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The valuation of special purpose vehicles by issuing structured credit-linked notes

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With the intersection of market and credit risk, the first contribution is to derive the analytic formulas of the Credit Linked Notes (CLNs) and the leveraged total return CLNs issued by an Special Purpose Vehicle (SPV) or the protection buyer. The second contribution is to prove that the values of structured CLNs issued by an SPV are higher than the ones issued by the protection buyer. When the credit quality of the reference obligation and protection buyer becomes worse or the leverage effect is higher, it is a superior solution for the structured CLNs issued through an SPV. Third, the empirical results of credit spreads do not incorporate the correlation coefficient of spot rate and market index into their regression models and show that they are positively correlated with the volatilities of spot rate and return on market index; however, we find that the relationship among them depends on the sign of correlation coefficient of spot rate and equity index market. Finally, using the differences in the maturities of the note and the reference obligation as the proxy for basis risk measure, we demonstrate that the purpose of the SPV is not used to eliminate the basis risk but the credit risk of protection buyer.

I. Introduction

Special Purpose Vehicles (SPVs) are business entities formed for the purpose of conducting a clearly-specified activity, such as collecting a specific group of accounts receivable or credit risks and so on. When investing in a project or financial instruments with well-defined risk and return, many investors may prefer the isolated and uniquely identifiable nature of an SPV to a more diffusely defined corporate form. Currently, SPVs are applied for the asset-based securities and similar financial products, such as collateralized debt obligations, mortgage-backed securities and structured Credit Linked Notes (CLNs).

According to 2002 survey by the British Bankers Association (BBA), credit-linked obligations are seen to be the second hot product followed by the credit default swaps, with 22% of market share by 2001 and 26% of market share by 2004. Besides, by the end of 2006 the size of global credit derivatives market

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will be \$20 trillion and BBA predicts that at the end of 2008 the global credit derivatives market will have expanded to \$33 trillion.

In practice, the CLNs can be issued either by the protection buyer or by an SPV. When the issuer is an SPV, the proceeds from the noteholders are used to buy high-quality collaterals that are held by the SPV. Hence the valuation of the note is only related to the credit event of reference obligation. Otherwise, when the issuer is the protection buyer, the proceeds are held on the balance sheet of the issuer as cash and hence the valuation of CLNs depends on both the credit events of reference obligation and its buyer (the issuer), and thus is different from the ones issued by SPVs. Consequently, the motivation of this article is to price structured CLNs such as the CLNs and the leveraged total return CLNs that are issued by an SPV or the buyer himself. Meanwhile, we also provide the fair fee charged by an SPV when issuing the structured CLNs. The fee is determined by the price difference of structured CLNs issued by the two entities.

Structured CLNs, such as the CLNs and the (leveraged) total return CLNs, are one type of credit derivatives that can be used by the protection buyer to hedge against the credit risk induced by some reference entities and invested by noteholders to link to the reference obligation, which is of noninvestment grade or illiquid. The payoff of a CLN is linked to the credit event of reference entities. If a credit event occurs, no further coupon payment is paid. At the termination date, the noteholders receive par value unless a credit event occurs, in which case they will be redeemed immediately for the credit event payment, which could be the nominal amount multiplied by the recovery rate of reference obligation.

The total return CLN could be also structured by incorporating the leverage factor. For a leveraged total return CLN, the payoff is linked to both the market price and credit event of reference obligation. For instance, if a financial institution issues \$5 million nominal of a 5-year leveraged total return CLN and the reference obligation is a zero coupon bond with \$25 million nominal, then the leverage factor is set to be five. The actual coupon payment also equals five times the coupon value. The coupon payment ceases immediately if a credit event occurs. Furthermore, at the termination date, the investors receive an additional payment called capital price adjustment. In this example, the payment equals five times \$5 million nominal times the change in the price of reference obligation.

There have been few studies of structured CLNs. Hui and Lo (2002) use, by extending, Merton (1974) corporate bond pricing model to value CLNs by incorporating the asset value of the reference entity as an addition variable. However, due to the unobservable parameters such as firm's value, their pricing methodology is difficult to implement. Another approach is the reduced-form model in which default time is a stopping time of some given hazard rate process, and the payoff upon credit event is specified exogenously. Hence, the probability of default in the next time partition is determined by a specified hazard rate. This approach has been widely considered by Lando (1994, 1998), Jarrow and Turnbull (1995, 2000), Duffie and Singleton (1999), Jarrow and Yu (2001) and so on. Jarrow and Turnbull (1995) uses the analogous cross-currency framework to evaluate the financial derivatives with credit risk by assuming a hazard function that is independent of spot interest rate. However, Kao (2000) documents that the interest rate level and the Russell 2000 index return have significant explanatory power for change in the credit spread index level. Campbell and Taksler (2003) demonstrate that the idiosyncratic equity volatility can explain about one-third of the variation in yield spreads. Janosi *et al.* (2002) find that the default intensity depends on the spot interest rate. Huang and Kong (2003) indicates that high interest rates and steep yield curves are usually associated with an expanding economy and low credit spread, whereas the higher interest rate volatility is usually associated with wider credit spread, especially for high-yield bond indexes. They also discover that a higher equity market index return will reduce credit spreads, and higher equity volatility will significantly widen credit spread. Athanassakos and Carayannopoulos (2001), Batten *et al.* (2005, 2006) and Batten and In (2006) indicate that relative credit spreads returns are negatively related to both changes in interest rate and changes in equity return. Therefore, the hazard process is dependent on spot interest rate and market index; and hence CreditMatrics, CreditRisk^{+} and KMV methodologies cannot be consistent with those empirical results, given their assumption of constant interest rate.

To incorporate the state variables such as spot interest rate and market index into the default intensity function, Lando (1998) uses the doubly stochastic Poisson processes of default that allow the hazard function to depend on state variables. Jarrow and Turnbull (2000) assume that the hazard function is dependent on the interest rate and index return volatility. Recently, the default model is extended to the multivariate underlying asset case to value a credit swap of the basket type. Duffie (1998) provides the valuation of a first-to-default type claim that depends on the first default time of a given list of credit events. Kijima (2000) and Kijima and

Muromachi (2000) consider the joint survival probability of occurrence times of a credit event under the assumption of conditional independence to value the default swaps of basket type.

We assume that the protection buyer can completely transfer the credit risk of reference obligation to the noteholders via the structured CLNs. Conditional independence between the default times of protection buyer and reference obligation with respect to the filtration generated by the spot interest rate and return of the market index is also assumed. In this situation, the first contribution of this article is to price the structured CLNs, such as CLNs and leveraged total return CLNs, which are issued by an SPV or the protection buyer, respectively. Further, in practice most of the protection buyers issue the structured CLNs through the SPVs. As a result, the second contribution of this article is to demonstrate the best timing for the structured CLNs issued by SPVs instead of protection buyers. In addition, we also derive the suitable fee that the protection buyer must pay to an SPV for the purpose of issuing the structured CLNs. From the numerical analyses, we investigate the characteristics for the credit spreads of structured CLNs directly issued by the protection buyers. Meantime, the empirical results, such as Kao (2000), Campbell and Taksler (2003) and Huang and Kong (2003), neglect the effect of the correlation coefficient of spot rate and market index, and they show the positive relationship among the credit spread, the volatilities of spot rate and return on market index. However, we demonstrate that the relationship among the values and credit spreads of structured CLNs, spot rate volatility and market index volatility depend on the sign of correlation coefficient of spot rate and equity index. Finally, the important sources of basis risk in credit derivatives discussed in the literature are the differences in the maturities of the hedging instrument and the reference obligation. In addition, Azarchs (2003) indicates that issuers could not eliminate basis risk by issuing credit derivatives, thus by defining the differences in the maturities of the note and the reference obligation as the proxy for the basis risk measure, we find that the value of an SPV is not an increasing function of the basis risk measure, but is positively related with the default intensity of the reference obligation and protection buyer. Therefore, we demonstrate that the purpose for issuing the structured CLNs through SPVs is to hedge the credit risk induced by the protection buyer but not the basis risk.

An outline of the article is as follows. In Section II, we introduce the trading economy. In Sections III and IV, we derive the analytic formula of CLNs and leveraged total return CLNs, respectively. We present the characteristics of the structured CLNs in Section V. Summary and conclusions is presented in Section VI.

II. Setup of Economy

Let the uncertainty in the economy be described by the filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T^*})$. We assume the existence and uniqueness of P, so that bond markets are complete. Let τ^A and τ^R stand for the default times of the protection buyer and reference entity and given as

$$\tau^i = \inf\left\{ t : \int_t^T \lambda^i(X_s)\mathrm{d}s \geq E_i \right\}, \quad \text{for } i = A \text{ or } R$$

where we assume that the construction of the doubly stochastic Poisson processes of default (also called a Cox process) with an intensity function $\lambda^i(X_t)$, $(X_t)_{t=0}^{T^*}$ is a right continuous with left limits R^d-valued process and represents d state variables underling the evolution of the economy, such as the spot rate, market index, credit ratings or other variables deemed relevant for predicting the likelihood of default. E_i is a unit exponential random variable which is independent of state variables and λ^i. The default time can be thought of as the first jump time of a Cox process with stochastic intensity process $\lambda^i(X_t)$ and are conditional independent with respect to the filtration generated by X under P.

Some empirical studies, such as Kao (2000), Campbell and Taksler (2003), Huang and Kong (2003), Batten *et al.* (2005, 2006), indicate that credit spreads are negatively related to both in interest rate and equity return. To describe the dependence of the default process on the state of the economy and incorporate the empirical results into our reduced-form model, we introduce the enlarged filtration F by setting

$$F_t = F_t^r \vee F_t^I \vee H_t^A \vee H_t^R$$

where $F_t^r = \sigma(r(s), 0 \leq s \leq t)$, $F_t^I = \sigma(I(s), 0 \leq s \leq t)$ and $H_t^i = \sigma(1_{\{\tau^i \leq s\}}, s \leq t)$, $i = A$ or R. $1_{\{\cdot\}}$ is the indicator function. $r(t)$ is the spot rate at time t. $I(t)$ denotes a time-t market index such as the Standard and Poor 500 stock index. As a result, $F_{T^*}^r \vee F_{T^*}^I$ contains complete information on the spot rate and the market index. In an economy that evolves according to the filtration $F_{T^*}^r \vee F_{T^*}^I$, it is possible to select a nonnegative, $F_{T^*}^r \vee F_{T^*}^I$-measurable process λ^i, and satisfies $\int_0^t \lambda^i[r(s), I(s)]\mathrm{d}s$ P–a.s. for all $t \in [0, T^*]$. An inhomogeneous point process can be defined, using the realized history of the process λ^i as its intensity function. Consequently, the conditional distributions of τ^i are given as

$$P(\tau^i > t \mid F^r_{T^*} \vee F^I_{T^*}) = \exp\left(-\int_0^t \lambda^i[r(s), I(s)]\mathrm{d}s\right)$$

$$t \in [0, T^*]$$

Hence, by the law of iterated expectation, we have

$$P(\tau^i > t) = E\left[\exp\left(-\int_0^t \lambda^i[r(s), I(s)]\mathrm{d}s\right)\right], \quad t \in [0, T^*]$$

Let $p(t, T)$ be the time-t price of a zero coupon bond paying one dollar at time T. $Y(t, T)$ is the yield-to-maturity of $p(t, T)$. $B(t)$ corresponds to the wealth accumulated by an initial one-dollar investment at short-term interest rate $r(t)$ in each subsequent period. Therefore,

$$p(t, T) = \exp[-y(t, T) \times (T - t)] = E\left(\frac{B(t)}{B(T)}\middle| F_t\right) \text{ and}$$

$$B(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right) \tag{1}$$

We assume that the point processes governing default for the spot rate and market index is

$$\lambda^i(u) = f[r(u), Z(u)]$$

where

$$z(u) \equiv \left[\log\left(\frac{I(u)}{I(0)}\right) - \log\left(\frac{B(u)}{B(0)}\right)\right]$$

Jarrow and Turnbull (1995, 2000), Lando (1998) and Jarrow and Yu (2001) assume the hazard rate function is linear when modelling their hazard rates. Thus this article follows their framework and our linear hazard rate function admits the following representation:

$$\begin{aligned}\lambda^i(u) &\equiv \psi^i_0 + \lambda^i_1 r(u) + \beta_i z(u)\\ &= \left[\psi^i_0 - \beta_i \log\left(\frac{I(0)}{B(0)}\right)\right] + \lambda^i_1 r(u) + \beta_i \log\left(\frac{I(u)}{B(u)}\right)\\ &= \lambda^i_0 + \lambda^i_1 r(u) + \beta_i \log\left(\frac{I(u)}{B(u)}\right), \quad \text{for } i = A \text{ or } R\end{aligned} \tag{2}$$

where λ^i_0 is the spontaneous default intensity of entity i, λ^i_1 measures the sensitivity of entity i to the level of spot rate and β_i represents the sensitivity of entity i to the excess return on market index.

