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Closed-Form Mortgage Valuation Using Reduced-Form Model

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Valuing mortgage-related securities is more complicated than valuing regular defaultable claims due to the borrower's prepayment behavior as well as the possibility of default. Some researchers use a structural-form model to obtain the closed-form formulas for the mortgage value. With this method, however, it is difficult to identify the critical region of early exercise. As an alternative, the reduced-form model developed in this article is able to value the mortgage without setting boundary conditions, and it can thereby accurately handle the multidimensional space of correlated state variables. The purpose of this article is to derive a closed-form solution of the mortgage valuation equation under a general reduced-form model that embeds relevant economic variables. This new approach enables portfolio managers to undertake sophisticated portfolio optimization and hedging analyses. An implementation procedure for the model is also provided to demonstrate how the valuation framework can be utilized in practical applications.

The mortgage market has grown rapidly from the 1980s to the present day, and mortgage valuation receives significant attention from both practitioners and academic researchers. Closed-form pricing formulas for mortgage-related securities are difficult to obtain because the borrower's prepayment and default behavior cause uncertainty in future cash flows. The closed-form formula helps fixed-income investors manage the duration and complexity of mortgage portfolios, as well as determine diversification strategies. The purpose of this article is to apply the reduced-form model to mortgage valuation and to conduct numerical analyses to show how sensitive closed-form solutions are to changes in the model's parameters.

The idea of pricing defaultable contracts first started with a structural-form approach (Merton 1974, Black and Cox 1976, Leland 1994). The structural-form approach models a financial contract with termination risk as an American-type

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option; the payoffs from stopping depend upon the underlying asset's path. Numerous studies have used partial differential equations with various boundary conditions to price such contracts. Recently, however, the reduced-form approach has been widely applied to price defaultable securities, and to calculate termination probabilities (Jarrow and Turnbull 1995, Jarrow 2001). With this approach one need not decide the optimal stopping time of a path-dependent payoff (as with the structural-form model) and can therefore derive a closed-form formula for the value of defaultable security. A reduced-form approach regards the default event as a Poisson process, which specifies the default probability at each time point. However, when using either a reduced-form or a structural-form approach in pricing the financial contracts with termination risk,[1] the property of path-dependence must be considered.[2]

The structural-form approach regards borrower prepayment and default as endogenous decisions that minimize the present value of the mortgage. More specifically, it models the borrower's behavior as having American-type options of prepayment (call) and default (put). The researcher usually specifies the relevant state variable processes, such as the processes of interest rate and housing price, in order to investigate the termination risk and to value the mortgage (Kau *et al*. 1993, Yang, Buist and Megbolugbe 1998, Ambrose and Buttimer 2000, Azevedo-Pereira, Newton and Paxson 2003). Because the pricing procedure involves solving the American-type options of prepayment and default, researchers usually resort to numerical techniques such as forward pricing (*e.g*., Monte Carlo simulations) or backward solution methods. Recently, Monte Carlo simulation methodology has become an important tool in the pricing and hedging of complicated financial contracts. It has been used to solve the second-order partial differential equation subject to the boundary and termination conditions and to implement sophisticated valuation procedures. Monte Carlo simulations are also used in risk management of financial intermediaries to generate termination and loss probability distributions (Schwartz and Torous 1989, Capone 2003, Goldberg and Harding 2003, Calem and LaCour-Little 2004).

The Monte Carlo approach in and of itself is able to solve most pricing problems and is flexible enough to cope with the practical challenges faced by market participants. When using this method, however, it is necessary to set the unknown

[1] For a more detailed discussion about the link between the two methods, refer to Bielecki and Rutkowski (2001).

[2] Path dependence occurs when the present terms of the contract depend not only on current values of the state variables but also on the past values (Kau and Keenan 1995).

critical regions of prepayment and default and to simulate the processes of relevant state variables (such as interest rate and housing price). As an alternative, this article provides a closed-form solution of the mortgage valuation equation derived under a general reduced-form model with relevant economic variables.

The reduced-form approach is based on using market information on hazard rates to evaluate the probabilities of prepayment and default. This approach assumes that the mortgage has a certain termination probability, conditional upon the survival of the mortgage, at each time point prior to maturity. Usually, these exogenous prepayment and default risks are assumed to follow Poisson processes. Moreover, because the risks of prepayment and default are mutually exclusive, researchers also use the concept of competing risks—where prepayment nullifies the default opportunity and vice-versa—to value mortgage risks more accurately (Ambrose and Sanders 2003, Ciochetti *et al.* 2003). Recently, several empirical studies have applied the Cox proportional hazard model (Cox and Oakes 1984) or Poisson regression to fit the shapes of observed prepayment and default data. These studies investigate the factors that most influence the probabilities of default and prepayment, such as the interest rate, loan-to-value ratio, housing price, loan type, debt service coverage ratio, household income and property type (Schwartz and Torous 1989, 1993, Quigley and Van Order 1990, 1995, Lambrecht, Perraudin and Satchell 2003). The Cox proportional hazard model and Poisson regression have been shown to reasonably model mortgage prepayment and default risks.

A closed-form solution of the mortgage valuation equation provides several advantages when analyzing certain problems associated with mortgage portfolio management. First, one can better understand how sensitive mortgage value is to the changes in relevant variables by conducting numerical analyses. Second, a closed-form solution can significantly increase the speed of calculation when the valuation of the mortgage is involved in more complicated analyses of investment decisions. Third, it provides a useful tool for the portfolio manager to undertake sophisticated portfolio optimizations and hedging analyses that would be infeasible under Monte Carlo simulations. Fourth, a closed-form solution of the mortgage valuation equation provides a basic building block for market participants to evaluate more complicated mortgage products. And finally, researchers can also improve the efficiency of investigating complex portfolio management problems by combining the closed-form formula with simulation techniques. As is well known, several studies have provided closed-form formulas derived under the structural-form model (Collin-Dufresne and Harding 1999, Ambrose and Buttimer 2000). However, it is difficult with this approach to identify the critical regions of mortgage prepayment and default because the related boundary conditions are unobservable. Furthermore, this approach has difficulty dealing with correlation among variables. Because of the difficulty in obtaining a closed-form solution for the structural-form model, numerical approximations such as the Monte Carlo simulation are usually employed.

This article extends Jarrow's (2001) model, which prices defaultable corporate bonds, and uses it to examine mortgage valuations as well. Furthermore, in order to price the mortgage with flexibility and accuracy, Jarrow's (2001) framework is extended into a multivariable model with correlated relevant state variables. More specifically, we specify Poisson processes for the prepayment and default risks and embed the relevant economic state variables, such as housing price and household income, into the hazard rates of prepayment and default to evaluate the mortgage under the reduced-form model.

The influence of relevant variables on mortgage values is quite complicated. For example, an increase in the housing price will produce an increase in prepayment probability but a decrease in default probability. Few studies discuss the influence of different variables (such as interest rate, housing price, household income and macroeconomic variables) on the mortgage value. By utilizing our model, however, one can now study the effects of individual variables and correlations, and accurately and efficiently value the complex mortgage.

In order to investigate the valuation and termination risk of mortgages, we conduct a sensitivity analysis and adopt parameter values taken from other studies to explore the impact of the relevant variables and their correlations on mortgage value. An implementation procedure for the model is also provided to demonstrate how this valuation framework can be used in practical applications. With historical market data, the implementation procedure can be used to estimate the parameters, and these estimated parameters can in turn be used to price mortgages.

The article is organized as follows: The next section presents the valuation framework that includes the identification of the mortgage contract components. Also covered is the definition of the value of mortgage payments to be made by the borrower, and the treatment of the prepayment and default risks of the mortgage. The third section is a demonstration of the model's implementation procedure, including the hazard rates of prepayment and default, the variances and correlations of state variables, and the coefficients of the linear regression model. This section also contains a steady-state analysis of the model's parameters and some numerical results. The final section summarizes our results and offers suggestions for future study.

The Model

Mortgage Contract

We consider a fully amortizing fixed-rate mortgage (FRM), having an initial mortgage principal balance of $M(0)$, a fixed coupon rate of c and T years until maturity. This implies a continuous payout rate Y equal to:[3]

$$Y = M(0) \times \frac{c}{1 - e^{-cT}}. \tag{1}$$

The principal outstanding at time t, $M(t)$, is given by

$$M(t) = M(0) \times \frac{1 - e^{-c(T-t)}}{1 - e^{-cT}}. \tag{2}$$

If it is not possible to terminate the mortgage contract before the maturity date, future cash flows can be determined and the value of the mortgage thereby equals the sum of the present values of the continuous payout. However, when the borrower has the option to prepay or default, the future cash flows and mortgage value become uncertain. To evaluate the mortgage including default and prepayment risks, the risk-neutral pricing method is used.

Model Structure

The borrower can decide to prepay, default or maintain the mortgage. Let us denote the random variables τ_P and τ_D as the time of prepayment and default during the period from t to T, respectively. Let $l(t)$ be the fractional loss rate if the default occurs at time t. We use τ to represent the time when the mortgage payment is first stopped, that is, $\tau = \min(\tau_P, \tau_D)$. If $\tau = \tau_P$, then the cash flow is $M(\tau_P)$. If $\tau = \tau_D$, then the cash flow is $M(\tau_D)(1 - l(\tau_D))$, which can be recovered from the balance of the mortgage at the time of default. Otherwise, if the borrower does not prepay or default on the mortgage, the cash flow is equal to the mortgage payment Y. Assuming no arbitrage and a complete market, the standard arbitrage pricing theory implies that a unique probability measure, Q, exists and that the value of the mortgage is the expectation of discounted future cash flows under Q (Jarrow and Turnbull 1995). The value of the mortgage is therefore:

[3] As we consider a fully amortizing fixed-rate mortgage in this article, the variable Y is specified as a constant in each time point (*i.e.*, Y). If an ARM is to be priced, the continuous payout rate will become a time-varying variable (*i.e.*, $Y(t)$). In addition, because the mortgage is priced under the continuous-time framework, the accrued interest is not considered before prepayment or default.

$$
\begin{aligned}
V(t) = {} & E_t^Q\left[M(\tau_P)e^{-\int_t^{\tau_P} r(u)\,du} I_{\{\tau=\tau_p,\tau\leq T\}}\right] \\
& + E_t^Q\left[(1-l(\tau_D))M_(\tau_D)e^{-\int_t^{\tau_D} r(u)\,du} I_{\{\tau=t_D,\tau\leq T\}}\right] \\
& + E_t^Q\left[\int_t^T Y e^{-\int_t^s r(u)\,du} I_{\{\tau>s\}}\,ds\right],
\end{aligned} \tag{3}
$$

where $r(t)$ is the short interest rate and $I_{\{\}}$ is the indicator function. The first term in Equation (3) is the expected value of the mortgage if prepaid before maturity. The second term is the expected value of a mortgage if default occurs before maturity. Finally, the third term is the expected value of the mortgage if held until maturity.

