generalized Morse index to traveling wave solutions of a competitive reaction diffusion model, Indiana Univ. math. J.33 (1984) 319-343.

[4] R.A. Gardner, Existence and stability of traveling wave solutions of competition models: a degree theoretic, J. Differential Equations 44 (1982), 343-362.

[5] M.M. Tang. P.C. Fife, Propagating fronts for competing species equations with diffusion, Arch. Ration. Mech. Anal. 73 (1980) 69-77.

[6] J.-S. Guo, C.-H. Wu, Traveling wave front for a two-component lattice dynamical system arising in competition models, J. Diff. Eqns 252 (2012) 4357-4391.