Lando (1998) uses the doubly stochastic Poisson processes of default that allow the linear hazard function to depend on state variables. Jarrow and Turnbull (2000) assume that the linear hazard function is dependent on the interest rate and index return volatility. Jarrow and Yu (2001) model the linear hazard function is only affected by the interest rate. Hence, we extend the model of Jarrow and Yu (2001) to incorporate the market index into our hazard rate function. Note that, Jarrow and Turnbull (2000) uses index return volatility to capture the effect on credit spread, but we consider the return of the market index which can describe the level of market index return and index return volatility. We also assume that the stochastic processes of $r(t)$ and $I(t)$ are given as follows:

$$\mathrm{d}r(t) = [\theta(t) - \alpha(t)r(t)]\mathrm{d}t + \sigma_r \mathrm{d}W^r_t \tag{3}$$

$$\frac{\mathrm{d}I(t)}{I(t)} = r(t)\mathrm{d}t + \sigma_I \mathrm{d}W^I_t \tag{4}$$

where $\theta(t)$ represents the long-term equilibrium value of the process; $\alpha(t)$ is a nonnegative mean reversion speed; and σ_r and σ_I are the volatilities of spot rate and return on market index, respectively. W^r_t and W^I_t are Brownian motions with respect to F_t and satisfy $E(W^r_t W^I_t) = \rho_{rI}\mathrm{d}t$, where ρ_{rI} is the correlation coefficient of spot rate and market index. The assumption of normality for hazard rate function allows the derivation of closed form solutions, such as expression (5) below. One of the disadvantages of this assumption is that the intensity function can be negative. However, in lattice-based models, this difficulty can be avoided via the use of nonlinear transformations, as in Jarrow and Turnbull (1997).

Let $\nu_A(t, T)$ and $\nu_R(t, T)$ be the prices of risky zero coupon for the protection buyer and reference obligation, respectively. δ_A and δ_R are correspondingly the recovery rates of the protection buyer and reference entity. Therefore, it is noteworthy that under the setup of equality (2), the valuations of risky zero coupon bonds $\nu_A(t,T)$ and $\nu_R(t, T)$ are presented in the following theorem.

Theorem 1: *The prices of risky zero coupon bond for entity i, i = A or R, admits the following representation*

$$\begin{aligned}\nu_i(t, T) = p(t, T)\Bigg[\delta_i + 1_{\{\tau^i > t\}}(1 - \delta_i)\exp\Bigg(&-\lambda^i_0(T - t)\\ &- \lambda^i_1 Y(t, T)(T - t) + \frac{\lambda^i_1(1 + \lambda^i_1)}{2}\sigma^2(t, T)\\ &+ \frac{(T - t)^2}{4}\beta_i\sigma^2_I - \frac{(T - t)^3}{6}\beta^2_i\sigma^2_I\\ &- (1 + \lambda^i_1)\beta_i\sigma_I\rho_{rI}\int_t^T (T - u)b(u, T)\mathrm{d}u\Bigg)\Bigg],\\ &i = A \text{ or } R\end{aligned} \tag{5}$$

where

$$\sigma^2(t,s)=V_t\left[\int_t^s r(u)\mathrm{d}u\right]=\int_t^s b(u,s)^2\mathrm{d}u$$

$$b(t,T)=-\sigma_r D(t,T),\ D(t,T)=\int_t^T \exp\{-[\Phi(s)-\Phi(t)]\}\mathrm{d}s$$

$$\Phi(t)=\int_0^t a(u)\mathrm{d}u,\ a(u)=\exp[-\alpha(t-u)]$$

$b(t,T)$ is denoted as the volatility of default-free zero coupon bond, and $V_t(\cdot)$ is the variance conditional on F_t.

We prove Theorem 1 in Appendix 1.

If the recovery rate is zero, we can rewrite equality (5) as

$$0 \le \frac{v_i(t,T)}{p(t,T)} \equiv 1_{\{\tau^i > t\}} \exp[-cs_i(t,T) \times (T-t)] \le 1,$$
$$i = A \text{ or } R$$

where $cs_i(t,T)$ is the credit spread function of risky zero coupon bond of the entity i and is defined as follows:

$$\begin{aligned} cs_i(t,T) = \Bigg[& \lambda_0^i + \lambda_1^i Y(t,T) - \frac{(T-t)\beta_i\sigma_I^2}{4} - \frac{(\lambda_1^i)^2+\lambda_1^i}{2} \\ & \times \frac{\sigma^2(t,T)}{T-t} + \frac{(T-t)^2\beta_i^2\sigma_I^2}{6} \\ & + \frac{(1+\lambda_1^i)}{T-t}\beta_i\sigma_I\rho_{rI}\int_t^T (T-u)b(u,T)\mathrm{d}u \Bigg] \end{aligned}$$

If $\beta_i = 0$, the credit spread function reduces to the result of Jarrow and Yu (2001), in which the hazard rate function is only affected by the level of interest rate. Thus $cs_i(t,T)$ is the generalization of Jarrow and Yu (2001).

Next, we discuss the properties of the credit spread function. Using the chain rule of differentiation, we have

$$\frac{\partial cs_i(t,T)}{\partial Y(t,T)} = \lambda_1^i, \quad i = A \text{ or } R$$

$$\frac{\partial cs_i(t,T)}{\partial \rho_{rI}} = \frac{(1+\lambda_1^i)\beta_i\sigma_r}{(T-t)}\int_t^T (T-u)b(u,T)\mathrm{d}u,$$
$$i = A \text{ or } R$$

$$\begin{aligned} \frac{\partial cs_i(t,T)}{\partial \sigma_r^2} = & \frac{(1+\lambda_1^i)}{2(T-t)\sigma_r^2} \\ & \times \left[-\lambda_1^i\sigma^2(t,T) + \beta_i\sigma_I\rho_{rI}\int_t^T (T-u)b(u,T)\mathrm{d}u\right], \end{aligned}$$
$$i = A \text{ or } R \tag{6}$$

$$\frac{\partial cs_i(t,T)}{\partial \sigma_r^2} = -\frac{(T-t)\beta_i}{4} + \frac{(T-t)^2\beta_i^2}{6} + \frac{(1+\lambda_1^i)\beta_i\rho_{rI}}{2\sigma_I(T-t)}$$
$$\int_t^T (T-u)b(u,T)\mathrm{d}u, \quad i = A \text{ or } R \tag{7}$$

Under the restrictions that

$$\lambda_0^i > 0,\ \beta_i < 0, \quad -1 < \lambda_1^i < 0$$

we have

$$\frac{\partial cs_i(t,T)}{\partial Y(t,T)} < 0, \quad \frac{\partial cs_i(t,T)}{\partial \rho_{rI}} > 0$$

As a result, a higher correlation coefficient of spot rate and market index or lower yields of default-free zero coupon bonds are associated with the wider credit spreads of risky zero coupon bonds. Furthermore, if $\rho_{rI} \ge 0$, we have

$$\frac{\partial cs_i(t,T)}{\partial \sigma_r^2} > 0 \quad \text{and} \quad \frac{\partial cs_i(t,T)}{\partial \sigma_I^2} > 0 \tag{8}$$

Equality (8) coincides with the empirical results such as Kao (2000), Campbell and Taksler (2003) and Huang and Kong (2003). Or equivalently, under the case that ρ_{rI} is nonnegative (usually corresponding to an expanding economy or recession), since the hazard rate function is an increasing function of the spontaneous default intensity and negatively sensitive to the level of spot rate as well as to the excess equity return, we can obtain that the credit spread of risky zero coupon bonds increases as the volatilities of spot rate or the volatility of return on market index increase. Hence, the characteristics of credit spread of risky zero coupon bonds match the empirical results for credit spreads.

However, the equalities (6) and (7) may be negative if ρ_{rI} is negative. As a result, the correlation coefficient of spot rate and market index plays an important role in the relationship among the credit spread, spot rate volatility and market index volatility. Nevertheless, the empirical results, such as Kao (2000), Campbell and Taksler (2003), Huang and Kong (2003), etc., do not incorporate the correlation coefficient of spot rate and market index into their regression models and then they may ignore the possibilities of negatively relationship among the credit spread, spot rate volatility and market index volatility. Consequently, for further research, we suggest that the dependent variable ρ_{rI} should be included in the empirical regressions to more exactly capture their relationship.

III. Pricing Model for CLNs

In this section, we first derive the analytic formula of CLNs when the issuers are respectively an SPV and the protection buyer. Then we provide the fair fee that the buyer should pay to the SPV for the purpose of issuing the CLNs. Let T be the maturity date of CLNs. C_s is the coupon payment[1] at time s, $s \in [t, T]$ and M is the nominal principle. δ_A and δ_R are the recovery rate of reference obligation and the issuer, respectively.

CLNs issued by an SPV

If the note issuer is an SPV, then the proceeds from the noteholders are used to buy some high-quality collateral that is held by the SPV, thus the valuation of the note is uncorrelated with the credit event of the protection buyer who owns the reference obligation. The payoff structures of CLN is as follows: (1) For coupon payment, if the reference entity does not default at the coupon payment date s, the payments are C_s. Otherwise, the payment is zero; (2) For the principal, if there is no default prior to time T, the payments are M. If there is a default event prior to time T, the credit event payment $\delta_R M$ is paid immediately at default time τ^R. Therefore, the valuation of the CLN equals

$$C_{\mathrm{SPV}}(t) = E\left[\int_t^T 1_{\{\tau^R > s\}} C_s \frac{B(t)}{B(s)} \mathrm{d}s + 1_{\{\tau^R > T\}} \frac{B(t)}{B(T)} M + 1_{\{t < \tau^R \le T\}} \frac{B(t)}{B(\tau^R)} \delta_R M \Big|_{F_t}\right]$$

The analytic formula of the CLN issued by an SPV is provided in the following theorem.

Theorem 2: *The analytic formula of the CLN is*

$$C_{\mathrm{SPV}}(t) = \begin{cases} \int_t^T C_s p(t,s) G_1(t,s)\mathrm{d}s + M\, p(t,T) G_1(t,T) \\ + \lambda_0^R \delta_R M \int_t^T p(t,s) G_1(t,s)\mathrm{d}s + \lambda_1^R \delta_R M \\ \int_t^T p(t,s) G_2(t,s)\mathrm{d}s \\ + \beta_R \delta_R M \int_t^T p(t,s) G_3(t,s)\mathrm{d}s & \text{if } \tau^R > t \\ \delta_R M & \text{if } \tau^R = t \\ 0 & \text{if } \tau^R < t \end{cases} \tag{9}$$

where

$$G_1(t,s) = \exp\Bigg(-\lambda_0^R (T-t) - \lambda_1^R Y(t,s)(s-t) + \frac{\lambda_1^R (1+\lambda_1^R)}{2} \sigma^2(t,s) + \frac{(s-t)^2}{4} \beta_R \sigma_1^2 + \frac{(s-t)^3}{6} \beta_R^2 \sigma_I^2 - (1+\lambda_1^R) \beta_R \sigma_I \rho_{rI} \int_t^s (s-y) b(y,s)\mathrm{d}y\Bigg) = \exp[-cs_R(t,s) \times (s-t)] \tag{10}$$

is a credit risk adjusted discount factor induced by spontaneous default intensity of reference obligation if the issuer is an SPV and

$$G_2(t,s) = G_1(t,s) \times \left[f(t,s) - \frac{\lambda_1^R}{2} b(t,s)^2 - \beta_R \sigma_I \rho_{rI} \int_t^s \Phi(u,s)(s-y)\mathrm{d}y\right]$$

is a credit risk adjusted discount factor induced by the sensitivity of reference obligation to the level of spot rate if the issuer is an SPV and $\mu_0(t,s) \equiv E_t[r(s)] = f(t,s) - \int_t^s \Phi(u,s) b(u,s)\mathrm{d}u \cdot G_3(t,s)$ *represents the credit risk adjusted discount factor induced by the sensitivity of reference obligation to the excess return on market index if the issuer is an SPV and is defined as follows*:

$$G_3(t,s) = G_1(t,s) \times \left[\frac{\sigma_I^2 (s-t)}{2} [-1 - \beta_R (s-t)] + (1+\lambda_1^R) \sigma_I \rho_{rI} \int_t^s b(y,s)\mathrm{d}y\right]$$

A detailed proof is sketched in Appendix 2.