Given the definition of the termination date of a mortgage, which represents the time that the mortgage is either defaulted or prepaid, the joint survival function of τ_P and τ_D can be defined as (Lancaster 1992):

$$S(t) = e^{-\int_0^t h(u)\,du}, \tag{4}$$

and

$$h(t) = \theta(t) + \pi(t),$$

$$\theta(t) = \lim_{dt\to 0} \frac{\Pr(t < \tau < t+dt, \tau = \tau_P | \tau > t)}{dt}, \tag{5}$$

$$\pi(t) = \lim_{dt\to 0} \frac{\Pr(t < \tau < t+dt, \tau = \tau_D | \tau > t)}{dt}, \tag{6}$$

where $h(t)$, $\theta(t)$ and $\pi(t)$ denote the hazard rates of the terminated mortgage, prepaid and defaulted at time t, respectively.

The distribution function of termination time is denoted as

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t (\theta(u)+\pi(u))\,du}. \tag{7}$$

From Equation (7), the probability density of the termination time of a mortgage is:

$$f(t)\,dt = dF(t) = \theta(t)e^{-\int_0^t (\theta(u)+\pi(u))\,du}\,dt + \pi(t)e^{-\int_0^t (\theta(u)+\pi(u))\,du}\,dt. \tag{8}$$

Because prepayment and default are mutually exclusive, the probability of termination is therefore equal to the probability of prepayment plus the probability of default. It follows that the probability of prepayment is $\theta(t)\exp(-\int_0^t (\theta(u)+\pi(u))\,du)\,dt$ and the probability of default is $\pi(t)\exp(-\int_0^t (\theta(u)+\pi(u))\,du)\,dt$. Thus, Equation (3) can be rewritten as:[4]

[4] For a more detailed derivation, please refer Bielecki and Rutkowski (2001).

$$
\begin{aligned}
V(t) = {} & E_t^Q\left[\int_t^T M(s)\theta(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\,ds\right] \\
& + E_t^Q\left[\int_t^T (1-l(s))M(s)\pi(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\,ds\right] \\
& + Y E_t^Q\left[\int_t^T e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\,ds\right]. \qquad (9)
\end{aligned}
$$

Based on Equation (9), it is clear that the reduced-form approach reflects the prepayment and default risks by the risk-adjusted short rate process $r(t) + \theta(t) + \pi(t)$. This is where, in the presence of prepayment and default risks, reduced-form models differ from structural-form models and the former have the advantage of being able to estimate the hazard rate from market data.

To obtain the closed-form mortgage valuation presented in Equation (9), we specify that the hazard rates of prepayment and default depend on the particular variables related to the termination risk, such as the interest rate, housing price and household income. We have four assumptions in our model. First, the extended Vasicek model is adopted as the short interest rate process $r(t)$.[5] Second, the other influential variables, such as housing price and household income, follow geometric Brownian motions.[6] Third, the fractional loss rate $l(t)$ is a deterministic process. Finally, the hazard rates are specified as a linear function of the explanatory variables. The following subsections will discuss these assumptions in detail.

Extended Vasicek Interest Rate Model

The extended Vasicek interest rate model is a single-factor model with deterministic volatility that can match an arbitrary initial forward-rate curve through the specification of the long-run short interest rate $\bar{r}(t)$ (Vasicek 1977 and Heath, Jarrow and Morton 1992). Under the risk-neutral measure Q, the term-structure evolution is described by the dynamics of the short interest rate:

$$dr(t) = a(\bar{r}(t) - r(t))\,dt + \sigma_r\,dZ_r(t), \qquad (10)$$

where a is the speed of adjustment (a positive constant), σ_r is the volatility of the short interest rate (a positive constant), $\bar{r}(t)$ is the long-run short interest rate

[5] Many interest rate models can be used to value mortgages. However, some studies have shown that the Vasicek model (and hence the extended Vasicek model) performs well in the pricing of mortgage-backed securities (Chen and Yang 1995).

[6] The lognormal process of a variable implies that the return of a variable is normally distributed. This assumption is common throughout the literature.

(a deterministic function of t) and $Z_r(t)$ is a standard Brownian motion under measure Q.

In Equation (10) the short interest rate follows a mean-reverting process under a risk-neutral measure. As shown in Heath, Jarrow and Morton (1992), to match an arbitrary initial forward-rate curve, one can set

$$\bar{r}(u) = f(t,u) + \frac{1}{a}\left(\frac{\partial f(t,u)}{\partial u} + \frac{\sigma_r^2(1-e^{-2a(u-t)})}{2a}\right), \tag{11}$$

where $f(t, u)$ is the instantaneous forward rate. Combining Equations (10) and (11), the evolution of the short interest rate can be rewritten as

$$r(u) = f(t,u) + \frac{\sigma_r^2(e^{-a(u-t)}-1)^2}{2a^2} + \int_t^u \sigma_r e^{-a(u-v)}\,dZ_r(v). \tag{12}$$

Lognormal Processes of Other Variables

For a practical but realistic empirical specification of the reduced-form model, we assume that the processes of other variables, such as housing price, household income and so on, satisfy:

$$dH_i(t) = H_i(t)(r(t)\,dt + \sigma_i\,dZ_i(t)), \quad \text{for } i = 1, 2, \ldots, n, \tag{13}$$

where σ_i is the volatility of variable i, and $Z_i(t)$ is a standard Brownian motion of variable i under Q.

As discussed above, the other variables are assumed to follow geometric Brownian motions with drift $r(t)$ and volatilities σ_i. The drift term is equal to the short interest rate under the risk-neutral measure. Without loss of generality we assume that various random variables are dependent. More specifically, let $Z_i(t)$ be correlated with $Z_r(t)$ as $E_t^Q[dZ_i(t)\,dZ_r(t)] = \phi_{rH_i}\,dt$. In addition, $Z_i(t)$ is correlated with $Z_j(t)$ as $E_t^Q[dZ_i(t)\,dZ_j(t)] = \phi_{H_iH_j}\,dt$, where ϕ_{rH_i} denotes the correlation between the short interest rate and the variable H_i, $\phi_{H_iH_j}$ denotes the correlation between variables H_i and H_j, and ϕ_{rH_i} and $\phi_{H_iH_j}$ are constants.

Using Equation (13) and Ito's Lemma we obtain

$$H_i(t) = H_i(0)\exp\left(\int_0^t r(u)\,du - \frac{1}{2}\sigma_i^2 t + \sigma_i Z_i(t)\right). \tag{14}$$

Given observations on dates 1, 2, ... t, Equation (14) can be solved for $\sigma_i Z_i(t)$ as a function of $\sigma_i Z_i(t-1)$ and this is given by

$$\sigma_i Z_i(t) = \sigma_i Z_i(t-1) + \left(\ln\left(\frac{H_i(t)}{H_i(t-1)} \right) - \int_{t-1}^{t} \left(r(u) - \frac{1}{2}\sigma_i^2 \right) du \right), \quad \text{for } t \geq 1,\ Z_i(0) = 0. \tag{15}$$

Jarrow (2001) indicates that $Z_i(t)$ is a measure of cumulative excess return per unit of risk on $H_i(t)$ and that $Z_i(t)$ is chosen as the state variable. Because the cumulative excess returns per unit of risk of all variables are equal in equilibrium, it is difficult to distinguish the influence of each individual variable. To deal with the multivariate cases, we assume that $e_i(t) = \sigma_i Z_i(t)$, the measure of cumulative excess return rates on $H_i(t)$, as the state variable, and we use the cumulative excess return rates of different variables $e_1(t), e_2(t), \ldots, e_n(t)$ throughout. The assumption that the volatilities of state variables affect default and prepayment hazard rates is supported by the empirical studies of Clapp *et al.* (2001) and Azevedo-Pereira, Newton and Paxson (2003).

Deterministic Loss Rate Process

Various processes have been used to determine the expected fractional loss rate $l(t)$ in previous studies. One can assume the fractional loss rate $l(t)$ is time-varying and path-dependent. The reason is that changes in the mortgage balance and collateral (house) value will result in a change in the expected fractional loss rate. However, specification of a path-dependent loss rate will result in a more complicated pricing procedure because the expected default value contains the result of the loss rate multiplied by the default hazard rate. For example, if we specify the loss rate to be linearly dependent on the state variables as well, the product of the loss rate and default hazard rate may become a Chi-square distribution in our model and as a result the expected mortgage value would become rather complicated to derive. As the loss rate can be estimated by using market data, it is usually assumed in previous studies to be an exogenous variable in reduced-form models. In other words, the loss rate process has been generally treated as a constant or a deterministic variable (Jarrow and Turnbull 1995). Nevertheless, some empirical evidence has shown that whether the loss rate is assumed to be stochastic or constant has no significant impact on mortgage pricing. For example, Jokivuolle and Peura (2003) specified the loss given default as a stochastic process to price the mortgage, and their qualitative results are consistent with Frye (2000a, 2000b), who assumes the expected loss given default is a constant. For simplicity, we will assume that the fractional loss rate $l(t)$ is a deterministic time-varying process and this will not significantly affect mortgage valuation.

Linear Functions of Explanatory Variables

Most studies use Cox proportional hazard model to specify the hazard function as a product of a baseline hazard and an exponential function of covariates. The adoption of Cox proportional hazard model in a pricing framework, however, results in a "double exponential" expression, that is, the exponential function is itself an argument of an exponential function. This gives rise to an infinite expectation of accumulation factors under the martingale measure if the influential variables are log-normally distributed (Miltersen, Sandmann and Sondermann 1997).