Using the chain rule of differentiation for $G_1(t, s)$, we have

$$\begin{aligned} \frac{\partial G_1(t,s)}{\partial \lambda_0^R} &= -(s-t) G_1(t,s) < 0, \quad \frac{\partial G_1(t,s)}{\partial \rho_{rI}} = -(s-t) G_1(t,s) \frac{\partial cs_R(t,s)}{\partial \rho_{rI}} < 0 \\ \frac{\partial G_1(t,s)}{\partial \sigma_r^2} &= -(s-t) G_1(t,s) \frac{\partial cs_R(t,s)}{\partial \sigma_r^2}, \quad \frac{\partial G_1(t,s)}{\partial \sigma_I^2} = -(s-t) G_1(t,s) \frac{\partial cs_R(t,s)}{\partial \sigma_I^2} \end{aligned} \tag{11}$$

[1] We use continuous time to calculate the accrued interest payment instead of discrete time, since the credit event may happen prior to the coupon payment date.

Hence, the discounted factor is decreasing functions of spontaneous default intensity of reference entity and ρ_{rI}. In addition, if $\rho_{rI}>0$ in view of the equality (8), we have

$$\frac{\partial G_1(t,T)}{\partial \sigma_r^2} < 0 \quad \text{and} \quad \frac{\partial G_1(t,T)}{\partial \sigma_I^2} < 0$$

This means that under the case for $\rho_{rI}>0$, higher volatilities of spot rate and market index return and default probability of reference obligation are usually associated with wider credit spread and lower credit risk discounted factor as well as the values of CLNs. However, particularly, if $\rho_{rI}<0$, the equalities (6) and (7) may be negative, which is different from the empirical results for credit spreads, and then the equality (11) may be positive. Meanwhile, it is also apparent that

$$\frac{\partial p(t,s)G_1(t,s)}{\partial Y(t,s)} = -(1+\lambda_1^R)(s-t)G_1(t,s) < 0$$

As a result, we conclude that higher yield of default-free zero coupon bond is likely to be accompanied by the lower values of CLNs.

Issued by the protection buyer

If the issuer is the protection buyer, the note value is related to the default intensities of both the protection buyer and reference obligation. To show its payoff structure, for the coupon payment, if both the reference entity and the protection buyer do not default at the coupon payment date s, the payments are C_s. If the protection buyer defaults prior to the coupon payment date s but the reference entity does not, the payment are $\delta_A C_s$. Otherwise, the payment is 0. For the principal of the CLN, if default does not occur prior to time T, the payments is M. If the buyer defaults prior to time T but the reference entity does not, similar to the case of the common bonds that are already in default, the default payment $\delta_A M$ is paid at time T. Besides, when the reference entity defaults prior to time T, two additional scenarios should be discussed. First, if the default time of reference entity is prior to the one of the buyer, the credit event payment $\delta_R M$ is paid immediately at default time τ^R whether or not the default time of the buyer is earlier than the date. If the first to default is the buyer, the payment $\delta_A\delta_R M$ is paid immediately at time τ^R, then the value of the CLN is as follows:

$$\begin{aligned}
C_{BP}(t) = E\Bigg[&\int_t^T 1_{\{\tau^A>s\}}1_{\{\tau^R>s\}}C_s\frac{B(t)}{B(s)}\mathrm{d}s \\
&+\int_t^T 1_{\{\tau^A\le s\}}1_{\{\tau^R>s\}}\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s + 1_{\{\tau^A>T\}}1_{\{\tau^R>T\}} \\
&\frac{B(t)}{B(T)}M + 1_{\{t<\tau^A\le T\}}1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\delta_A M \\
&+1_{\{\tau^A>\tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_R M + 1_{\{\tau^A\le\tau^R\}}1_{\{t<\tau^R\le T\}} \\
&\frac{B(t)}{B(\tau^R)}\delta_A\delta_R M\Bigg|F_t\Bigg]
\end{aligned}$$

The analytic formula of the CLN issued by the protection buyer is provided in Theorem 3.

Theorem 3: *The analytic formula of the CLN issued by the protection buyer is as follows*:

$$C_{PB}(t) = \begin{cases}
\int_t^T C_s p(t,s)[\delta_A G_1(t,s)+(1-\delta_A)G_4(t,s)]\mathrm{d}s & \\
+Mp(t,T)[\delta_A G_1(t,T)+(1-\delta_A)G_4(t,T)] & \\
+\lambda_0^R\delta_R M\int_t^T p(t,s)[\delta_A G_1(t,s)+(1-\delta_A)G_4(t,s)]\mathrm{d}s & \\
+\lambda_1^R\delta_R M\int_t^T p(t,s)[\delta_A G_2(t,s)+(1-\delta_A)G_5(t,s)]\mathrm{d}s & \\
+\beta_R\delta_R M\int_t^T p(t,s)[\delta_A G_3(t,s)+(1-\delta_A)G_6(t,s)]\mathrm{d}s & \textit{if } \tau^A>t,\ \tau^R>t \\
\delta_A C_{\mathrm{SPV}}(t) & \textit{if } \tau^A\le t,\ \tau^R>t \\
0 & \textit{otherwise}
\end{cases} \tag{12}$$

where

$$\begin{aligned}
G_4(t,s)=\exp\Bigg(&-(\lambda_0^R+\lambda_0^A(s-t)-(\lambda_1^R+\lambda_1^A)Y(t,s)(s-t) \\
&+\frac{(\lambda_1^R+\lambda_1^A)(1+\lambda_1^R+\lambda_1^A)}{2}\sigma^2(t,s) \\
&+\frac{(s-t)^2}{4}(\beta_R+\beta_A)\sigma_1^2+\frac{(s-t)^3}{6}(\beta_R+\beta_A)^2\sigma_I^2 \\
&-(1+\lambda_1^R+\lambda_1^A)(\beta_R+\beta_A)\sigma_I\rho_{rI} \\
&\times\int_t^s(s-u)b(u,s)\mathrm{d}u\Bigg)
\end{aligned}$$

is a discount factor induced by spontaneous default intensity of reference obligation if the issuer is a protective buyer, and

$$G_5(t,s)=G_3(t,s)\times\left[f(t,s)-\frac{(\lambda_1^R+\lambda_1^A)}{2}b(t,s)^2\right.$$
$$\left.-(\beta_R+\beta_A)\sigma_I\rho_{rI}\int_t^s\Phi(u,s)(s-u)\mathrm{d}u\right]$$

is a credit risk adjusted discount factor induced by the sensitivity of reference obligation to the level of spot rate if the issuer is a protective buyer. $G_6(t, s)$ *expresses the credit risk adjusted discount factor induced by the sensitivity of reference obligation to the excess return on market index if the issuer is a protective buyer and is displayed as follows*:

$$G_6(t,s)=G_3(t,s)\times\left[\frac{(\sigma_I^2(s-t)}{2}[-1-(\beta_R+\beta_A)(s-t)]\right.$$
$$\left.+(1+\lambda_1^R+\lambda_1^A)\sigma_I\rho_{rI}\int_t^s b(u,s)\mathrm{d}u\right]$$

We prove Theorem 3 in Appendix 3.

It is apparent that equality (9) is a special case of equality (11) by substituting $\delta_A=\lambda_0^A=\lambda_1^A=0$. Or equivalently, a CLN is free of credit exposure for the protection buyer if the issuer is an SPV. Moreover, we can rewrite $G_4(t, s)$ as follows:

$$\begin{aligned}G_4(t,s)=\exp\Bigg[&-(\lambda_0^H)(s-t)-\lambda_1^H Y(t,s)(s-t)\\&+\frac{\lambda_1^H(1+\lambda_1^H)}{2}\sigma^2(t,s)+\frac{(s-t)^2}{4}\beta_H\sigma_I^2\\&+\frac{(s-t)^3}{6}\beta_H^2\sigma_I^2-(1+\lambda_1^H)\beta_H\sigma_I\rho_{rI}\\&\times\int_t^s(s-u)b(u,s)\mathrm{d}u\Bigg]\\\equiv{}&\exp[-(cs_H(t,s)(s-t)]\end{aligned}\tag{13}$$

where $\lambda_0^H\equiv\lambda_0^A+\lambda_0^R, \lambda_1^H\equiv\lambda_1^A+\lambda_1^R$ and $\beta_H=\beta_A+\beta_R$. In view of (2), the default hazard rate function[2] $\lambda^H(u)$ of entity H defined in this way is indeed the sum of $\lambda^A(u)$ and $\lambda^R(u)$, where $u\in[t,T]$.

Hence, we have

$$\begin{aligned}&\frac{\partial G_4(t,s)}{\partial\lambda_0^i}=-(s-t)G_4(t,s)<0,\quad i=A\text{ or }R\\&\frac{\partial G_4(t,s)}{\partial\rho_{rI}}=-(s-t)G_4(t,s)\frac{\partial cs_H(t,s)}{\partial\rho_{rI}}<0\\&\frac{\partial G_4(t,s)}{\partial\sigma_r^2}=-(s-t)G_4(t,s)\frac{\partial cs_H(t,s)}{\partial\sigma_r^2},\quad\frac{\partial G_4(t,s)}{\partial\sigma_I^2}\\&\qquad=-(s-t)G_4(t,s)\frac{\partial cs_H(t,s)}{\partial\sigma_I^2}\end{aligned}\tag{14}$$

Similar to the case for a CLN issued by an SPV, the discounted factor is increasing with spontaneous default intensity of the reference entity, spontaneous default intensity of the protective buyer and ρ_{rI}. Additionally, if $\rho_{rI}>0$, we can see that $G_4(t, s)$ is decreasing functions of spontaneous default intensities for both the protection buyer and reference entity and the volatilities of spot rate and return on market index. However, it is possible to obtain the opposite outcomes when $\rho_{rI}<0$. Moreover, it is also clear that

$$\frac{\partial p(t,s)G_4(t,s)}{\partial Y(t,s)}=-(1+\lambda_1^H)(s-t)G_4(t,s)<0,\quad i=A\text{ or }R$$

Consequently, this in turn shows that higher yields of default-free zero coupon bonds are also related to lower values of CLNs no matter who the issuers are.

Fair fee charged by an SPV

In practice, the CLNs can be issued either by the protection buyer or by an SPV. When the issuer is an SPV, the proceeds from the noteholder are used to buy high-quality collateral that is held by the SPV. Otherwise, the proceeds are held on the balance sheet of the protection buyer as cash. Thus, when it comes to investing in CLNs, many noteholders may prefer the isolated and uniquely identifiable nature of an SPV to a more diffusely defined corporate form such as the protection buyer. Since the SPVs play an important role in practice, when financial institutions who own the reference obligation issue CLNs through SPVs, it is imperative to determine the values or fair fees charged by SPVs with issuing the CLNs, especially for accounting purposes. Given the pricing formulas in equalities (9) and (11), the fair

[2] The default time of entity H can be explained as the minimum of τ^A and τ^R and hence entity H is the first-to-default contingent claim. The readers can refer to Bielecki and Rutkowski (2002) for a full treatment.

fee charged by an SPV with issuing a CLN is equal to equality (9) minus equality (12) and is presented at the following corollary.

Corollary 1: *From Theorems* 2 *and* 3, *the fair fee charged by an SPV issuing a CLN, defined as* SPV_{fee}, *is given as follows*:

$$SPV_{\text{fee}}(t) = \begin{cases} (1-\delta_A)\int_t^T C_s p(t,s)[G_1(t,s)-G_4(t,s)]\mathrm{d}s \\ +(1-\delta_A)Mp(t,T)[G_1(t,T)-G_4(t,T)] \\ +(1-\delta_A)\lambda_0^R\delta_R M\int_t^T p(t,s)[G_1(t,s)-G_4(t,s)]\mathrm{d}s \\ +(1-\delta_A)\lambda_1^R\delta_R M\int_t^T p(t,s)[G_2(t,s)-G_5(t,s)]\mathrm{d}s \\ +(1-\delta_A)\beta_R\delta_R M\int_t^T p(t,s)[G_3(t,s)-G_6(t,s)]\mathrm{d}s & \text{if } \tau^A>t,\ \tau^R>t \\ (1-\delta_A)C_{SPV}(t) & \text{if } \tau^A\le t,\ \tau^R>t \\ 0 & \text{otherwise} \end{cases}$$

From Corollary 1, if $G_1(t,s)>G_4(t,s)$, $G_2(t,s)>G_5(t,s)$ and $G_3(t,s)>G_6(t,s)$, we can see that the fee charged by an SPV is always greater than zero. In view of (10) and (13), $G_4(t,s)$ consider both default risks of reference obligation and protection buyer, but $G_1(t,s)$ only consider default risk of reference obligation. Hence, it is obvious to obtain that $G_1(t,s)>G_4(t,s)$. Alone line as spot rate and index are positive, it can be shown that $G_2(t,s)>G_5(t,s)$ and $G_3(t,s)>G_6(t,s)$. Therefore, we conclude that the appropriate fee charged by an SPV with issuing the CLNs is a positive amount.

IV. Pricing the Leveraged Total Return CLNs

Issued by an SPV

For pricing the leveraged total return CLNs, we assume that the maturity date of a leveraged total return CLN is T, L is the leverage factor, LC_s is the coupon payment at time s, $s\in[t,T]$ and M is the principal amount. In addition, $v^R(t_0,U)$ is the price at time t of a risky zero coupon bond that pays one dollar at time U and issues by the reference obligation, where $t_0\le t\le T\le U\le T^*$. t_0 is the launching date of the leveraged total return notes. Similarly, if the issuer is an SPV, the note value is uncorrelated with the credit event of the protection buyer. For their payoff structure, if the reference entity does not default at the coupon payment date s, the coupon payment is LC_s. Otherwise, the coupon payment is zero. In view of the principal amount, if there is no default prior to time T, the noteholders receive the leveraged principal, which is defined as the sum of the principal plus the principal multiplied by both the leverage factor and the return on the price of risky zero coupon bond of reference obligation. Otherwise, the noteholders receive the leveraged principal immediately at time τ^R. Hence, the valuation of the leveraged total return CLN is as follows:

$$\begin{aligned}\mathrm{TC}_{\mathrm{SPV}}(t) = E\Bigg[&\int_t^T 1_{\{\tau^R>s\}}\mathrm{LC}_s\frac{B(t)}{B(s)}\mathrm{d}s + 1_{\{\tau^R>T\}}\frac{B(t)}{B(T)} \\ &\times\left[M\left(1+L\frac{v^R(T,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right] \\ &+1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)} \\ &\times\left[M\left(1+L\frac{\delta_R p(\tau^R,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]\Bigg|F_t\Bigg]\end{aligned}$$

The pricing formula for the leveraged total return CLN issued by an SPV is shown in Theorem 4.

Theorem 4: *The analytic formula of the leveraged total return CLN is*

$$\mathrm{TC}_{\mathrm{SPV}}(t) = \begin{cases} \int_t^T \mathrm{LC}_s p(t,s)G_1(t,s)\mathrm{d}s + M\Big\{(1-L) \\ \Big[p(t,T)G_1(t,T)+\int_t^T p(t,s)[\lambda_0^R G_1(t,s) \\ +\lambda_1^R G_2(t,s)+\beta_R G_3(t,s)]\mathrm{d}s\Big] \\ +\frac{L}{v^R(t_0,U)}\Big[\delta_R p(t,T)G_7(t,T,U) \\ +(1-\delta_R)p(t,U)G_1(t,U) \\ +\int_t^T \delta_R p(t,s)[\lambda_0^R G_7(t,s,U)+\lambda_1^R G_8(t,s,U) \\ +\beta_R G_9(t,s,U)]\mathrm{d}s\Big]\Big\} & \text{if } \tau^R>t \\ M\left(1+L\frac{\delta_R p(t,U)-v^R(t_0,U)}{v^R(t_0,U)}\right) & \text{if } \tau^R=t \\ 0 & \text{if } \tau^R<t \end{cases} \tag{15}$$

where

$$G_7(t,T,U)=\exp\Bigg[-\lambda_0^R(T-t)-Y(t,U)(U-t)-(\lambda_1^R-1)Y(t,T)(T-t)+\frac{(T-t)^2}{4}\beta_R\sigma_1^2+\frac{(\lambda_1^R)^2-\lambda_1^R}{2}\sigma^2(t,T)+\frac{(T-t)^3}{6}\times\beta_R^2\sigma_1^2+\lambda_1^R\rho_1(t,T,U)+\beta_R\sigma_1\rho_2(t,T,U)+\lambda_1^R\beta_R\sigma_1\rho_3(t,T)\Bigg]$$

is a credit risk adjusted discount factor induced by spontaneous default intensity of reference obligation if the leveraged total return CLN is issued by an SPV and

$$G_8(t,s,U)=G_7(t,s,U)\times\left[\mu_0(t,s)+\int_t^s\Phi(y,s)b(y,U)\mathrm{d}y-\frac{\lambda_1^R}{2}b(t,s)^2-\beta_R\sigma_I\rho_{rI}\int_t^s\Phi(y,s)\mathrm{d}y\right]$$

is a credit risk adjusted discount factor induced by the sensitivity of reference obligation to the level of spot rate if the leveraged total return CLN is issued by an SPV. $G_9(t,T,U)$ expresses the credit risk adjusted discount factor induced by the sensitivity of reference obligation to the excess return on market index if the leveraged total return CLN is issued by an SPV and is defined as follows:

$$G_9(t,s,U)=G_7(t,s,U)\times\left[\frac{\sigma_1^2(s-t)}{2}[-1-\beta_R(s-t)]+\sigma_I\rho_{rI}\int_t^s b(y,U)\mathrm{d}y+\lambda_1^R\sigma_I\rho_{rI}\int_t^s b(y,s)\mathrm{d}y\right]$$

$$\rho_1(t,T,U)=\int_t^T b(y,U)b(y,T)\mathrm{d}y$$

$$\rho_2(t,T,U)=-\rho_{rI}\int_t^T(T-y)b(y,U)\mathrm{d}y,\ \rho_3(t,T)=-\rho_{rI}\int_t^T(T-y)b(y,T)\mathrm{d}y$$

The detailed proof for Theorem 4 is given by Appendix 4.

By virtue of (13), if U equals T, we obtain $G_7(t,T,U)=G_1(t,T)$ and thereby $G_7(t,T,U)$ is composed of decreasing functions of spontaneous default intensity, the interest rate volatility and the volatility of return on market index when $p_{rI}>0$. Conversely, the relationship among $G_7\ (t,T,U)$, spot rate volatility and market index volatility may not be consistent with the empirical results when $p_{rI}<0$. Meanwhile, it is also apparent that

$$\frac{\partial p(t,T)G_5(t,T,U)}{\partial Y(t,T)}=\frac{\partial p(t,T)G_1(t,T)}{\partial Y(t,T)}=-(1+\lambda_1^R)(T-t)G_1(t,T)<0$$

Therefore, we can document that higher level of yield of default-free zero coupon bond is associated with lower values of the leveraged total return CLNs under the condition of $U=T$.

Issued by the protection buyer

Similarly, if the issuer is the protection buyer, the note value is related with the default intensities of the protection buyer and the reference obligation. For the coupon payment, if neither the reference entity nor the protection buyer default at the coupon payment date s, the payments are LC_s. If the protection buyer defaults at the coupon payment date s but the reference entity does not, the payments are $\delta_A LC_s$. For the leveraged principal, the following four cases should be discussed. First, if there is no default prior to maturity date T, the noteholders receive the leveraged principal at the maturity. Second, if the reference entity does not default prior to maturity T but the protection buyer does default, similar to the case that risky bond defaults, the noteholders receive the amount equal to leveraged principal multiplied by the recovery rate of the buyer at the maturity. Third, if both the entities default prior to maturity T but the first-to-default is the reference entity, the noteholders receive the leveraged principal immediately at time τ^R. Finally, if both the entities default prior to maturity T but the first-to-default is the protection buyer, the noteholders receive the amount equal to leveraged principal multiplied by the recovery rate of the protection buyer immediately at time τ^R. In brief, the payoff of a leveraged total return CLN is as follows:

$$\mathrm{TC_{PB}}(t)=E\Bigg\{\int_t^T 1_{\{\tau^A>s\}}1_{\{\tau^R>s\}}\mathrm{LC}_s\frac{B(t)}{B(s)}\mathrm{d}s+\int_t^T 1_{\{\tau^A\le s\}}1_{\{\tau^R>s\}}L\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s+1_{\{\tau^A>T\}}1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\times\left[M\left(1+L\frac{v^R(T,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]+1_{\{t<\tau^A\le T\}}1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\delta_A\times\left[M\left(1+L\frac{v^R(T,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]$$

$$+1_{\{\tau^A>\tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\times\left[M\left(1+L\frac{\delta_R P(\tau^R,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]+1_{\{\tau^A\le\tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_A\times\left[M\left(1+L\frac{\delta_R p(\tau^R,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]\Bigg|F_t\Bigg\}$$

We provide the analytic formula of a leveraged total return CLN issued by the protection buyer in Theorem 5.

Theorem 5: *The analytic formula of the leveraged total return CLN is as follows*:

$$\mathrm{TC}_{PB}(t)=\begin{cases}\int_t^T \mathrm{LC}_s p(t,s)[(1-\delta_A)G_4(t,s)+\delta_A G_1(t,s)]\mathrm{d}s+M\Bigg\{(1-L)\Bigg(p(t,T)[(1-\delta_A)G_A(t,T)\\ \quad+\delta_A G_1(t,T)]+\Bigg[(1-\delta_A)\int_t^T p(t,s)\left[\lambda_0^R G_4(t,s)+\lambda_1^R G_5(t,s)+\beta_R G_6(t,s)\right]\mathrm{d}s+\delta_A\\ \int_t^T p(t,s)\left[\lambda_0^R G_1(t,s)+\lambda_1^R G_2(t,s)+\beta_R G_3(t,s)\right]\mathrm{d}s\Bigg]\Bigg)+\frac{L}{v^R(t_0,U)}\Bigg(\delta_R p(t,T)[(1-\delta_A)G_{10}(t,T,U).\\ \quad+\delta_A G_7(t,T,U)]+(1-\delta_R)p(t,U)\times[\delta_A G_1(t,U)+(1-a\delta_A)G_{13}(t,T,U)]\\ \quad+\delta_R\Bigg[\delta_A.\int_t^T p(t,s)\left[\lambda_0^R G_7(t,s,U)+\lambda_1^R G_8(t,s,U)+\beta_R G_9(t,s,U)\right]\mathrm{d}s+(1-\delta_A)\\ \int_t^T p(t,s)[\lambda_0^R G_{10}(t,s,U)+\lambda_1^R G_{11}(t,s,U)\\ +\beta_R G_{12}(t,s,U)]\mathrm{d}s\Bigg]\Bigg)\Bigg\} & \text{if } \tau^A>t,\tau^R>t\\ \delta_A \mathrm{TC}_{SPV}(t) & \text{if } \tau^A\le t,\tau^R>t\\ 0 & \text{otherwise}\end{cases}\tag{16}$$

where

$$G_{10}(t,T,U)=\exp\Bigg[-\lambda_0^H(T-t)-Y(t,U)(U-t)-(\lambda_1^H-1)Y(t,T)(T-t)+\frac{(T-t)^2}{4}\beta_H\sigma_I^2+\frac{(\lambda_1^H)^2-\lambda_1^H}{2}\sigma^2(t,T)+\frac{(T-t)^3}{6}\beta_H^2\sigma_I^2+\lambda_1^H\rho_1(t,T,U)+\beta_H\sigma_I\rho_2(t,T,U)+\lambda_1^H\beta_H\sigma_I\rho_3(t,T)\Bigg]$$

is a discount factor induced by spontaneous default intensity of reference obligation is leveraged total return CLN is issued by a protection buyer.

$$G_{11}(t,s,U)=G_{10}(t,s,U)\times\left[\mu_0(t,s)+\int_t^s b(y,U)\Phi(y,s)\mathrm{d}y-\frac{\lambda_1^H}{2}b(t,s)^2-\beta_H\sigma_I\rho_{rI}\int_t^s\Phi(y,s)\mathrm{d}y\right]$$

is a credit risk adjusted discount factor induced by the sensitivity of reference obligation to the level of spot rate if the leveraged total return CLN is issued by a protection buyer.