We assume the hazard rates of prepayment and default are linear functions of the short-term interest rate and the cumulative excess return rates of other influential variables in our model. This specification has been assumed in many studies, including Duffee (1999), Driessen (2002), Jarrow (2001), Janosi, Jarrow and Yildirim (2003), Capone (2003) and Calem and LaCour-Little (2004). This assumption facilitates the pricing procedure of the mortgage value but it implies that a negative hazard rate is possible.[7] Nonetheless, Duffee (1999) argues that this problem can be largely ignored if the model accurately prices the relevant instruments. The hazard rates of prepayment $\theta(t)$ and default $\pi(t)$ are specified as

$$\theta(t) = \lambda_0(t) + \lambda_r r(t) + \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t), \tag{16}$$

$$\pi(t) = k_0(t) + k_r r(t) + k_1 e_1(t) + \cdots + k_n e_n(t), \tag{17}$$

where $\lambda_0(t)$ and $k_0(t)$ are deterministic, $\lambda_r, \lambda_1, \ldots, \lambda_n$ and $k_r, k_1, \ldots,$ and k_n are constants. We define $\lambda_0(t)$ and $k_0(t)$ as the baseline hazard rates of prepayment and default at time t. λ_r and k_r are the relative magnitudes of the interest rate effect on prepayment and default hazard rates. In addition, λ_i and k_i are the relative magnitudes of state variable i's effect on prepayment and default hazard rates.

Risk-Neutral Mortgage Valuation

Given the above specification, one can obtain the mortgage value by solving the moment generating function of the risk-adjusted short rate process $r(t) + \theta(t) + \pi(t)$.[8] More specifically, the valuation of the mortgage in Equation (9) can be rewritten as

[7] Some studies assume that the hazard rate follows a stochastic process in investigating credit risk. This assumption implies that a negative hazard rate is possible (Lando 1998, Duffee 1999, Jarrow and Turnbull 2000 and Jarrow 2001).

[8] The key point about deriving the closed-form solution is to solve the moment generating function of risk-adjusted short rate process $r(t) + \theta(t) + \pi(t)$. Using other

$$\begin{aligned}V(t) &= Y\int_t^T E_t^Q\left[e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right]ds\\&+\int_t^T M(s)E_t^Q\left[\theta(s)e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right]ds\\&+\int_t^T M(s)(1-l(s))E_t^Q\left[\pi(s)e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right]ds,\end{aligned} \tag{18}$$

where $g_0(u) = \lambda_0(u) + k_0(u)$, $g_r = 1 + \lambda_r + k_r$, $g_1 = \lambda_1 + k_1, \ldots, g_n = \lambda_n + k_n$. The first term of Equation (18) can be expressed as follows:[9]

$$\begin{aligned}&\int_t^T E_t^Q\left[e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right]ds\\&\quad=\int_t^T e^{-\int_t^s g_0(u)\,du + A'\mu_X + \frac{A'\Omega_X A}{2}}\,ds,\end{aligned} \tag{19}$$

where

$$A = [-g_r, -g_1, \ldots, -g_n]',$$

$$X = [X_0, X_1, \ldots, X_n]',$$

$$X_0 = \int_t^s r(u)\,du,\ X_i = \int_t^s e_i(u)\,du,$$

$$\mu_X = \left[\mu_{X_0}(t,s), \mu_{X_1}(t,s), \ldots, \mu_{X_n}(t,s)\right]',$$

$$\Omega_X = \begin{bmatrix} \sigma_{X_0}^2(t,s) & \sigma_{X_1X_0}(t,s) & \cdots & \sigma_{X_nX_0}(t,s)\\ \sigma_{X_0X_1}(t,s) & \sigma_{X_1}^2(t,s) & & \\ \vdots & & \ddots & \\ \sigma_{X_0X_n}(t,s) & \cdots & & \sigma_{X_n}^2(t,s)\end{bmatrix},$$

distributional types, we can obtain the mortgage value as well, however, the pricing procedure becomes more complicated when their moment generating functions are not normal type.

[9] The derivation is shown in Appendix A.

$$\mu_{X_0}(t,s) = E_t^Q\left[\int_t^s r(u)\,du\right]$$
$$= f(t,s)(s-t) + \frac{\sigma_r^2}{2a^2}\left((s-t) - \frac{2}{a}(1-e^{-a(s-t)})\right.$$
$$\left. + \frac{1}{2a}(1-e^{-2a(s-t)})\right),$$

and

$$\sigma_{X_0}^2(t,s) = Var_t\left[\int_t^s r(u)\,du\right] = \frac{\sigma_r^2}{a^2}\left((s-t) - \frac{2}{a}(1-e^{-a(s-t)})\right.$$
$$\left. + \frac{1}{2a}(1-e^{-2a(s-t)})\right).$$

For $i = 1, 2, \ldots, n$,

$$\mu_{X_i}(t,s) = E_t^Q(X_i) = \sigma_i Z_i(t)(s-t),$$

$$\sigma_{X_i}^2(t,s) = Var_t(X_i) = \sigma_i^2 \frac{(s-t)^3}{3},$$

$$\sigma_{X_0 X_i}(t,s) = \phi_{rH_i}\sigma_i \eta(t,s),$$

where

$$\eta(t,s) = \sigma_r\left(-\frac{1}{a^3}(1-e^{-a(s-t)}) + \frac{1}{a^2}e^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2\right),$$

and

$$\sigma_{X_i X_j}(t,s) = \frac{1}{3}\sigma_i\sigma_j\phi_{H_i H_j}(s-t)^3.$$

Similarly, the second term of Equation (18) can be shown to be

$$\int_t^T M(s) E_t^Q\left[\theta(s) e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right] ds$$
$$= \int_t^T M(s)(\lambda_0(s) + B'\mu_W + B'\Omega_{XW}A) e^{-\int_t^s g_0(u)\,du + A'\mu_x + \frac{A'\Omega_X A}{2}}\,ds, \qquad (20)$$

and the third term of Equation (18) is equal to

$$\int_t^T M(s)(1-l(s))E_t^Q\left[\pi(s)e^{-\int_t^s (g_0(u)+g_r r(u)+g_1 e_1(u)+\cdots+g_n e_n(u))\,du}\right] ds$$
$$= \int_t^T M(s)(1-l(s))(k_0(s)+C'\mu_W + C'\Omega_{XW}A)e^{-\int_t^s g_0(u)\,du + A'\mu_x + \frac{A'\Omega_X A}{2}}\, ds, \tag{21}$$

where

$B = [\lambda_r, \lambda_1, \ldots, \lambda_n]'$,

$C = [k_r, k_1, \ldots, k_n]'$,

$W = [r(s), e_1(s), \ldots, e_n(s)]'$,

$\mu_W = \left[\mu_r(t,s), \mu_{e_1}(t,s), \ldots, \mu_{e_n}(t,s)\right]'$,

$$\Omega_{XW} = \begin{bmatrix} \sigma_{rX_0}(t,s) & \sigma_{X_0e_1}(t,s) & \cdots & \sigma_{X_0e_n}(t,s) \\ \sigma_{rX_1}(t,s) & \sigma_{X_1e_1}(t,s) & \cdots & \sigma_{X_1e_n}(t,s) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{rX_n}(t,s) & \sigma_{X_ne_1}(t,s) & \cdots & \sigma_{X_ne_n}(t,s) \end{bmatrix},$$

$\mu_r(t,s) = E_t^Q[r(s)] = f(t,s) + \dfrac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2$,

and

$\sigma_{rX_0}(t,s) = \dfrac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2$.

For $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, i \neq j$,

$\sigma_{rX_i}(t,s) = \sigma_i\sigma_r\phi_{rH_i}\widehat{\eta}(t,s)$,

where

$\widehat{\eta}(t,s) = \dfrac{1}{a^2}(1-e^{-a(s-t)}) - \dfrac{1}{a}e^{-a(s-t)}(s-t)$,

$\mu_{e_i}(t,s) = E_t^Q[e_i(s)] = \sigma_i Z_i(t)$,

$$\sigma_{X_0e_i}(t,s) = \mathrm{Cov}_t\left(e_i(u), \int_t^s r(u)\,du\right)$$
$$= \sigma_i\phi_{rH_i}\left(\frac{\sigma_r}{a}(s-t) - \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})\right),$$

$$\sigma_{X_i e_i}(t,s) = \mathrm{Cov}_t\left(\int_t^s e_i(u)\,du, e_i(s)\right) = \frac{1}{2}\sigma_i^2(s-t)^2,$$

and

$$\sigma_{X_j e_i}(t,s) = \mathrm{Cov}_t\left(\int_t^s e_j(u)\,du, e_i(s)\right) = \frac{1}{2}\sigma_i\sigma_j\phi_{H_i H_j}(s-t)^2.$$

Finally, substituting Equations (19), (20) and (21) into Equation (18), we obtain the mortgage value with default and prepayment risks under the risk-neutral measure Q:

$$\begin{aligned} V(t) = {} & Y\int_t^T e^{-\int_t^s g_0(u)\,du + A'\mu_X + \frac{1}{2}A'\Omega_X A}\,ds \\ & + \int_t^T M(s)(\lambda_0(s) + B'\mu_W + B'\Omega_{X,W}A)e^{-\int_t^s g_0(u)\,du + A'\mu_X + \frac{1}{2}A'\Omega_X A}\,ds \\ & + \int_t^T M(s)(1-l(s))(k_0(s) + C'\mu_W \\ & + C'\Omega_{X,W}A)e^{-\int_t^s g_0(u)\,du + A'\mu_X + \frac{1}{2}A'\Omega_X A}\,ds. \end{aligned} \tag{22}$$

In Equation (22), the first term represents the expected value of a mortgage that does not terminate until maturity. The second and third terms represent the expected values of mortgages that have been prepaid and defaulted before maturity, respectively. In this formulation, the mortgage value is jointly determined by the values of the initial yield curve, $f(t, s)$ and the term-structure evolution parameters a and σ_r. The volatility parameters of the variables σ_i, the correlation parameters ϕ_{rH_i} and $\phi_{H_i H_j}$, and the parameters of the linear hazard functions $\lambda_0(t)$, $k_0(t)$, λ_i and k_i are also key factors in this calculation.

Implementation of the Model

In this section, we demonstrate a procedure for applying the model. The first subsection describes the procedure of estimating parameters including the hazard rates of prepayment and default, the variances and correlations of the state variables, and the coefficients of the linear regression model. The following subsections demonstrate the influence of all the model's parameters on mortgage value with the steady-state analysis and the numerical analysis.