$G_{12}(t,T,U)$ *means the credit risk adjusted discount factor induced by the sensitivity of reference obligation to the excess return on market index if the leveraged total return CLN is issued by a protection buyer and is shown as follows*:

$$G_{12}(t,s,U)=G_{10}(t,s,U)\times\left[\frac{\sigma_I^2(s-t)}{2}[-1-\beta_H(s-t)]+\sigma_I\rho_{rI}\int_t^s b(y,s)\mathrm{d}y+\lambda_1^H\sigma_I\rho_{rI}\int_t^s b(y,s)\mathrm{d}y\right]$$

$$G_{13}(t,T,U)=\exp\Bigg[-\lambda_0^R(U-t)-\lambda_0^A(T-t)-\lambda_1^R Y(t,U)(U-t)-\lambda_1^A Y(t,T)(T-t)+\frac{(T-t)^2}{4}\beta_A\sigma_I^2+\frac{(U-t)^2}{4}\beta_R\sigma_I^2$$

$$+\frac{(\lambda_1^A)^2-\lambda_1^A}{2}\sigma^2(t,T)+\frac{(\lambda_1^R)^2+\lambda_1^R}{2}\sigma^2(t,U)$$
$$+\frac{(U-t)^3}{6}\beta_R^2\sigma_I^2+\frac{(T-t)^3}{6}\beta_A^2\sigma_I^2$$
$$+(1+\lambda_1^R)\lambda_1^A\rho_1(t,T,U)+(1+\lambda_1^R)$$
$$\beta_A\sigma_I\rho_2(t,T,U)+\lambda_1^A\beta_A\sigma_I\rho_3(t,T)$$
$$+(1+\lambda_1^R)\beta_R\sigma_I\rho_4(t,U)$$
$$+\lambda_1^A\beta_R\sigma_I\rho_5(t,T,U)+\beta_A\beta_R\sigma_I^2\rho_6(t,T,U)\Big]$$

is a discount factor induced by spontaneous default intensity of reference obligation if the leveraged total return CLN is issued by a protection buyer. Being different with $G_{10}(t,T,U)$, $G_{10}(t,T,U)$ reflects that the reference obligation still survive after T, yet $G_{13}(t,T,U)$ represents that reference obligation is still alive after U.

$$\rho_4(t,U)=-\rho_{rI}\int_t^U(U-y)b(y,U)\mathrm{d}y$$
$$\rho_5(t,T,U)=-\rho_{rI}\int_t^T(U-y)b(y,T)\mathrm{d}y,\ \ \rho_6(t,T,U)=\int_t^T(U-y)(T-y)\mathrm{d}y$$

The detailed proof for Theorem 5 is given by Appendix 5.

By virtue of equalities (15) and (16), if we assume that $\delta_A=\lambda_0^A=\lambda_1^A=\beta_A=0$, it can be seen that equality (15) is a special case of equality (16), i.e., the leveraged total return CLN is free of credit exposure for the protection buyer since the issuer is an SPV. Meanwhile, without loss of generality, if we assume that $U=T$, we obtain $G_{10}(t,T,U)=G_4(t,T)=G_{13}(t,T,U)$, $G_{11}(t,T,U)=G_5(t,T)$ and $G_{12}(t,T,U)=G_6(t,T)$. In addition, the relationship among the spot rate volatility, the market index volatility and $G_{10}(t,T,U)$ or $G_{13}(t,T,U)$ is similar to the case of $G_4(t,T)$. In addition, it is also clear that

$$\frac{\partial_p(t,T)G_{10}(t,T,U)}{\partial Y(t,T)}=\frac{\partial_p(t,T)G_4(t,T)}{\partial Y(t,T)}=\frac{\partial_p(t,T)G_{13}(t,T,U)}{\partial Y(t,T)}=-(1+\lambda_1^H)(T-t)G_4(t,T)<0$$

Hence, the higher yields of risk-free zero coupon bonds are also correlative with lower values of the leveraged total return CLNs no matter who the issuers are.

Fair fee charged by an SPV

In practice, the leveraged total return CLNs can also be issued either by the protection buyer or by an SPV, and it is important to determine the values or fair fees charged by SPVs with issuing the leveraged total CLNs. Given the pricing formulas of equalities (15) and (16), the fair fee that the SPV charges with issuing the leveraged total return CLN is equal to equality (15) minus equality (16), as presented in the following corollary.

Corollary 2: *The fair fee charged by an SPV with issuing the leveraged total return CLN, defined as $TCSPV_{\text{fee}}$, is given as follows*:

$$\mathrm{TCSPV}_{\text{fee}}(t)=\begin{cases}\displaystyle\int_t^T \mathrm{LC}_s p(t,s)(1-\delta_A)[G_1(t,s)-G_4(t,s)]\mathrm{d}s+M\Bigg\{(1-L)(1-\delta_A)\\ \displaystyle\times[G_1(t,T)-G_4(t,T)]+\lambda_0^R(1-\delta_A)\int_t^T p(t,s)[G_1(t,s)-G_4(t,s)]\mathrm{d}s\\ \displaystyle+\lambda_1^R(1-\delta_A)\int_t^T p(t,s)[G_2(t,s)-G_5(t,s)]\mathrm{d}s+\beta_R(1-\delta_A)\int_t^T p(t,s)[G_3(t,s)-G_6(t,s)]\mathrm{d}s\\ \displaystyle+\frac{L}{v^R(t_0,U)}\delta_R(1-\delta_A)p(t,T)[G_7(t,T,U)-G_{10}(t,T,U)]+(1-\delta_R)(1-\delta_A)p(t,U)\\ \displaystyle\times[G_1(t,U)-G_{13}(t,T,U)]+(1-\delta_A)\delta_R\lambda_0^R\int_t^T p(t,s)[G_7(t,s,U)-G_{10}(t,s,U)]\mathrm{d}s\\ \displaystyle+(1-\delta_A)\delta_R\lambda_1^R\int_t^T p(t,s)[G_8(t,s,U)-G_{11}(t,s,U)\mathrm{d}s]\\ \displaystyle+(1-\delta_A)\delta_R\beta_R\int_t^T p(t,s)[G_9(t,s,U)-G_{12}(t,s,U)\mathrm{d}s]\Bigg\} & \text{if } \tau^A>t,\ \ \tau^R>t\\ (1-\delta_A)\mathrm{TC}_{\mathrm{SPV}}(t) & \text{if } \tau^A\le t,\ \ \tau^R>t\\ 0 & \text{otherwise}\end{cases}$$

From Corollary 2, similar to the case for the CLNs, if $G_7(t,s,U) > G_{10}(t,s,U)$, $G_1(t,U) > G_{13}(t,s,U)$, $G_8(t,s,U) > G_{11}(t,s,U)$ and $G_9(t,s,U) > G_{10}(t,s,U)$, we can see that the fee charged by an SPV is always greater than zero. Since $G_1(t,U)$, $G_7(t,s,U)$, $G_8(t,s,U)$ and $G_9(t,s,U)$ are the values that consider only the default event of reference obligation, but $G_{10}(t,s,U)$, $G_{13}(t,s,U)$, $G_{11}(t,s,U)$ and $G_{12}(t,s,U)$ are the values that consider both the default events of reference obligation and protection buyer, it is reasonable that the fair fee charged by an SPV for issuing the leveraged total return CLNs is positive.

V. Numerical Analyses of Structured CLNs

In this section, we investigate the properties of the CLNs and the leveraged total return CLNs, which are correspondingly issued by an SPV or the protection buyer. Besides, we also demonstrate the appropriate fee that a protection buyer should pay to an SPV for issuing the CLNs or the leveraged total return CLNs, and then show the appropriate timing for the CLNs and the leveraged total return CLNs issued through an SPV. Finally, we examine the properties of their required yields (or credit spreads).

CLNs

We assume that the principal amount is AUD 40 000 000 and the maturity date is 2 years. The coupon rate is 5%. If a credit event occurs, no further interest will be paid at the coupon payment date. At the maturity, the noteholders receive AUD 40 000 000 unless a credit event occurs, in which case they receive the amount equal to the recovery rate δ_R multiplied by the principal. We assume that $\alpha = 0.0254$, $\delta_R = 0.3$, $\delta_A = 0.4$, $\lambda_1^A = \lambda_1^R = -0.01$, $\beta_A = \beta_R - 0.05$ and the initial term structure is flat and satisfies $p(t,T) = \exp[-0.05 \times (T-t)]$.

Characteristics of the CLNs

We report some numerical values of the CLN by varying the different levels of spontaneous default intensities, spot rate volatility, market index volatility and correlation coefficient of spot rate and market index in Exhibit 1. From Table 1, the numerical results demonstrate that declining the spontaneous default intensity of reference entity λ_0^R and ρ_{rI} may rise up the values of the CLN, which is issued by an SPV. Since the higher level of spontaneous default intensity of reference entity is associated with wider credit spread as well as higher default probabilities, which make intuitive sense, it results in lower values of CLNs. It is noteworthy that when $\rho_{rI} > 0$, the values of the CLN are decreasing functions of σ_r and σ_I. Consequently, higher spot rate volatility and market index volatility widen the credit spread and increase the default probability. This result is consistent with the empirical results, such as Kao (2000), Campbell and Taksler (2003), Huang and Kong (2003), etc. Nevertheless, if $\rho_{rI} < 0$, we can see that the values of the CLN are not definitely decreasing functions of σ_r and σ_I. The numerical results indicate that ρ_{rI} is an important factor that determines the relationship among the values of CLNs, interest volatility and market index volatility.

Table 1. The values of the CLN issued by an SPV

			λ_0^R	
ρ_{rI}	σ_I	σ_r	0.01	0.03
−0.5	0.2	0.02	98.5024	96.6495
		0.05	98.5101	96.657
	0.5	0.02	98.7328	96.9003
		0.05	98.7729	96.9226
0	0.2	0.02	98.7328	96.8825
		0.05	98.7317	96.8819
	0.5	0.02	98.4955	96.6422
		0.05	98.4932	96.6403
0.5	0.2	0.02	98.716	96.8648
		0.05	98.6908	96.8414
	0.5	0.02	98.4886	96.635
		0.05	98.4763	96.6236

Notes: This table reports the price of the CLN as functions of spontaneous default intensity, interest volatility, market index volatility and correlation coefficient between interest rate and return on market index. The numerical results show that the value of the CLN, which issued by an SPV, is a decreasing function of λ_0^R and ρ_{rI}. In addition, the values of the CLN is a decreasing function of the volatility σ_I and the volatility σ_r when $\rho_{rI} > 0$. However, if $\rho_{rI} < 0$, we find that the relationship among them may be negative and is different from the empirical results.

In Table 2, when the issuer is the protection buyer, the properties are similar to the ones issued by an SPV, as mentioned above. In addition, since higher levels of λ_0^A increase the default probability of the protection buyer, it is conceivable that the values of the CLNs are also negatively correlated with the spontaneous default intensity of the protection buyer.

Appropriate fees charged by an SPV for issuing CLNs

When the issuer is an SPV, the proceeds from the noteholders are used to buy high-quality collaterals held by the SPV, and hence the CLNs are free of the credit exposure of the protection buyer. As a result,

Table 2. The values of the CLN issued by the protection buyer

			$(\lambda_0^R, \lambda_0^A)$			
ρ_{rI}	σ_I	σ_r	(0.03, 0.03)	(0.03, 0.05)	(0.05, 0.03)	(0.05, 0.05)
−0.5	0.2	0.02	96.0811	95.5284	94.2257	91.7464
		0.05	96.1370	95.5890	94.2809	91.7993
	0.5	0.02	96.3029	95.6249	92.7029	90.2948
		0.05	95.6483	94.1643	92.8435	90.4297
0	0.2	0.02	96.0392	95.4892	94.1857	91.7081
		0.05	96.0350	95.4851	94.1817	91.7043
	0.5	0.02	94.4029	93.9291	92.5049	90.2009
		0.05	94.3981	93.9245	92.5006	90.1967
0.5	0.2	0.02	95.9984	95.4501	94.1257	91.6697
		0.05	95.9330	95.3874	94.0827	91.6093
	0.5	0.02	94.3031	93.8334	92.4071	90.1071
		0.05	94.1487	93.6854	92.2583	89.9644

Notes: This table reports the values of the CLN, issued by the protection buyer, as functions of spontaneous default intensities of reference obligation and protection buyer, spot rate volatility, market index volatility and ρ_{rI}. The properties are similar to the ones issued by an SPV. Moreover, the results also show that values of the CLN are dereasing function of the spontaneous default intensity of protection buyer.

Table 3. Fair fees charged by an SPV with issuing the CLN

			$(\lambda_0^R, \lambda_0^A)$			
ρ_{rI}	σ_I	σ_r	(0.01, 0.01)	(0.01, 0.03)	(0.03, 0.01)	(0.03, 0.03)
−0.5	0.2	0.02	2.4213	2.974	2.4238	4.9031
		0.05	2.3731	2.9211	2.3761	4.8577
	0.5	0.02	2.4299	3.1079	4.1974	6.6055
		0.05	3.1246	4.6086	4.0791	6.4929
0	0.2	0.02	2.6936	3.2436	2.6968	5.1744
		0.05	2.6967	3.2466	2.7002	5.1776
	0.5	0.02	4.0926	4.5664	4.1373	6.4413
		0.05	4.0951	4.5687	4.1397	6.4436
0.5	0.2	0.02	2.7176	3.2659	2.7391	5.1951
		0.05	2.7578	3.3034	2.7587	5.2321
	0.5	0.02	4.1855	4.6552	4.2279	6.5279
		0.05	4.3276	4.7909	4.3653	6.6592

Notes: This table reports the suitable fee for an SPV by the differences in values between an SPV and the protection. We find that the fair fee is as a function of spontaneous default intensity of reference obligation and issuer, interest volatility, equity index return volatility and correlation between interest rate level and a market index. The numerical results show that the fair fees charged by an SPV is increasing functions of σ_I, λ_0^R and λ_0^A.

from Table 3, it is obvious that the price differences between the values of CLNs issued by an SPV and the protection buyer are definitely positive. These results explain why investors prefer the isolated and uniquely identifiable nature of an SPV to a more diffusely defined corporate form such as the protection buyer.