The Estimation of Parameters

To estimate the prepayment and default hazard rates under the assumption of a well-diversified portfolio, we use the product-limit estimator (Kaplan and

Meier 1958).[10] Let the sample period be decomposed into n time intervals, $n = \frac{T}{\Delta t}$. The estimated hazard rates of prepayment and default are defined as follows:

$$\hat{\theta}(i) = \frac{\hat{f}_p(i)}{\hat{S}(i)}, \tag{23}$$

and

$$\hat{\pi}(i) = \frac{\hat{f}_d(i)}{\hat{S}(i)}. \tag{24}$$

where

$\hat{S}(i)$ is the estimated survival probability in the ith time interval, $\hat{S}(u) = \prod_{i=1}^{u}(1 - \frac{c(i)+o(i)}{m(i)})$, $u = 1, 2, \ldots, n$,
$\hat{f}(i)$ is the estimated termination probability, $\hat{f}(i) = \frac{c(i)}{m(i)} + \frac{o(i)}{m(i)} = \hat{f}_p(i) + \hat{f}_d(i)$,
$m(i)$ is the survival number at the beginning of the time interval, $i = 1, 2, \ldots, n$,
$c(i)$ is the number of prepayment events in the ith time interval, and
$o(i)$ is the number of default events in the ith time interval.

Market data on zero coupon bond yields can be used to estimate the forward rate and short-term interest rate parameters (a and σ_r^2). For the estimation of forward rate, note that

$$f(t, T) = -\frac{\partial \log(P(t, T))}{\partial T} \text{ and } r(t) = f(t, t),$$

where $P(t, T)$ denotes the price at time t of a riskless zero coupon bond with the maturity date T. Based on a set of zero coupon bonds with various maturity dates, the maximum smoothness technique can be used to estimate $\hat{f}(t, T)$ and $\hat{r}(t)$ (Adams and van Deventer 1994). For the estimation of a and σ_r^2, the procedure follows a formula for the variance of the riskless zero coupon bond (denoted as σ_P^2), which is (Heath, Jarrow and Morton 1992):

$$\sigma_P^2 = \left(\frac{\sigma_r^2(e^{-a(T-t)} - 1)^2}{a^2}\right). \tag{25}$$

[10] In previous studies, the hazard rates of prepayment and default are calculated based on the realized prepayment and default data (Schwartz and Torous 1989 and Stanton 1995). They are defined as physical hazard rates. The estimated hazard rates need to be martingale hazard rates, which means that the hazard rates are under martingale probability, in this article. However, the physical and martingale hazard rates are equivalent under the assumption of a well-diversifiable portfolio (Jarrow, Lando and Yu 2005).

Fixing a time to maturity $T - t$, by rolling estimation, we obtain the sample variance of the riskless zero coupon bond, $\hat{\sigma}_P^2 = \mathrm{Var}(\log \frac{P(t+\Delta t,T)}{P(t,T)} - r(t)\Delta t)\frac{1}{\Delta t}$. The estimated values of $\hat{\sigma}_r$ and $\hat{a}$ can be obtained by running a nonlinear regression based on Equation (25).

To estimate the variances, correlations and cumulative excess returns of the other model parameters, such as housing prices and household incomes, one can use historical market data to calculate the sample variance and correlation coefficients. These are shown as follows:

$$\hat{\sigma}_i^2 = \mathrm{Var}\left(\frac{H_i(t) - H_i(t-\Delta t)}{H_i(t-\Delta t)}\right)\frac{1}{\Delta t}, \tag{26}$$

$$\hat{\phi}_{rH_i} = \mathrm{Corr}\left(\frac{H_i(t) - H_i(t-\Delta t)}{H_i(t-\Delta t)}, r(t) - r(t-\Delta t)\right), \tag{27}$$

$$\hat{\phi}_{H_iH_j} = \mathrm{Corr}\left(\frac{H_i(t) - H_i(t-\Delta t)}{H_i(t-\Delta t)}, \frac{H_j(t) - H_j(t-\Delta t)}{H_j(t-\Delta t)}\right). \tag{28}$$

This procedure also involves a rolling estimation of the parameters using available information for a given period. Given the estimates of the market volatility and interest rate at each period, the cumulative excess return process is computed using Equation (15), starting the series from initial time t. The time-series of cumulative excess returns of various variables are computed by a rolling estimation from period to period.[11]

Following the above estimation procedure, one can obtain the estimated time-series of interest rate ($r(t)$), cumulative excess returns of various variables ($e_i(t)$) and hazard rates of prepayment and default ($\theta(t)$ and $\pi(t)$), respectively. The parameters of $\lambda_0(t)$, $k_0(t)$, λ_i and k_i can be estimated by a linear regression model as described in Equations (16) and (17). After these parameters have been estimated, one can then use our valuation framework to price mortgages, and undertake hedging analyses. Although our hazard rate specification implies that a negative hazard rate is possible, one can add some restrictions to ensure that $P(\theta(s) < 0) \approx 0$ and $P(\pi(s) < 0) \approx 0$ when estimating the parameters of $\lambda_0(t)$, $k_0(t)$, λ_r, λ_i, k_r and k_i.

It is worth noting that when using Monte Carlo simulations to value mortgage contracts, practitioners also need to estimate the related parameters, such as $f(t, T)$, a, σ_r, σ_i and $\phi_{i,j}$. Furthermore, mortgage market practitioners usually utilize mortgage data to calculate the values of prepayment and default hazard

[11] For a detailed description of the estimation method of interest rate and all the variable's parameters, please refer Janosi, Jarrow and Yildirim (2003).

rates ($\theta(t)$ and $\pi(t)$). These estimated parameter values can be employed in our model. Our method only needs to additionally estimate the parameter values of $\lambda_0(t)$, $k_0(t)$, λ_r, k_r, λ_i and k_i. These parameters are easily estimated with rolling linear regressions.

Sensitivity Analyses of State Variable's Parameters

In this section, we provide the steady-state analysis of the model's parameters to investigate how the various relevant variables will influence the mortgage value. For clarity, we rewrite the mortgage value in the following form:

$$V(t) = \int_t^T \Phi(s)\, ds, \tag{29}$$

where

$$\Phi(s) = \Gamma_1(s)\, e^{\Gamma_2(s)},$$

$$\begin{aligned}\Gamma_1(s) = {} & (Y + M(s)(\lambda_0(s) + B'\mu_W + B'\Omega_{X,W} A) \\ & + M(s)(1 - l(s))(k_0(s) + C'\mu_W + C'\Omega_{X,W} A)),\end{aligned}$$

and

$$\Gamma_2(s) = -\int_t^s g_0(u)\, du + A'\mu_X + \frac{1}{2} A'\Omega_X A.$$

Denote ξ as a parameter, which can be $f(t, s)$, σ_r, a, σ_i, ϕ_{rH_i} or $\phi_{H_i H_j}$. The partial derivative of the mortgage value with respect to ξ is[12]

$$\frac{\partial V(t)}{\partial \xi} = \int_t^T \frac{\partial \Phi(s)}{\partial \xi} ds, \tag{30}$$

where

$$\begin{aligned}\frac{\partial \Phi(s)}{\partial \xi} = {} & \Phi(s)\Gamma_1^{-1}(s) M(s) \\ & \times \left((B + (1 - l(s))C)' \frac{\partial \mu_W}{\partial \xi} + (B + (1 - l(s))C)' \frac{\partial \Omega_{XW}}{\partial \xi} A \right) \\ & + \Phi(s) \left(A' \frac{\partial \mu_X}{\partial \xi} + \frac{1}{2} A' \frac{\partial \Omega_X}{\partial \xi} A \right).\end{aligned}$$

For example, the impact of the correlation between the interest rate and variable i on the mortgage value is

[12] The derivation of $\frac{\partial \Phi(s)}{\partial \xi}$ is shown in Appendix B.

$$\frac{\partial V(t)}{\partial \phi_{rH_i}} = \int_t^T \frac{\partial \Phi(s)}{\partial \phi_{rH_i}} ds,$$

where

$$\frac{\partial \Phi(s)}{\partial \phi_{rH_i}} = -\Phi(s)\Gamma_1^{-1}(s)M(s)((\lambda_i + (1 - l(s))k_i)g_r \eta_2^P(t, s)$$
$$+ (\lambda_r + (1 - l(s))k_r)g_i \eta_3^P(t, s)) + \Phi(s)g_r g_i \eta_1^P(t, s).$$

According to Equation (30), one cannot directly judge whether the impact of the parameter on mortgage value is positive or negative by observing the above partial derivative. Thus, we provide a condition to judge the direction of the influence of the parameter on the mortgage value. For example, one can get $\frac{\partial \Phi(s)}{\partial \xi} \geq 0$, if the following condition holds:

$$\Gamma_1^{-1}(s)M(s)\left((B + (1 - l(s))C)' \frac{\partial \mu_W}{\partial \xi} + (B + (1 - l(s))C)' \frac{\partial \Omega_{XW}}{\partial \xi} A\right)$$
$$\geq -\left(A' \frac{\partial \mu_X}{\partial \xi} + \frac{1}{2} A' \frac{\partial \Omega_X}{\partial \xi} A\right).$$

To investigate the influence of the parameters of the linear hazard functions on the mortgage value, we show the following partial derivatives of mortgage value with respect to the relevant parameters:

$$\frac{\partial \Phi(s)}{\partial \lambda_0(s)} = M(s)e^{\Gamma_2(s)} - \Phi(s) \leq 0, \tag{31}$$

$$\frac{\partial \Phi(s)}{\partial k_0(s)} = M(s)(1 - l(s))e^{\Gamma_2(s)} - \Phi(s) \leq 0, \tag{32}$$

$$\frac{\partial \Phi(s)}{\partial B'} = \Phi(s)\Gamma_1^{-1}(s)M(s)(\mu_W + (\Omega_{XW}A - \Omega'_{XW}B) - (1 - l(s))\Omega'_{XW}C)$$
$$- \Phi(s)(\mu_X + \Omega_X A), \tag{33}$$

$$\frac{\partial \Phi(s)}{\partial C'} = \Phi(s)\Gamma_1^{-1}(s)M(s)(-\Omega'_{XW}B + (1 - l(s))(\mu_W + \Omega_{XW}A - \Omega'_{XW}C))$$
$$- \Phi(s)(\mu_X + \Omega_X A). \tag{34}$$

The impact of the parameters for the linear hazard functions on the mortgage value can be evaluated according to Equations (31), (32), (33) and (34). For example, one can obtain $\frac{\partial \Phi(s)}{\partial B'} \geq 0$, if the following condition is satisfied:

$$\Gamma_1^{-1}(s)M(s)(\mu_W + (\Omega_{XW}A - \Omega'_{XW}B) - (1 - l(s))\Omega'_{XW}C) \geq \mu_X + \Omega_X A.$$

In addition, if the following condition holds, we can obtain $\frac{\partial \Phi(s)}{\partial C'} \geq 0$:

$$\Gamma_1^{-1}(s)M(s)(-\Omega'_{XW}B + (1 - l(s))(\mu_W + \Omega_{XW}A - \Omega'_{XW}C)) \geq \mu_X + \Omega_X A.$$

The following subsection contains a numerical example showing the calculation of our mortgage model and the partial derivatives. It also analyzes how the interest rate, housing price and household income influence mortgage value.