In addition, the fair fees charged by an SPV are also increasing functions of λ_0^R, λ_0^A and σ_I, then in some circumstances such as inferior credit qualities of the reference obligation and protection buyer or higher uncertainty in the equity return, the higher fee should be paid by the protection buyer to an SPV. An immediate implication is that it is better to issue the CLNs through an SPV under the circumstances that the credit qualities of the protection buyer become worse or the return on market index changes dramatically.

Credit spreads of CLNs issued by the protection buyer

We define the required yield of a CLN as the coupon rate such that the price of a CLN is at par. From Table 4, as expected, we can see that the required yields of the CLNs are increasing functions of λ_0^R and λ_0^A. Meanwhile, with the positive ρ_{rI}, the increment of σ_r and σ_I may urge the required yields of the CLNs to go up. In addition, from the pricing function of default-free zero coupon bonds as defined in equality (1), the yield to maturity is equal to 5%. Hence, the credit spreads of the CLN are equal to the required yields of the CLN minus 5%. It is apparent that the credit spreads of the CLN are also positively correlated with λ_0^R and λ_0^A.

When ρ_{rI} is nonnegative, the credit spreads of the CLN are positively correlated with σ_r and σ_I and are consistent with the empirical evidence. When ρ_{rI} is negative, however, the credit spreads may be negatively related with the volatilities of spot rate and market index. These results demonstrate that the correlation coefficient of spot rate and market index plays a crucial role in determining the relationship between them.

The leveraged total return CLNs

For leverage total return CLNs, we assume that the principal amount is USD five million and the maturity date is 3 years later. The actual coupon rate equals leverage factor multiplied by 6%, where the leverage factor is set as five. The coupon payment ease immediately if a credit event occurs. At the termination date, the investors can receive par plus capital price adjustment. Capital price adjustment equals leverage factor times USD 5 million times the change in the price of the reference obligation. For simplicity, we assume that $\alpha = 0.0254$, $\delta_R = 0.3$, $\delta_A = 0.4$, $\lambda_1^A = \lambda_1^R = -0.01, \beta_A = \beta_R = -0.05$ and the initial term structure is flat and satisfies $p(t, T) = \exp[-0.06 \times (T - t)]$.

Characteristics of the leveraged total return CLN

We provide some numerical values of the leveraged total return CLN by varying the different levels of spontaneous default intensity, interest rate volatility, equity index return volatility, correlation coefficient of spot rate and market index, leverage factor and the basis risk measure (the difference in maturity between note and reference obligation) in Table 5.

From Table 5, we discover that values of the leveraged total return CLN, which is issued by an SPV, are decreasing functions of λ_0^R and the basis risk measure$(U - T)$, but have positive relationship with leverage factor L. Because that the higher levels of spontaneous default intensity of reference obligation as well as difference in maturity of the note and the reference obligation respectively result in higher credit risk and basis risk, the values of the leveraged total returns CLN are lower. Meanwhile, the higher leverage factor means higher payoff of the reference obligation, and hence the values of the leveraged total return CLN are higher.

Table 4. The required yields of the CLN

			$(\lambda_0^R, \lambda_0^A)$			
ρ_{rI}	σ_I	σ_r	(0.01, 0.01)	(0.011, 0.03)	(0.03, 0.01)	(0.03, 0.03)
−0.5	0.2	0.02	6.9940%	7.7634%	8.9189%	9.2323%
		0.05	6.9834%	7.7531%	8.9079%	9.2256%
	0.5	0.02	6.9635%	7.7318%	9.1374%	9.3713%
		0.05	7.7432%	8.8993%	9.1328%	9.3698%
0	0.2	0.02	6.9953%	7.7856%	8.9403%	9.2397%
		0.05	6.9956%	7.7895%	8.9411%	9.2407%
	0.5	0.02	8.8245%	9.0623%	9.1379%	9.3803%
		0.05	8.8441%	9.0645%	9.1398%	9.3915%
0.5	0.2	0.02	6.9865%	8.0295%	8.9192%	9.2414%
		0.05	7.0221%	8.1184%	9.0196%	9.2491%
	0.5	0.02	8.8713%	9.1053%	9.1487%	9.3957%
		0.05	8.9057%	9.1187%	9.1599%	9.4026%

Notes: This table reports the characteristics of required yields of the CLN. We assume that the initial term structure is flat and the yield to maturity is 5%, the credit spreads are equal to required yields of the CLN minus 5%. The numerical results show that the required yields (or credit spreads) of the CLNs are increasing functions of λ_0^R and λ_0^A. They are also increasing functions of σ_r and σ_I under the nonnegative ρ_{rI}. However, if $\rho_{rI} < 0$, some of them are negatively related with σ_r and σ_I and difference from the empirical results.

Table 5. The values of the leveraged total return CLN issued by an SPV

ρ_{rI}	L	Basis risk measure $(U-T)$	σ_I	σ_r	λ_0^R 0.01	0.03
−0.5	1	0	0.2	0.02	98.213	94.836
				0.05	98.351	94.869
			0.5	0.02	98.325	94.921
				0.05	98.768	91.547
		10	0.2	0.02	82.293	73.772
				0.05	78.612	70.722
			0.5	0.02	75.257	67.066
				0.05	75.763	63.738
	5	0	0.2	0.02	147.225	126.988
				0.05	147.301	126.974
			0.5	0.02	141.812	122.23
				0.05	141.962	122.202
		10	0.2	0.02	85.359	84.381
				0.05	81.223	76.253
			0.5	0.02	78.171	72.508
				0.05	76.121	67.273
0	1	0	0.2	0.02	98.154	94.778
				0.05	98.036	94.77
			0.5	0.02	98.027	94.687
				0.05	98.001	91.486
		10	0.2	0.02	82.269	73.68
				0.05	78.496	70.465
			0.5	0.02	75.134	67.053
				0.05	75.089	63.697
	5	0	0.2	0.02	147.202	126.973
				0.05	147.185	126.957
			0.5	0.02	141.752	122.215
				0.05	141.741	122.141
		10	0.2	0.02	85.341	84.329
				0.05	81.176	76.179
			0.5	0.02	78.12	72.49
				0.05	76.046	67.259
0.5	1	0	0.2	0.02	98.035	94.659
				0.05	97.917	94.651
			0.5	0.02	98.004	94.528
				0.05	97.884	91.365
		10	0.2	0.02	82.150	73.561
				0.05	78.377	70.346
			0.5	0.02	74.977	66.934
				0.05	74.865	63.577
	5	0	0.2	0.02	147.091	126.854
				0.05	147.083	126.838
			0.5	0.02	141.633	122.096
				0.05	141.630	122.022
		10	0.2	0.02	85.222	84.205
				0.05	81.057	76.060
			0.5	0.02	78.001	72.331
				0.05	75.924	67.140

Notes: This table reports the values of the leveraged total return CLN as functions of spontaneous default intensity of reference obligation, spot rate volatility, market index volatility, correlation coefficient of spot rate and market index, leverage factor, and the basis risk measure $(U-T)$. The numerical results show the values of the leveraged total return CLN issued by an SPV is a increasing function of leverage factor and decreasing functions of λ_0^R and basis risk measure. When $\rho_{rI}>0$, the values of the leveraged total return CLN are decreasing functions of σ_I and σ_r. However, similar to CLN, when $\rho_{rI}<0$, we also find that the values of the leveraged total return CLN may be decreasing functions of σ_I and σ_r.

When $\rho_{rI} > 0$, upward σ_I and σ_r will sink the values of the leveraged total return CLN. The outcomes are consistent with the prior empirical results. However, similar to the case for CLN, we also find that the values of the leveraged total return CLN may be decreasing functions of σ_I and σ_r, if $\rho_{rI} < 0$, this experience reflects that ρ_{rI} is also an important factor for the relationship among the values of leveraged total return CLNs, interest volatility and volatility of return on market index.

From Table 6, as expected, it can be seen that the properties of values of the leveraged total return CLN issued by the protection buyer is the same as the ones issued by an SPV. In addition, increment in λ_0^A is relative with the declining values of note. This is because that the higher levels of λ_0^A is associated with wider credit spread and higher default probabilities, it is reasonable that the higher spontaneous default intensity of protection buyer results in lower values of the leveraged total return CLNs.

Appropriate fees charged by an SPV for issuing the leveraged total return CLNs

From Table 7, we find that the value of the leveraged total return CLN issued by an SPV is always larger than the one issued by the protection buyer. The results are consistent with the market practice and explain why in practice the leveraged total return CLNs are usually issued by an SPV. Importantly, it also shows that the fee charged by the SPV increases with the growing λ_0^R, λ_0^A, σ_r and the leverage factor. Hence, it is better to issue the leveraged total return CLNs through an SPV under the circumstances that the credit qualities of the reference obligation and protection buyer become worse, the short-term interest rate changes dramatically or the leverage effect is higher.

In addition, it is obvious that the relationship between the fair fees and the basis risk measure $(U - T)$ does not have a constant tendency and this implies that the main purpose of the SPV is to hedge the credit risk induced by the protection buyer but not to hedge the basis risk.

Credit spreads of the leveraged total return CLNs issued by the protection buyer

From Table 8, the numerical results show that the required yields of the leveraged total return CLN are increasing functions of spontaneous default intensities of reference obligation and protection buyer and the basis risk measure. Likewise, the difference between the required yield and 6% imply the credit spreads of the leveraged total return CLN. Hence, the higher levels of spontaneous default intensities and the basis risk measure result in higher credit risk and basis risk, the required yields or credit spreads of the leveraged total returns CLN are lower. Similar to the case for CLN, we also find that the credit spreads of the leveraged total return CLN diminish progressively as σ_I and σ_r raise up, if $\rho_{rI} < 0$, hence this reflects that ρ_{rI} is also an important factor for the relationship among the credit spreads of leveraged total return CLNs, interest volatility and market index volatility.

VI. Summary and Conclusions

In this article, with the intersection of market and credit risk, we first derive the analytic formulas of the CLN and the leveraged total return CLN, which are issued by an SPV and protection buyer, respectively. The suitable fee that the protection buyer pays to an SPV with issuing the financial products such as the structured CLNs is also provided, which is an aspect that has not been discussed in the literature.

From the numerical analyses of structured CLNs, we find that the suitable fee charged by an SPV is an increasing function of spontaneous default intensity of the protection buyer, thus it is better to issue the leveraged total return CLN through an SPV under the circumstance that the credit qualities of the reference obligation and protection buyer become worse. Meanwhile, for the leveraged total return CLNs, the fees are not definitely increasing function of the basis risk measure. As a result, we conclude that the purpose of the SPVs is to hedge the credit risk of the protection buyer but not the basis risk.