Numerical Results

Recent studies have found that interest rates, housing prices and household incomes significantly affect the borrower's prepayment and default decisions (Yang, Buist and Megbolugbe 1998, Clapp *et al.* 2001, Azevedo-Pereira, Newton and Paxson 2003). We therefore choose these variables as the major sources of valuation uncertainty. For simplicity, we assume that the baseline hazard rates and the loss rate are constants. The parameter values we have adopted to operationalize our model are as follows: $M_0 = \$100$, $c = 5\%$, $l = 0.1$, $f(t, s) = 4\%$, $a = 0.2$, $\sigma_r = 0.01$, $\sigma_H = 0.1$, $\sigma_Y = 0.1$, $\phi_{rH} = 0.37$, $\phi_{rY} = 0.67$, $\phi_{HY} = 0.58$, $\lambda_0(u) = \lambda_0 = 0.176$, $\lambda_r = -0.51339$, $\lambda_1 = 3.96 \times 10^{-5}$, $\lambda_2 = 1.144 \times 10^{-2}$, $k_0(u) = k_0 = 5.19 \times 10^{-6}$, $k_r = -1.12 \times 10^{-7}$, $k_1 = -0.675 \times 10^{-8}$ and $k_2 = -0.716 \times 10^{-6}$, where λ_1 and k_1 and λ_2 and k_2 denote the relative magnitudes of the housing price and household income effects on the prepayment and default hazard rates, respectively. Putting these values into the model, the value of mortgage $V(t)$ is \$104.546 related to a mortgage balance of \$100.[13]

The borrower's incentive to prepay or default will be affected by the spread between the current interest rate and the contract rate as well as his/her expectation about future interest rates. Consequently, the term structure could influence the probability of mortgage termination. Under the above parameter values, we obtain $\frac{\partial V(t)}{\partial f(t,s)} = -471.296$ and $\frac{\partial V(t)}{\partial a} = 0.411$. According to these results, we can infer that there is a positive relationship between a and mortgage value and a negative relationship between $f(t, s)$ and the mortgage value. Moreover, the housing price, the level of household income and the market interest rate all influence a borrower's decision. A higher housing price, for example, will increase the likelihood of prepayment and decrease the probability of default,

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[13] Here, we assume that the baseline hazard rates and loss rate are constants. The parameters of λ_0, k_0, λ_r, k_r, λ_1 and k_1 are taken from Schwartz and Torous (1993). The parameters of λ_2 and k_2 are taken from Deng, Quigley and Van Order (1996). As the values of αx and βx are quite small from empirical evidence, we use the property of $\theta_0(t)e^{\alpha x} \approx \theta_0(t)(1 + \alpha x)$ and $\pi_0(t)e^{\beta x} \approx \pi_0(t)(1 + \beta x)$ in specifying the hazard rates of prepayment and default. We choose the estimator, which is labeled as the age 1 baseline hazard rate in Schwartz and Torous (1993), as λ_0, therefore, λ_r is the product of λ_0 and the estimator of the coefficient of the refinancing rate. λ_1 is the product of λ_0 and the estimator of the coefficient of the housing return. λ_2 is the product of λ_0 and the estimator of the coefficient of household income. The coefficients of default regression are computed in the same way.

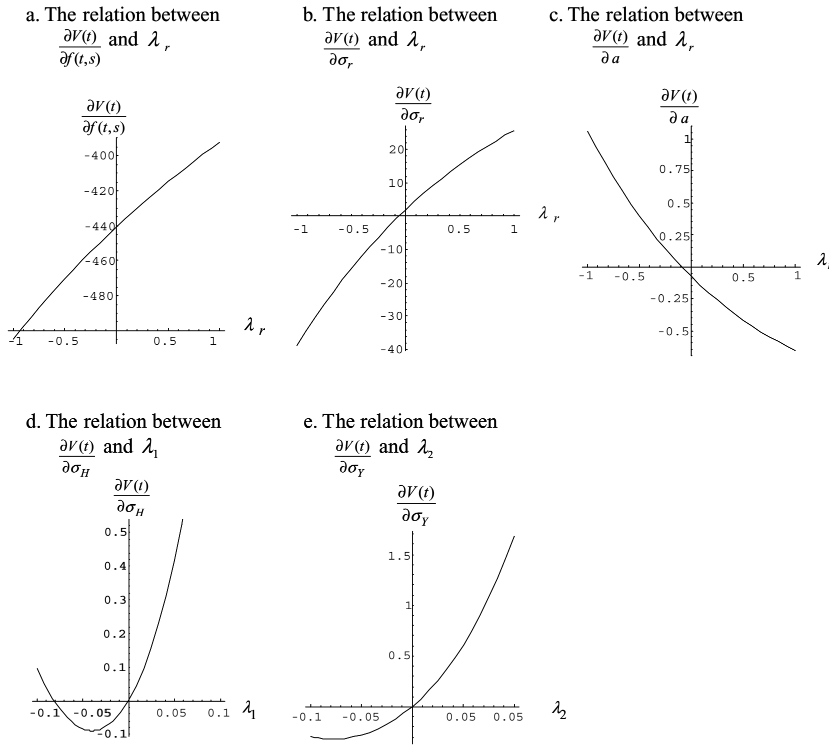
as a borrower will make the decision that offers the greatest benefit. Given the above parameter values, we obtain $\frac{\partial V(t)}{\partial \sigma_r} = -16.306$, $\frac{\partial V(t)}{\partial \sigma_H} = 1.849 \times 10^{-4}$ and $\frac{\partial V(t)}{\partial \sigma_Y} = 0.106$. Therefore, a change in σ_r affects the value of the mortgage in the negative direction, whereas changes in σ_H and σ_Y affect the value of the mortgage in the positive direction. The influence of a change in σ_H on the value of the mortgage is small due to the rather small assumed values of parameters λ_1 and k_1.

Several studies have shown that correlation among the state variables affects the value of FRM. For example, Yang, Buist and Megbolugbe (1998) demonstrate that correlations between interest rates, housing prices and household incomes are all able to influence the termination probability. One can analyze the effect of changing correlations among the relevant variables by examining the corresponding partial derivative that we have provided. From the results of $\frac{\partial V(t)}{\partial \phi_{rH}} = 1.485 \times 10^{-4}$, $\frac{\partial V(t)}{\partial \phi_{rY}} = 0.031$ and $\frac{\partial V(t)}{\partial \phi_{HY}} = 2.876 \times 10^{-6}$, we find that ϕ_{rY}, ϕ_{rH} and ϕ_{HY} do not influence the value of the mortgage significantly, but do have a small positive effect.

We next conduct a sensitivity analysis to investigate the impact of the parameters in the linear regression model. Given the above parameter values, we obtain $\frac{\partial V(t)}{\partial \lambda_0} = -19.823$, $\frac{\partial V(t)}{\partial \lambda_r} = -0.615$, $\frac{\partial V(t)}{\partial \lambda_1} = 1.390$, $\frac{\partial V(t)}{\partial \lambda_2} = 1.849$, $\frac{\partial V(t)}{\partial k_0} = -65.988$, $\frac{\partial V(t)}{\partial k_r} = -2.469$, $\frac{\partial V(t)}{\partial k_1} = 1.401$ and $\frac{\partial V(t)}{\partial k_2} = 2.008$. These results show that the baseline hazard rates have a significant impact on mortgage value. An increase in the parameters of baseline hazard rates and the coefficients of interest rate (λ_0, k_0, λ_r and k_r) will produce a decline in the mortgage value. By contrast, an increase in the coefficients on housing price levels and household incomes (λ_1, k_1, λ_2 and k_2) will increase the mortgage value.

The impact of the parameters on the mortgage value depends on the values of parameters λ_r, k_r, λ_1, k_1, λ_2 and k_2. According to Equation (30), we can check the robustness of the relationship to changing parameters. The influence of parameters $f(t,s)$, σ_r, a, σ_i, ϕ_{rH_i} and $\phi_{H_iH_j}$ on mortgage value when there are changes in λ_r, k_r, λ_1, k_1, λ_2 and k_2 are displayed in Figures 1 and 2. Figure 1(a), (b) and (c) shows how $\frac{\partial V(t)}{\partial f(t,s)}$, $\frac{\partial V(t)}{\partial \sigma_r}$ and $\frac{\partial V(t)}{\partial a}$ change when $-1 \leq \lambda_r \leq 1$. Figure 1(a) reveals that $\frac{\partial V(t)}{\partial f(t,s)}$ is always negative no matter whether the value of $\lambda_r + k_r$ is positive or negative. This, in turn, implies that there is a negative relationship between $f(t, s)$ and the mortgage value. According to Figure 1(b), the value of $\frac{\partial V(t)}{\partial \sigma_r}$ is positive (negative) if $\lambda_r > -k_r$ ($\lambda_r < -k_r$). We can also infer that there is a positive (negative) relationship between σ_r and mortgage value when $\lambda_r > -k_r$ ($\lambda_r < -k_r$). Figure 1(c) shows that the relation between mortgage value and a is contrary to the relation between mortgage value and σ_r, and that there is a positive (negative) relationship between mortgage value and a when $\lambda_r > -k_r$ ($\lambda_r < -k_r$).