It is noteworthy that the correlation coefficient of spot rate and market index plays an important role in determining the relationship among the interest rate volatility, market index volatility and the credit spreads of the structured CLNs. If ρ_{rI} is nonnegative, similar to the empirical results such as Das and Tufano (1996), Duffee (1999), Kao (2000) and Huang and Kong (2003), upward the volatilities of spot rate and return on market index may prompt the accession of credit spreads. Nevertheless, the credit spreads may be negatively related with volatilities of spot rate and return on market index under the case for negative ρ_{rI}. Consequently, our suggestion is that the empirical regression models for studying the characteristics of credit spreads should incorporate the correlation coefficient of spot rate and market index into the regression models as a control variable. This model can be extended to price the credit derivatives issued by an SPV and the protection

Table 6. The values of the leveraged total return CLN issued by the protection buyer

					$(\lambda_0^R, \lambda_0^A)$			
ρ_{rI}	L	Basis risk measure $(U-T)$	σ_I	σ_R	(0.01, 0.01)	(0.01, 0.03)	(0.03, 0.01)	(0.03, 0.03)
−0.5	1	0	0.2	0.02	97.7225	94.7915	93.1315	90.1885
				0.05	97.7255	94.7945	93.1345	90.1935
			0.5	0.02	97.8275	94.8975	93.5385	90.8975
				0.05	97.7455	94.902	88.3395	85.4735
		10	0.2	0.02	81.1875	78.8585	72.2165	70.1035
				0.05	76.7605	74.5805	68.8005	66.4195
			0.5	0.02	73.6295	71.3475	65.3825	62.5785
				0.05	73.7025	71.4725	61.665	58.9865
	5	0	0.2	0.02	127.4085	121.1495	100.2395	93.5825
				0.05	127.4175	119.1375	99.5595	91.8955
			0.5	0.02	124.2025	117.7535	98.2665	92.3865
				0.05	124.3015	115.6505	96.5465	91.6585
		10	0.2	0.02	82.5485	81.7235	81.1315	76.0105
				0.05	78.3695	75.5485	71.5635	66.8495
			0.5	0.02	75.9485	74.2145	69.1835	65.7505
				0.05	74.6255	71.9035	62.9785	60.4565
0	1	0	0.2	0.02	97.6380	94.7160	93.0360	90.0450
				0.05	97.3810	94.5960	93.0210	90.0310
			0.5	0.02	97.4420	94.7052	93.0270	90.0270
				0.05	97.3550	94.5410	88.1940	85.3280
		10	0.2	0.02	81.1760	78.6360	72.0690	69.9110
				0.05	76.6670	74.4290	68.6320	66.2680
			0.5	0.02	73.5670	71.1502	65.3110	62.4030
				0.05	73.4030	71.0240	61.5480	58.7320
	5	0	0.2	0.02	127.4050	121.1350	100.1160	93.3400
				0.05	127.3230	119.1360	99.4180	91.6510
			0.5	0.02	124.1430	117.5890	98.1056	92.1440
				0.05	124.1290	115.4752	96.5080	91.5110
		10	0.2	0.02	82.5350	81.6650	81.1140	75.9972
				0.05	78.3680	75.4480	71.3460	66.7250
			0.5	0.02	75.6350	74.1940	69.0023	65.6240
				0.05	73.5584	71.6270	62.7940	60.3670
0.5	1	0	0.2	0.02	97.4989	94.5783	92.8979	89.9070
				0.05	97.2429	94.4579	92.8829	89.8931
			0.5	0.02	97.3039	94.5670	92.8889	89.8899
				0.05	97.2169	94.4429	88.0559	85.1902
		10	0.2	0.02	81.0379	78.4979	71.9279	69.7729
				0.05	76.5289	74.2909	68.4931	66.1299
			0.5	0.02	73.4249	71.0121	65.1729	62.2649
				0.05	73.2649	70.8859	61.4099	58.5939
	5	0	0.2	0.02	127.2799	121.0149	99.9779	93.2019
				0.05	127.1849	118.9982	99.2799	91.5129
			0.5	0.02	124.0049	117.4509	97.9670	92.0059
				0.05	123.9989	115.3371	96.3699	91.3729
		10	0.2	0.02	82.3969	81.5269	80.9761	75.8591
				0.05	78.2299	75.3099	71.2079	66.5869
			0.5	0.02	75.4969	74.0559	68.8252	65.4859
				0.05	73.4171	71.4866	62.6559	60.2261

Notes: This table reports the characteristics of the values of the leveraged total return CLN which is issued by the protection buyer. The numerical results show that the properties of values of the leveraged total return CLN issued by the protection buyer is the same as the ones issued by an SPV. In addition, the value of note is a decreasing function of spontaneous default intensity of issuer λ_0^A.

Table 7. Fair fees charged by an SPV with issuing the leveraged total return CLN

ρ_{rI}	L	Basis risk measure $(U-T)$	σ_I	σ_r	$(\lambda_0^R, \lambda_0^A)$ (0.01, 0.01)	(0.01, 0.03)	(0.03, 0.01)	(0.03, 0.03)
−0.5	1	0	0.2	0.02	0.4905	3.4215	1.7045	4.6475
				0.05	0.6255	3.5565	1.7345	4.6755
			0.5	0.02	0.4975	3.4275	1.3825	4.0235
				0.05	1.0225	3.866	3.2075	6.0735
		10	0.2	0.02	1.1055	3.4345	1.5555	3.6685
				0.05	1.8515	4.0315	1.9215	4.3025
			0.5	0.02	1.6275	3.9095	1.6835	4.4875
				0.05	2.0605	4.2905	2.073	4.7515
	5	0	0.2	0.02	19.8165	26.0755	26.7485	33.4055
				0.05	19.8835	28.1635	27.4145	35.0785
			0.5	0.02	17.6095	24.0585	23.9635	29.8435
				0.05	17.6605	26.3115	25.6555	30.5435
		10	0.2	0.02	2.8105	3.6355	3.2495	8.3705
				0.05	2.8535	5.6745	4.6895	9.4035
			0.5	0.02	2.2225	3.9565	3.3245	6.7575
				0.05	3.4955	5.2175	4.2945	6.8165
0	1	0	0.2	0.02	0.5160	3.4380	1.7420	4.7330
				0.05	0.6550	3.4400	1.7490	4.7390
			0.5	0.02	0.5850	3.3218	1.6600	4.6600
				0.05	0.6460	3.4600	3.2920	6.1580
		10	0.2	0.02	1.0930	3.6330	1.6110	3.7690
				0.05	1.8290	4.0670	1.8330	4.1970
			0.5	0.02	1.5670	3.9838	1.7420	4.6500
				0.05	1.6860	4.0650	2.1490	4.9650
	5	0	0.2	0.02	19.7970	26.0670	26.8570	33.6330
				0.05	19.8620	28.0490	27.5390	35.3060
			0.5	0.02	17.6090	24.1630	24.1094	30.0710
				0.05	17.6120	26.2658	25.6330	30.6300
		10	0.2	0.02	2.8060	3.6760	3.2150	8.3318
				0.05	2.8080	5.7280	4.8330	9.4540
			0.5	0.02	2.4850	3.9260	3.4877	6.8660
				0.05	2.4876	4.4190	4.4650	6.8920
0.5	1	0	0.2	0.02	0.5361	3.4567	1.7611	4.7520
				0.05	0.6741	3.4591	1.7681	4.7579
			0.5	0.02	0.7011	3.3170	1.6351	4.6341
				0.05	0.7871	3.4411	3.3091	6.1748
		10	0.2	0.02	1.1121	3.6521	1.6331	3.7881
				0.05	1.8481	4.0861	1.8529	4.2161
			0.5	0.02	1.5521	3.9649	1.7611	4.6691
				0.05	1.6001	3.9791	2.1671	4.9831
	5	0	0.2	0.02	19.8111	26.0761	26.8761	33.6521
				0.05	19.8981	28.0848	27.5581	35.3251
			0.5	0.02	17.6281	24.1821	24.1290	30.0901
				0.05	17.6315	26.2933	25.6521	30.6491
		10	0.2	0.02	2.8251	3.6951	3.2289	8.3459
				0.05	2.8271	5.7471	4.8521	9.4731
			0.5	0.02	2.5041	3.9451	3.5058	6.8451
				0.05	2.5069	4.4374	4.4841	6.9139

Notes: This table reports the fees charged by an SPV as functions of spontaneous default intensities of reference obligation and protection buyer, interest rate volatility, market index volatility, correlation coefficient of spot rate level and market index, leverage factor and the basis risk measure. We find that the fees are large to zero and hence it explains why the leveraged total return CLNs is always issued through an SPV. The numerical results also show that the fee is increasing function of $\lambda_0^R, \lambda_0^A, \sigma_r$ and leverage factor. In addition, the relationship between the fee and basis risk measure does not have a constant tendency. It implies that the main purpose of the SPV is to hedge the credit risk induced by the protection buyer but not the basis risk.

Table 8. The required yields leveraged total return CLNs

ρ_{rI}	L	Basis risk measure $(U-T)$	σ_I	σ_r	$(\lambda_0^R, \lambda_0^A)$ (0.01, 0.01)	(0.01, 0.03)	(0.03, 0.01)	(0.03, 0.03)
−0.5	1	0	0.2	0.02	7.5521%	8.9971%	9.1431%	10.1401%
				0.05	7.5520%	8.9970%	9.1424%	10.1400%
			0.5	0.02	7.5515%	8.9969%	9.1419%	10.1399%
				0.05	7.5519%	8.9968%	10.4090%	11.7990%
		10	0.2	0.02	13.9450%	14.9832%	19.4890%	20.6510%
				0.05	15.6430%	17.0950%	21.5120%	23.3650%
			0.5	0.02	18.5980%	20.2507%	23.8593%	25.6430%
				0.05	18.5550%	20.5458%	25.8099%	25.9890%
	5	0	0.2	0.02	3.4210%	4.4580%	5.9820%	8.0410%
				0.05	3.4150%	4.7670%	6.2680%	9.8850%
			0.5	0.02	4.4060%	4.9490%	6.8780%	9.7270%
				0.05	4.4020%	5.0310%	7.9530%	9.8863%
		10	0.2	0.02	13.6510%	13.9370%	13.9451%	15.6834%
				0.05	14.9880%	16.1570%	20.0457%	23.3520%
			0.5	0.02	15.9861%	17.2256%	21.1550%	24.2260%
				0.05	17.1134%	20.0312%	25.7950%	25.8267%
0	1	0	0.2	0.02	7.5591%	9.0041%	9.1651%	10.1721%
				0.05	7.5597%	9.0053%	9.1667%	10.1731%
			0.5	0.02	7.5595%	9.0042%	9.1664%	10.1739%
				0.05	7.5599%	9.0061%	10.4110%	11.8000%
		10	0.2	0.02	13.9470%	14.9852%	19.4910%	20.6530%
				0.05	15.6450%	17.0960%	21.5140%	23.3670%
			0.5	0.02	18.6001%	20.2528%	23.8613%	25.6450%
				0.05	18.6370%	20.5878%	25.8120%	25.9910%
	5	0	0.2	0.02	3.4234%	4.4586%	5.9890%	8.0430%
				0.05	3.4370%	4.7873%	6.2700%	9.8870%
			0.5	0.02	4.6580%	4.9494%	6.8821%	9.7290%
				0.05	4.4085%	5.0316%	7.9570%	9.8883%
		10	0.2	0.02	13.6530%	13.9490%	13.9791%	15.6954%
				0.05	14.9993%	16.1630%	20.0477%	23.3540%
			0.5	0.02	15.9902%	17.2311%	21.1670%	24.2980%
				0.05	18.5152%	20.0410%	25.7980%	25.8291%
0.5	1	0	0.2	0.02	7.6401%	9.0841%	9.2251%	10.2221%
				0.05	7.6340%	9.0850%	9.2244%	10.2220%
			0.5	0.02	7.6355%	9.1246%	9.2669%	10.2619%
				0.05	7.6340%	9.1189%	10.5320%	11.9100%
		10	0.2	0.02	14.0270%	15.1072%	19.6110%	20.7730%
				0.05	15.7652%	17.2160%	21.6340%	23.4870%
			0.5	0.02	18.6501%	20.3138%	23.9813%	25.7450%
				0.05	18.6770%	20.6678%	25.9320%	26.1110%
	5	0	0.2	0.02	3.5434%	4.4590%	6.1090%	8.1630%
				0.05	3.5370%	4.7678%	6.3905%	10.0070%
			0.5	0.02	4.5180%	4.9498%	7.0023%	9.8490%
				0.05	4.4743%	5.0319%	8.0770%	9.9983%
		10	0.2	0.02	13.7230%	13.9990%	14.0541%	15.8064%
				0.05	15.0603%	16.2370%	20.1677%	23.4740%
			0.5	0.02	15.9987%	17.2476%	21.2870%	24.3181%
				0.05	18.6561%	20.0466%	25.7990%	25.8316%

Notes: This table reports the characteristics of required yields of the leverage total return CLN. We assume that the initial term structure is flat and the yield to maturity is 6%, the credit spreads are equal to required yields of the CLN minus 6%. The numerical results show that the required yields (or credit spreads) are decreasing function of leverage factor and increasing functions of spontaneous default intensities of reference obligation and protection buyer and basis risk measure. The relationship among spot rate volatility, market index volatility and the credit spreads is dependent on the sign of ρ_{rI}.

buyer itself. It is useful to determine the value of an SPV for issuing the credit derivatives.