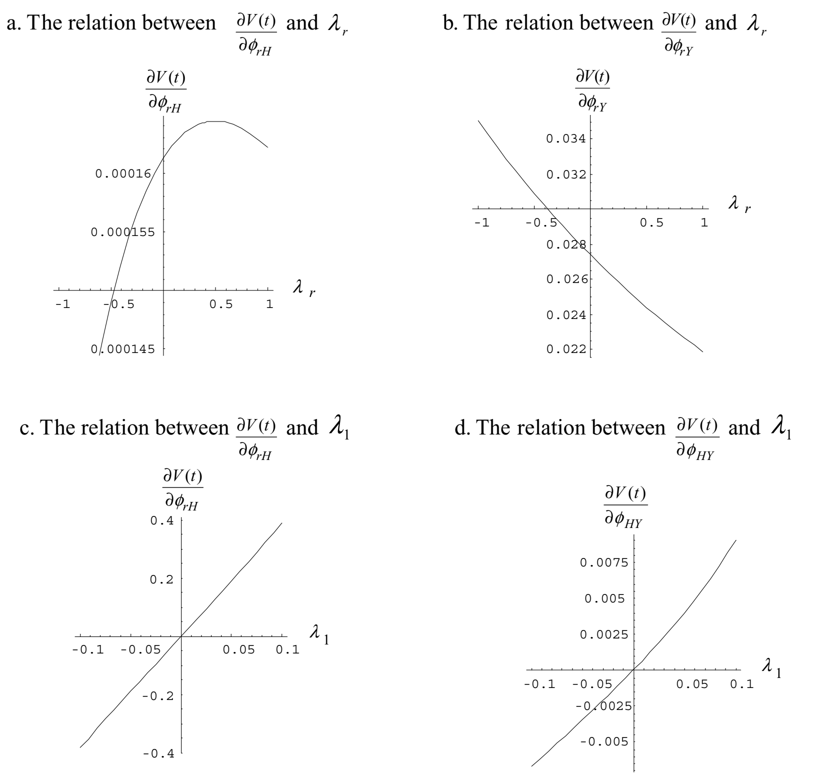
Figure 1 ■ The partial derivatives of the mortgage value with respect to $f(t, s)$, a, σ_r, σ_H and σ_Y for different values of λ_r, λ_1 and λ_2.



Note: These figures represent the influences of $f(t, s)$, a, σ_r, σ_H and σ_Y on the mortgage value when the parameter values λ_r, λ_1 and λ_2 change. The results are obtained according to Equation (30) under the cases of $-1 < \lambda_r < 1$, $-0.1 < \lambda_1, < 0.1$ and $-0.1 < \lambda_2 < 0.1$.

Figure 1(d) and (e) shows how values of $\frac{\partial V(t)}{\partial \sigma_H}$ and $\frac{\partial V(t)}{\partial \sigma_Y}$ change when $-0.1 < \lambda_i < 0.1, i = 1, 2$, respectively. These figures demonstrate that quadratic-form relationships exist between $\frac{\partial V(t)}{\partial \sigma_H}$ and λ_1, and between $\frac{\partial V(t)}{\partial \sigma_Y}$ and λ_2. This may be due to the fact that the effects of variances in the moment generating function are squared forms. We find a positive relationship between $\sigma_H(\sigma_Y)$ and mortgage value when $\lambda_1 > 0$ $(\lambda_2 > 0)$, or when $\lambda_1(\lambda_2)$ is a large negative value. Figure 2 shows the influence of parameter correlations on mortgage value based on changes in λ_r and λ_1. From Figure 2(a) and (b), we see that $\frac{\partial V(t)}{\partial \phi_{rH}} > 0$ and $\frac{\partial V(t)}{\partial \phi_{rY}} > 0$ when $-1 < \lambda_r < 1$. This implies that there are positive relationships between ϕ_{rH} and mortgage value, and between ϕ_{rY} and mortgage value regardless of whether $\lambda_r + k_r$ is positive or negative. As displayed in Figure 2(c) and (d), both ϕ_{rH} and ϕ_{HY} are positively (negatively) related to mortgage value when $\lambda_1 < 0$ $(\lambda_1 > 0)$. The same results hold for the influence of λ_2 on $\frac{\partial V(t)}{\partial \phi_{rH}}$ and $\frac{\partial V(t)}{\partial \phi_{HY}}$.

Figure 2 ■ The partial derivatives of the mortgage value with respect to ϕ_{rH}, ϕ_{rY} and ϕ_{HY} for different values of λ_r and λ_1.



Note: These figures represent the influences of ϕ_{rH}, ϕ_{rY} and ϕ_{HY} on the mortgage value when the parameter values λ_r and λ_1 change. The results are obtained according to Equation (30) under the cases of $-1 < \lambda_r < 1$ and $-0.1 < \lambda_1, 0.1$.

Moreover, the influences of k_r, k_1 and k_2 on the changes in $\frac{\partial V(t)}{\partial f(t,s)}$, $\frac{\partial V(t)}{\partial \sigma_r}$, $\frac{\partial V(t)}{\partial a}$, $\frac{\partial V(t)}{\partial \sigma_H}$, $\frac{\partial V(t)}{\partial \sigma_Y}$, $\frac{\partial V(t)}{\partial \phi_{rY}}$, $\frac{\partial V(t)}{\partial \phi_{rH}}$ and $\frac{\partial V(t)}{\partial \phi_{HY}}$ are similar to the above results.

The numerical results suggest that the major factors that influence mortgage value are the interest rate ($f(t, s)$ and σ_r) and baseline hazard rates (λ_0 and k_0). However, other major factors may emerge for different sets of parameter values. In fact, whether a variable is a major determinant of mortgage value depends on the estimated parameters of the linear hazard function. In our numerical example, the interest rate is a major factor because the value of $1 + \lambda_r + k_r$ is large. However, if $1 + \lambda_r + k_r \approx 0$, the interest rate has a very small influence on the mortgage value. Moreover, λ_0 and k_0 are major valuation factors because the conditions $\sum_{i=1}^{n} \lambda_i X_i < \lambda_0$ and $\sum_{j=1}^{n} k_j X_j < k_0$ lead to $\frac{\partial V(t)}{\partial \lambda_0} > \frac{\partial V(t)}{\partial \lambda_i}$ and $\frac{\partial V(t)}{\partial k_0} > \frac{\partial V(t)}{\partial k_i}$. Although our numerical results are sensitive to assumed parameter values, we believe they help us to understand the impact of state variables and their correlations on hazard rates of prepayment and default and, therefore, mortgage value.[14]

Conclusion

The valuation and hedging of mortgage-related securities requires a model that includes the borrower's prepayment and default behavior. Many researchers have proposed models for valuing the mortgage based on the contingent-claim pricing theory. However, with this approach the boundary conditions of prepayment and default are difficult to identify. Alternatively, the reduced-form approach estimates the hazard rates using market data. The default and prepayment risks are then calculated using hazard functions, and as a result, the mortgage contract is more easily valued.

This article proposes a methodology for implementing the reduced-form model and embeds the relevant variables on hazard rates to derive a closed-form valuation solution for a fixed-rate mortgage. The default and prepayment risks in our mortgage-pricing model are represented by the risk-adjusted discount rate. Our approach differs from the structural-form approach in how it treats prepayment and default risks and provides an efficient valuation method for the pricing and hedging of more complex mortgages. The closed-form formula is therefore a valuable tool for investors and mortgage portfolio managers to determine their diversification and hedging strategies.

Using our model for sensitivity analyses, one can investigate how the various relevant variables and their correlations influence mortgage value. We conduct a numerical analysis to investigate the effects of interest rate, housing price and household income parameters on mortgage value. We find that the interest rate parameters and baseline hazard rates are the most important factors influencing mortgage value. We also discuss the influence of the correlations of the influential variables on mortgage value, a topic that is not thoroughly investigated in the previous literature. Our results show that changes in correlation coefficients affect the mortgage value, although not to a significant degree.

In future research, this model could be combined with actual market data for the production of empirical studies. Additionally, our closed-form formula

[14] For reasons of comparison, we adopt this set of parameter values to perform our model and show the effects of the different variables on the mortgage value.

can be used to investigate the duration and convexity of a mortgage portfolio and determine an optimal diversification strategy. Also, the fractional loss rate $l(t)$ can be generalized to include time-varying and path-dependent specifications.

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Appendix A

In this appendix, we provide the derivation of Equations (19), (20) and (21).

From Equation (12),

$$r(u) = f(t,u) + \frac{\sigma_r^2(e^{-a(u-t)}-1)^2}{2a^2} + \int_t^u \sigma_r e^{-a(u-v)}\,dZ_r(v).$$

Let

$$\rho(v,u) = \sigma_r e^{-a(u-v)},\ b(t,u) = \int_t^u \rho(t,v)\,dv = \sigma_r \frac{1-e^{-a(u-t)}}{a}.$$

Define

$$\begin{aligned} X_0 = \int_t^s r(u)\,du = \int_t^s f(t,u)\,du + \int_t^s \frac{b(t,u)^2}{2}\,du \\ + \int_t^s \int_t^u \rho(v,u)\,dZ_r(v)\,du. \end{aligned} \tag{A1}$$

After changing the order of integration, a direct computation yields

$$\int_t^s \frac{b(t,u)^2}{2}\,du = \int_t^s \frac{b(v,s)^2}{2}\,dv,$$

and

$$\begin{aligned} \int_t^s \int_t^u \rho(v,u)\,dZ_r(v)\,du &= \int_t^s \left(\int_v^s \rho(v,s)\,du\right) dZ_r(v) \\ &= \int_t^s b(v,s)\,dZ_r(v). \end{aligned} \tag{A2}$$

Substituting Equation (A2) into Equation (A1) gives

$$\int_t^s r(u)\,du = \int_t^s f(t,u)\,du + \int_t^s \frac{b(v,s)^2}{2}\,dv + \int_t^s b(v,s)\,dZ_r(v).$$

Define

$$X_i(s) = \int_t^s e_i(u)\,du, i = 1, 2, \ldots, n,$$

where

$$e_i(u) = \sigma_i Z_i(u) = \sigma_i \left(Z_i(t) + \int_t^u dZ_i(v) \right).$$

Then

$$\begin{aligned}
X_i(s) &= \int_t^s e_i(u)\,du \\
&= \int_t^s \sigma_i \left(Z_i(t) + \int_t^u dZ_i(v) \right) du, i = 1, 2, \ldots, n, \\
&= \sigma_i \int_t^s Z_i(t)\,du + \sigma_i \int_t^s \int_t^u dZ_i(v)\,du \\
&= \sigma_i \int_t^s Z_i(t)\,du + \sigma_i \int_t^s \left(\int_v^s du \right) dZ_i(v) \\
&= \sigma_i Z_i(t)(s-t) + \sigma_i \int_t^s (s-v)\,dZ_i(v).
\end{aligned}$$

We assume the initial yield curve is flat. A direct computation gives

$$\begin{aligned}
\mu_{X_0}(t,s) = E_t^Q \left[\int_t^s r(u)\,du \right] &= \int_t^s f(t,u)\,du + \int_t^s \frac{b(t,u)^2}{2}\,du \\
&= f(t,s)(s-t) + \frac{\sigma_r^2}{2a^2} \\
&\quad \times \left((s-t) - \frac{2}{a}(1 - e^{-a(s-t)}) + \frac{1}{2a}(1 - e^{-2a(s-t)}) \right).
\end{aligned}$$

By Ito's Lemma, we obtain the following:

$$\begin{aligned}
E_t^Q[dZ_r(t)\,dZ_r(t)] &= E_t^Q[dZ_i(t)\,dZ_i(t)] = dt, \\
E_t^Q[dZ_r(t)\,dZ_r(u)] &= E_t^Q[dZ_r(t)\,dZ_i(u)] \\
&= E_t^Q[dZ_i(t)\,dZ_i(u)] \\
&= E_t^Q[dZ_i(t)\,dZ_j(u)] \\
&= 0, \\
E_t^Q[dZ_r(t)\,dZ_i(t)] &= \phi_{rH_i}\,dt, \\
E_t^Q[dZ_i(t)\,dZ_j(t)] &= \phi_{H_iH_j}\,dt, \; i, j = 1, 2, \ldots, n, i \neq j.
\end{aligned}$$

Then

$$
\begin{aligned}
\sigma^2_{X_0}(t,s) &= Var_t\left(\int_t^s r(u)\,du\right)\\
&= E_t^Q\left[\int_t^s b(v,s)\,dZ_r(v) \times \int_t^s b(v,s)\,dZ_r(v)\right]\\
&= \int_t^s b(v,s)^2\,dv\\
&= \frac{\sigma_r^2}{a^2}\left((s-t) - \frac{2}{a}(1-e^{-a(s-t)}) + \frac{1}{2a}(1-e^{-2a(s-t)})\right).
\end{aligned}
$$

As $X_i(s) = \sigma_i Z_i(t)(s-t) + \sigma_i \int_t^s (s-v)\,dZ_i(v)$, we have the following:

$$
\mu_{X_i}(t,s) = E_t^Q[X_i(s)] = \int_t^s E_t^Q[e_i(u)]\,du = \sigma_i \int_t^s Z_i(t)\,du = \sigma_i Z_i(t)(s-t),
$$

$$
\begin{aligned}
\sigma^2_{X_i}(t,s) &= Var_t(X_i(s))\\
&= \sigma_i^2 E_t^Q\left[\int_t^s (s-v)\,dZ_i(v) \times \int_t^s (s-v)\,dZ_i(v)\right]\\
&= \sigma_i^2 \int_t^s (s-v)^2\,dv\\
&= \sigma_i^2 \frac{(s-t)^3}{3},
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{X_0X_i}(t,s) &= \mathrm{Cov}_t(X_0(s), X_i(s))\\
&= E_t^Q\left[\int_t^s b(v,s)\,dZ_r(v) \times \sigma_i \int_t^s (s-v)\,dZ_i(v)\right]\\
&= \sigma_i \phi_{rH_i} \int_t^s b(v,s)(s-v)\,dv,\\
&= \sigma_i \phi_{rH_i} \eta(t,s),
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(t,s) &= \int_t^s b(v,s)(s-v)\,dv\\
&= \sigma_r\left(-\frac{1}{a^3}(1-e^{-a(s-t)}) + \frac{1}{a^2}e^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2\right).
\end{aligned}
$$

Similarly,

$$\sigma_{X_i X_j}(t,s) = E_t^Q\left[\sigma_i \int_t^s (s-v)\,dZ_i(v) \times \sigma_j \int_t^s (s-v)\,dZ_j(v)\right]$$
$$= \sigma_i\sigma_j\phi_{H_iH_j}\int_t^s (s-v)^2\,dv = \sigma_i\sigma_j\phi_{H_iH_j}\frac{(s-t)^3}{3}.$$

Furthermore, we can obtain

$$\mu_r(t,s) = E_t^Q[r(s)] = f(t,s) + \frac{b(t,s)^2}{2} = f(t,s) + \frac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2,$$
$$\sigma_{rX_0}(t,s) = E_t^Q\left[r(s)\int_t^s r(u)\,du\right] - E_t^Q[r(s)]E_t^Q\left[\int_t^s r(u)\,du\right]$$
$$= E_t^Q\left[\int_t^s \rho(v,u)\,dZ_r(v) \times \int_t^s b(v,s)\,dZ_r(v)\right]$$
$$= \int_t^s \rho(v,u)b(v,s)\,dv$$
$$= \frac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2,$$
$$\sigma_{rX_i}(t,s) = E_t^Q\left[r(s)\int_t^s e_i(u)\,du\right] - E_t^Q[r(s)]E_t^Q\left[\int_t^s e_i(u)\,du\right]$$
$$= E_t^Q\left[\int_t^s \rho(v,u)\,dZ_r(v) \times \int_t^s (s-v)\,dZ_i(v)\right]$$
$$= \sigma_i\phi_{r_sH_i}\int_t^s \rho(v,s)(s-v)\,dv$$
$$= \sigma_i\phi_{r_sH_i}\widehat{\eta}(t,s),$$

where

$$\widehat{\eta}(t,s) = \int_t^s \rho(v,s)(s-v)\,dv = \frac{\sigma_r}{a^2}(1-e^{-a(s-t)}) - \frac{\sigma_r}{a}e^{-a(s-t)}(s-t),$$
$$\mu_{e_i}(t,s) = E_t^Q[e_i(s)] = \sigma_i Z_i(t),$$
$$\sigma_{X_0e_i}(t,s) = \mathrm{Cov}_t\left(e_i(\mathrm{s}), \int_t^s r(u)\,du\right)$$
$$= E_t^Q\left[e_i(\mathrm{s})\int_t^s r(u)\,du\right] - E_t^Q[e_i(s)]E_t^Q\left[\int_t^s r(u)\,du\right]$$

$$
\begin{aligned}
&= E_t^Q\left[\int_t^s \sigma_i\, dZ_i(v) \times \int_t^s b(v,s)\, dZ_r(v)\right] \\
&= \sigma_i \int_t^s b(v,s)\, dv \\
&= \sigma_i \phi_{r_s H_i}\left(\frac{\sigma_r}{a}(s-t) - \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{X_i e_i}(t,s) &= \mathrm{Cov}_t\left(\int_t^s e_i(u)\, du, e_i(s)\right) \\
&= E_t^Q\left[\int_t^s e_i(u)e_i(s)\, du\right] - E_t^Q\left[\int_t^s e_i(u)\, du\right] E_t^Q[e_i(s)] \\
&= E_t^Q\left[\int_t^s \sigma_i dZ_i(v) \times \int_t^s \sigma_i(s-v)\, dZ_i(v)\right] \\
&= \sigma_i^2 \int_t^s (s-v)\ dv \\
&= \frac{1}{2}\sigma_i^2 (s-t)^2.
\end{aligned}
$$

For $i \neq j$,

$$
\begin{aligned}
\sigma_{X_j e_i}(t,s) &= \mathrm{Cov}_t\left(\int_t^s e_j(u)\, du, e_i(s)\right) \\
&= E_t^Q\left[\int_t^s e_j(u)e_i(s)\, du\right] - E_t^Q\left[\int_t^s e_j(u)\, du\right] E_t^Q[e_i(s)] \\
&= E_t^Q\left[\int_t^s \sigma_j(s-v)\, dZ_j(v) \times \int_t^s \sigma_i\, dZ_i(v)\right] \\
&= \sigma_i \sigma_j \phi_{H_i H_j} \int_t^s (s-v)\ dv \\
&= \frac{1}{2}\sigma_i \sigma_j \phi_{H_i H_j} (s-t)^2.
\end{aligned}
$$

Given that (x, y) is a bivariate normal, we have

$$
E_t[e^{Ax+By}] = e^{A\mu_X + B\mu_y + {}^1\!/\!{}_2\left(A^2\sigma_x^2 + 2\sigma_{xy}AB + B^2\sigma_y^2\right)}. \tag{A3}
$$

Then, for the first term of Equation (18), using Equation (A3), we obtain

$$\int_t^T E_t^Q\left[e^{-\int_t^s ((\lambda_0(u)+k_0(u))+(1+\lambda_r+k_r)r(u)+(\lambda_1+k_1)e_i(u)+\cdots(\lambda_n+k_n)e_n(u))\,du}\right]ds$$
$$= \int_t^T e^{-\int_t^s g_0(u)\,du} E_t^Q[e^{A'X}]\,ds$$
$$= \int_t^T e^{-\int_t^s g_0(u)\,du} e^{A'\mu_X + \frac{A'\Omega_X A}{2}}\,ds.$$

This is Equation (19).

Also,

$$\frac{\partial E_t[e^{A'X+B'W}]}{\partial B'}$$
$$= E_t[We^{A'X+B'W}]$$
$$= e^{A'\mu_X+B'\mu_W+1/2(A'\Omega_X A+2B'\Omega_{XW}A+B'\Omega_W B)}(\mu_W + \Omega_W B + \Omega_{XW}A). \qquad \text{(A4)}$$

Using Equation (A4) for the second term and the third term of Equation (18), we obtain

$$E_t^Q\left[\int_t^T M(s)\theta(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\,ds\right]$$
$$= \int_t^T M(s)E_t^Q\left[\theta(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\right]ds$$
$$= \int_t^T M(s)e^{-\int_t^s g_0(u)\,du}E_t^Q\left[(\lambda_0(s)+\lambda_r r(s)+\lambda_1 e_1(s)+\cdots+\lambda_n e_n(s))e^{A'X}\right]ds$$
$$= \int_t^T M(s)e^{-\int_t^s g_0(u)\,du}E_t^Q[\lambda_0(s)e^{A'X} + B'We^{A'X}]\,ds$$
$$= \int_t^T M(s)e^{-\int_t^s g_0(u)\,du}(\lambda_0(s)E_t^Q[e^{A'X}] + B'E_t^Q[We^{A'X}])ds.$$

As

$$E_t[We^{A'X}] = \left.\frac{\partial E_t[e^{A'X+B'W}]}{\partial B'}\right|_{B=0} = e^{A'\mu_X+1/2(A'\Omega_X A)}(\mu_W + \Omega_{XW}A).$$

We obtain the following:

$$\int_t^T M(s)e^{-\int_t^s g_0(u)\,du}(\lambda_0(s)E_t^Q[e^{A'X}] + B'E_t^Q[We^{A'X}])ds$$
$$= \int_t^T M(s)e^{-\int_t^s g_0(u)\,du}(\lambda_0(s) + B'\mu_W + B'\Omega_{XW}A)e^{A'\mu_X + \frac{A'\Omega_X A}{2}}\,ds$$
$$= \int_t^T M(s)(\lambda_0(s) + B'\mu_W + B'\Omega_{XW}A)e^{-\int_t^s g_0(u)\,du + A'\mu_x + \frac{A'\Omega_X A}{2}}\,ds.$$

This is Equation (20). In the same way, we obtain the following:

$$E_t^Q\left[\int_t^T M(s)(1-l(s))\pi(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\,ds\right]$$
$$= \int_t^T M(s)(1-l(s))E_t^Q\left[\pi(s)e^{-\int_t^s (r(u)+\theta(u)+\pi(u))\,du}\right]ds$$
$$= \int_t^T M(s)(1-l(s))e^{-\int_t^s g_0(u)\,du}$$
$$\times E_t^Q\left[(k_0(s) + k_r r(s) + k_1 e_1(s) + \cdots + k_n e_n(s))e^{A'X}\right]ds$$
$$= \int_t^T M(s)(1-l(s))e^{-\int_t^s g_0(u)\,du}E_t^Q\left[k_0(s)e^{A'X} + C'We^{A'X}\right]ds$$
$$= \int_t^T M(s)(1-l(s))e^{-\int_t^s g_0(u)\,du}(k_0(s)E_t^Q[e^{A'X}] + E_t^Q[C'We^{A'X}])\,ds$$
$$= \int_t^T M(s)(1-l(s))e^{-\int_t^s g_0(u)\,du}(k_0(s) + C'\mu_W + C'\Omega_{XW}A)e^{A'\mu_X + \frac{A'\Omega_X A}{2}}\,ds$$
$$= \int_t^T M(s)(1-l(s))(k_0(s) + C'\mu_W + C'\Omega_{XW}A)e^{-\int_t^s g_0(u)\,du + A'\mu_x + \frac{A'\Omega_X A}{2}}\,ds.$$

This is Equation (21).

Appendix B

Let ξ denote the parameters of state variables which include $f(t, s)$, σ_r, a, σ_i, ϕ_{rH_i} and $\phi_{H_iH_j}$. In this appendix, we provide the expressions for $\frac{\partial\mu_X}{\partial\xi}$, $\frac{\partial\mu_W}{\partial\xi}$, $\frac{\partial\Omega_X}{\partial\xi}$ and $\frac{\partial\Omega_{XW}}{\partial\xi}$. The partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to ξ are not shown here because the values of these terms are zero.

The partial derivatives of the elements of μ_X and μ_W with respect to $f(t, s)$ are

$$\frac{\partial\mu_{X_0}(t,s)}{\partial f(t,s)} = (s-t), \text{ and } \frac{\partial\mu_r(t,s)}{\partial f(t,s)} = 1.$$

Then, one can substitute the above results into Equation (30) to obtain the results of $\frac{\partial V(t)}{\partial f(t,s)}$.

The following expressions show the partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to a:

$$\frac{\partial \mu_{X_0}(t,s)}{\partial a} = \sigma_r^2\left(-\frac{1}{a^3}(s-t) + \frac{3}{a^4}(1-e^{-a(s-t)}) - \frac{1}{a^3}(e^{-a(s-t)}(s-t)) - \frac{3}{4a^4}(1-e^{-2a(s-t)}) + \frac{1}{2a^3}(e^{-2a(s-t)}(s-t))\right),$$

$$\frac{\partial \mu_r(t,s)}{\partial a} = \frac{\sigma_r^2}{a^2}(1-e^{-a(s-t)})e^{-a(s-t)}(s-t) - \frac{\sigma_r^2}{a^3}(1-e^{-a(s-t)})^2,$$

$$\frac{\partial \sigma^2_{X_0}(t,s)}{\partial a} = \sigma_r^2\left[-\frac{2}{a^3}(s-t) + \frac{6}{a^4}(1-e^{-a(s-t)}) - \frac{2}{a^3}(e^{-a(s-t)}(s-t)) - \frac{3}{2a^4}(1-e^{-2a(s-t)}) + \frac{1}{a^3}(e^{-2a(s-t)}(s-t))\right],$$

$$\frac{\partial \sigma_{rX_0}(t,s)}{\partial a} = \frac{\sigma_r^2}{a^2}(1-e^{-a(s-t)})e^{-a(s-t)}(s-t) - \frac{\sigma_r^2}{a^3}(1-e^{-a(s-t)})^2,$$

$$\frac{\partial \sigma_{X_0X_i}(t,s)}{\partial a} = \sigma_i\phi_{rH_i}\sigma_r\left(\frac{3}{a^4}(1-e^{-a(s-t)}) - \frac{1}{a^3}e^{-a(s-t)}(s-t) - \frac{2}{a^3}e^{-a(s-t)}(s-t) - \frac{1}{a^2}e^{-a(s-t)}(s-t)^2 - \frac{1}{2a^2}(s-t)^2\right),$$

$$\frac{\partial \sigma_{rX_i}(t,s)}{\partial a} = \sigma_i\phi_{rH_i}\left(-\frac{2\sigma_r}{a^3}(1-e^{-a(s-t)}) + \frac{\sigma_r}{a^2}e^{-a(s-t)}(s-t), + \frac{\sigma_r}{a^2}e^{-a(s-t)}(s-t) + \frac{\sigma_r}{a}e^{-a(s-t)}(s-t)^2\right),$$

$$\frac{\partial \sigma_{X_0e_i}(t,s)}{\partial a} = \sigma_i\phi_{rH_i}\left(-\frac{\sigma_r}{a^2}(s-t) + \frac{2\sigma_r}{a^3}(1-e^{-a(s-t)}) - \frac{\sigma_r}{a^2}e^{-a(s-t)}(s-t)\right).$$

Substituting the above results into Equation (30), we can obtain the results of $\frac{\partial V(t)}{\partial a}$. The partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to σ_r are

$$\frac{\partial \mu_{X_0}(t,s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}\left((s-t) - \frac{2}{a}(1-e^{-a(s-t)}) + \frac{1}{2a}(1-e^{-2a(s-t)})\right),$$

$$\frac{\partial \mu_r(t,s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})^2,$$

$$\frac{\partial \sigma_{X_0}^2(t,s)}{\partial \sigma_r} = \frac{2\sigma_r}{a^2}\left[(s-t) - \frac{2}{a}(1-e^{-a(s-t)}) + \frac{1}{2a}(1-e^{-2a(s-t)})\right],$$

$$\frac{\partial \sigma_{rX_0}(t,s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})^2,$$

$$\frac{\partial \sigma_{X_0X_i}(t,s)}{\partial \sigma_r} = \sigma_i\phi_{rH_i}\left(-\frac{1}{a^3}(1-e^{-a(s-t)}) + \frac{1}{a^2}e^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2\right),$$

$$\frac{\partial \sigma_{rX_i}(t,s)}{\partial \sigma_r} = \sigma_i\phi_{rH_i}\left(\frac{1}{a^2}(1-e^{-a(s-t)}) - \frac{1}{a}e^{-a(s-t)}(s-t)\right) \text{ and}$$

$$\frac{\partial \sigma_{X_0e_i}(t,s)}{\partial \sigma_r} = \sigma_i\phi_{rH_i}\left[\frac{1}{a}(s-t) - \frac{1}{a^2}(1-e^{-a(s-t)})\right].$$

Substituting the above results into Equation (30), we can obtain the result of $\frac{\partial V(t)}{\partial \sigma_r}$. The partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to σ_i are

$$\frac{\partial \mu_{X_i}(t,s)}{\partial \sigma_i} = (s-t)Z_i(t),$$

$$\frac{\partial \mu_{e_i}(t,s)}{\partial \sigma_i} = Z_i(t),$$

$$\frac{\partial \sigma_{X_i}^2(t,s)}{\partial \sigma_i} = \frac{2\sigma_i(s-t)^3}{3},$$

$$\frac{\partial \sigma_{X_0X_i}(t,s)}{\partial \sigma_i} = \sigma_i\phi_{rH_i}\left(-\frac{1}{a^3}(1-e^{-a(s-t)}) + \frac{1}{a^2}e^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2\right),$$

$$\frac{\partial \sigma_{X_iX_j}(t,s)}{\partial \sigma_i} = \sigma_j\phi_{H_iH_j}\frac{(s-t)^3}{3},$$

$$\frac{\partial \sigma_{rX_i}(t,s)}{\partial \sigma_i} = \phi_{rH_i}\frac{\sigma_r}{a^2}(1-e^{-a(s-t)}) - \frac{\sigma_r}{a}e^{-a(s-t)}(s-t),$$

$$\frac{\partial \sigma_{X_ie_i}(t,s)}{\partial \sigma_i} = \sigma_i(s-t)^2,$$

$$\frac{\partial \sigma_{X_0e_i}(t,s)}{\partial \sigma_i} = \phi_{rH_i}\left(\frac{\sigma_r}{a}(s-t) - \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})\right) \text{ and}$$

$$\text{for } i \neq j,\ \frac{\sigma_{X_je_i}(t,s)}{\partial \sigma_i} = \frac{1}{2}\sigma_j\phi_{H_iH_j}(s-t)^2.$$

Substituting the above results into Equation (30), we can obtain the result of $\frac{\partial V(t)}{\partial \sigma_i}$. Furthermore, the partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to ϕ_{rH_i} are as follows:

$$\frac{\partial \sigma_{X_0 X_i}(t,s)}{\partial \phi_{rH_i}} = \sigma_i \sigma_r \left(-\frac{1}{a^3}(1-e^{-a(s-t)}) + \frac{1}{a^2} e^{-a(s-t)}(s-t) + \frac{1}{2a}(s-t)^2 \right),$$

$$\frac{\partial \sigma_{r X_i}(t,s)}{\partial \phi_{rH_i}} = \sigma_i \sigma_r \left(\frac{1}{a^2}(1-e^{-a(s-t)}) - \frac{1}{a} e^{-a(s-t)}(s-t) \right) \text{ and}$$

$$\frac{\partial \sigma_{X_0 e_i}(t,s)}{\partial \phi_{rH_i}} = \sigma_i \sigma_r \left(\frac{1}{a}(s-t) - \frac{1}{a^2}(1-e^{-a(s-t)}) \right).$$

Finally, the partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to $\phi_{H_i H_j}$ are

$$\frac{\partial \sigma_{X_i X_j}(t,s)}{\partial \phi_{H_i H_j}} = \sigma_i \sigma_j \frac{(s-t)^3}{3} \text{ and}$$

$$\frac{\sigma_{X_j e_i}(t,s)}{\partial \phi_{H_i H_j}} = \frac{1}{2} \sigma_i \sigma_j (s-t)^2.$$

Substituting the above results into Equation (30), we can obtain the results of $\frac{\partial V(t)}{\partial \phi_{rH_i}}$ and $\frac{\partial V(t)}{\partial \phi_{H_i H_j}}$.