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Appendix 1

Proof of Theorem 1: By using the law of the iterated conditional expectations and the fact that the default times are conditional independent with respect to $F^r_{T^*} \vee F^I_{T^*}$, we have

$$\begin{aligned} v^i(t,T) &= \mathrm{E}\left[\frac{B(t)}{B(T)}[\delta_i 1_{\{\tau^i \le T\}} + 1_{\{\tau^i > T\}}]\middle| F_t\right] \\ &= \mathrm{E}\left[\frac{B(t)}{B(T)}[\delta_i + (1-\delta_i)1_{\{\tau^i > T\}}]\middle| F_t\right] \\ &= \delta_i p(t,T) + (1-\delta_i)\mathrm{E} \\ &\quad \times \left[\frac{B(t)}{B(T)} P\left(\tau^i > T \middle| F^r_{T^*} \vee F^I_{T^*} \vee H^i_t\right) \middle| F_t\right] \end{aligned} \tag{A1}$$

Due to the fact that the conditional expectation is clearly 0 on the set $\{\tau^i \le t\}$, we obtain

$$\begin{aligned} &P\left(\tau^i > T \middle| F^r_{T^*} \vee F^I_{T^*} \vee H^i_t\right) \\ &\quad = 1_{\{\tau^i > t\}} P\left(\tau^i > T \middle| F^r_{T^*} \vee F^I_{T^*} \vee H^i_t\right) \\ &\quad = 1_{\{\tau^i > t\}} \frac{P\left(\tau^i > T \middle| F^r_{T^*} \vee F^I_{T^*}\right)}{P\left(\tau^i > t \middle| F^r_{T^*} \vee F^I_{T^*}\right)} \\ &\quad = 1_{\{\tau^i > t\}} \frac{\exp\left(-\int_0^T \lambda^i(u)\mathrm{d}u\right)}{\exp\left(-\int_0^t \lambda^i(u)\mathrm{d}u\right)} \\ &\quad = 1_{\{\tau^i > t\}} \exp\left(-\int_t^T \lambda^i(u)\mathrm{d}u\right) \end{aligned} \tag{A2}$$

Therefore, we have

$$v^i(t,T) = \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)E\left[\exp\left(-\int_t^T [r(u)+\lambda^i(u)]\mathrm{d}u\right)\Big|F_t\right]$$

Substitution of the linear intensity $\lambda^i(u) = \lambda_0^i + \lambda_1^i r(u) + \beta_i \log[I(u)/B(u)]$ into (A1), we obtain

$$\begin{aligned}v^i(t,T) &= \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)E\left\{\exp\left(-\left[\int_t^T \left(r(u) + \lambda_0^i + \lambda_1^i r(u) + \beta_i \log\left[\frac{I(u)}{B(u)}\right]\right)\mathrm{d}u\right]\right)\Big|F_t\right\}\\ &= \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)\exp[-\lambda_0^i(T-t)]E\\ &\quad\times\left\{\exp\left(-\left[\int_t^T \left((1+\lambda_1^i)r(u) + \beta_i\log\left[\frac{I(u)}{B(u)}\right]\right)\mathrm{d}u\right]\right)\Big|F_t\right\}\end{aligned}$$

Let $X_1 \equiv -\int_t^T (1+\lambda_1^i)r(u)\mathrm{d}u$ and $X_2 \equiv -\int_t^T \beta_i \log[I(u)/B(u)]\mathrm{d}u$. In view of (4), we have

$$\begin{aligned}X_2 &= -\int_t^T \beta_i \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u\\ &= -\int_t^T \beta_i\left(\log\left[\frac{I(t)}{B(t)}\right] - \frac{1}{2}\sigma_I^2(u-t) + \sigma_I\int_t^u \mathrm{d}W_v^I\right)\mathrm{d}u\\ &= -\beta_i\log\left[\frac{I(t)}{B(t)}\right](T-t) + \frac{1}{2}\beta_i\sigma_I^2\int_t^T (u-t)\mathrm{d}u - \beta_i\sigma_I\int_t^T\int_t^u \mathrm{d}W_v^I\mathrm{d}u\\ &= -\beta_i\log\left[\frac{I(t)}{B(t)}\right](T-t) + \frac{1}{2}\beta_i\sigma_I^2\int_t^T (u-t)\mathrm{d}u - \beta_i\sigma_I\int_t^T\int_v^T \mathrm{d}u\mathrm{d}W_v^I\\ &= -\beta_i\log\left[\frac{I(t)}{B(t)}\right](T-t) + \frac{1}{2}\beta_i\sigma_I^2\int_t^T (u-t)\mathrm{d}u - \beta_i\sigma_I\int_t^T (T-v)\mathrm{d}W_v^I\end{aligned}$$

Without loss of generality, we assume that $I(t) = B(t) = 1$, we obtain

$$X_2 = \frac{(T-t)^2}{4}\beta_i\sigma_I^2 - \beta_i\sigma_I\int_t^T (T-v)\mathrm{d}W_v^I$$

Furthermore, by virtue of (3), the process of spot rate satisfies:[3]

$$\begin{aligned}\int_t^T r(u)\mathrm{d}u &= -\ln p(t,T) + 0.5\int_t^T b(u,T)^2\mathrm{d}u - \int_t^T b(u,T)\mathrm{d}W_u^r\\ \mu(t,T) \equiv E_t\left[\int_t^T r(u)\mathrm{d}u\right] &= -\ln p(t,T) + 0.5\int_t^s b(u,s)^2\mathrm{d}u = -\ln p(t,T) + \frac{1}{2}\sigma^2(t,T)\\ \sigma^2(t,T) \equiv V_t\left[\int_t^T r(u)\mathrm{d}u\right] &= \int_t^T b(u,T)^2\mathrm{d}u\end{aligned} \quad \text{(A3)}$$

where $E_t(\cdot)$ and $V_t(\cdot)$ are respectively the conditional expectation and variance with respect to F_t, respectively. Hence, we have

$$v^i(t,T) = \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)\exp[-\lambda_0^i(T-t)]E_t[\exp(X_1+X_2)]$$

[3] Under the time-s forward rate curve, the spot rate is given by the following expression:

$$\begin{aligned}r(s) &= f(t,s) - \int_t^s \Phi(u,s)b(u,s)\mathrm{d}u + \int_t^s \Phi(u,s)\mathrm{d}W^r(u)\\ &= f(t,s) + \frac{b(t,s)^2}{2} + \int_t^s \Phi(u,s)\mathrm{d}W^r(u)\end{aligned} \quad \text{(A1a)}$$

Integrating Equation A1a from time t to T has:

$$\begin{aligned}\int_t^T r(s)\mathrm{d}s &= \int_t^T f(t,s)\mathrm{d}s + \int_t^T \frac{b(t,s)^2}{2}\mathrm{d}s + \int_t^T\int_t^s \Phi(u,s)\mathrm{d}W^r(u)\mathrm{d}s = \int_t^T f(t,s)\mathrm{d}s\\ &\quad + \int_t^T \frac{b(t,s)^2}{2}\mathrm{d}s - \int_t^T b(t,s)\mathrm{d}W^r(u)\end{aligned} \quad \text{(A1b)}$$

Since $p(t,T) = \exp(-\int_t^T f(t,s)\mathrm{d}s)$, Equation A1b can be rewritten as: $\int_t^T r(s)\mathrm{d}s = -\ln p(t,T) + \int_t^T \frac{b(t,s)^2}{2}\mathrm{d}s - \int_t^T b(t,s)\mathrm{d}W^r(u)$. For a detail expression, see Jarrow and Yu (2001).

$$= \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)\exp[-\lambda_0^i(T-t)]$$
$$\times \exp\left[E_t(X_1) + E_t(X_2) + \frac{1}{2}V_t(X_1) + \frac{1}{2}V_t(X_2) + \mathrm{Cov}_t(X_1, X_2)\right]$$
$$= \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)\exp[-\lambda_0^i(T-t)]$$
$$\times \exp\Bigg[-(1+\lambda_1^i)\mu(t,T) + \frac{(T-t)^2}{4}\beta_i\sigma_I^2 + \frac{(1+\lambda_1^i)^2}{2}\sigma^2(t,T) + \frac{(T-t)^3}{6}\beta_i^2\sigma_I^2 + (1+\lambda_1^R)\beta_i\sigma_I \mathrm{Cov}_t \times \left[\int_t^T r(u)\mathrm{d}u, \int_t^T (T-v)\mathrm{d}W^1(v)\right]\Bigg] \tag{A4}$$

where $\mathrm{Cov}_t(\cdot,\cdot)$ are the covariance conditional on F_t. As a result, we have

$$\mathrm{Cov}_i\left[\int_t^T r(u)\mathrm{d}u, \int_t^T (T-v)\mathrm{d}W^1(v)\right]$$
$$= \mathrm{Cov}_t\left[-\int_t^T b(u,T)\mathrm{d}W^r(u), \int_t^T (T-v)\mathrm{d}W^I(v)\right]$$
$$= -\rho_{rI}\int_t^T (T-u)b(u,T)\mathrm{d}u$$

Combining the last equality with (A4), we obtain

$$v^i(t,T) = \delta_i p(t,T) + 1_{\{\tau^i > t\}}(1-\delta_i)\exp[-\lambda_0^i(T-t)]\exp \left[(1+\lambda_1^i)\ln p(t,T) + \frac{\lambda_1^i(1+\lambda_1^i)}{2}\sigma^2(t,T) + \frac{(T-t)^2}{4}\beta_i\sigma_I^2 + \frac{(T-t)^3}{6}\beta_i^2\sigma_I^2 - (1+\lambda_1^i)\beta_i\sigma_I\rho_{rI}\int_t^T (T-u)b(u,T)\mathrm{d}u\right]$$
$$= p(t,T)\Big[\delta_i + 1_{\{\tau^i > t\}}(1-\delta_i)\exp \Big(-\lambda_0^i(T-t) - \lambda_1^i Y(t,T)(T-t) + \frac{\lambda_1^i(1+\lambda_1^i)}{2} \sigma^2(t,T) + \frac{(T-t)^2}{4}\beta_i\sigma_I^2 + \frac{(T-t)^3}{6}\beta_i^2\sigma_I^2 - (1+\lambda_1^i)\beta_i\sigma_I\rho_{rI}\int_t^T (T-u)b(u,T)\mathrm{d}u\Big)$$

This completes the proof of the Theorem 1.

Appendix 2

Proof of Theorem 2: We define that

$$C_{SPV}(t) = E\left[\int_t^T 1_{\{\tau^R > s\}} C_s \frac{B(t)}{B(s)}\mathrm{d}s + 1_{\{\tau^R > T\}}\frac{B(t)}{B(T)}M + 1_{\{t \le \tau^R < T\}}\frac{B(t)}{B(\tau^R)}\delta_R M \middle| F_t\right] \equiv I_{B1} + I_{B2} + I_{B3}$$

To compute I_{B1}, using the law of the iterated conditional expectations and the fact that the default times are conditional independent with respect to $F^r_{T^*} \vee F^I_{T^*}$, we have

$$I_{B1} = E\left[\int_t^T 1_{\{\tau^R > s\}} C_s \frac{B(t)}{B(s)}\mathrm{d}s | F_t\right]$$
$$= E\left(E\left[\int_t^T 1_{\{\tau^R > s\}} C_s \frac{B(t)}{B(s)} | F^r_{T^*} \vee F^I_{T^*} \vee H^A_t \vee H^R_t\right]\mathrm{d}s \middle| F_t\right)$$
$$= E\left(E\left[\int_t^T 1_{\{\tau^R > s\}} C_s \frac{B(t)}{B(s)} | F^r_{T^*} \vee F^I_{T^*} \vee H^R_t\right]\mathrm{d}s \middle| Ft\right)$$
$$= 1_{\{\tau^R > t\}} E\left[\int_t^T C_s \frac{B(t)}{B(s)}\exp[-\int_t^s \lambda^R(u)\mathrm{d}u]\mathrm{d}s \middle| F_t\right] \tag{B1}$$

Since that

$$1_{\{\tau^R > t\}} C_s \frac{B(t)}{B(s)}\exp\left[-\int_t^s \lambda^R(u)\mathrm{d}u\right] > 0$$

using Fubini's theorem again and substitution of equality (2) into (B1), we have

$$I_{B1} = 1_{\{\tau^R > t\}}\int_t^T C_s E\left\{\exp\left(-\int_t^s \left[r(u) + \lambda_0^R + \lambda_1^R r(u). + \beta_R \log\left[\frac{I(u)}{B(u)}\right]\right]\mathrm{d}u\right)\middle| F_t\right\}\mathrm{d}s$$
$$= 1_{\{\tau^R > t\}}\int_t^T C_s \exp[-\lambda_0^R(S-t)]E \left\{\exp\left(-\int_t^s \left[(1+\lambda_1^R)r(u) + \beta_R\log\left[\frac{I(u)}{B(u)}\right]\right]\mathrm{d}u\right)\middle| F_t\right\}\mathrm{ds}$$
$$= 1_{\{\tau^R > t\}}\int_t^T C_s \exp\Big[-\lambda_0^R(s-t) - (1+\lambda_1^R)\mu(t,s). + \frac{(s-t)^2}{4}\beta_R\sigma_I^2 + \frac{(1+\lambda_1^R)^2}{2}\sigma^2(t,s) + \frac{(s-t)^3}{6}\beta_R^2\sigma_I^2 - (1+\lambda_1^R)\beta_R\sigma_I\rho_{rI}\int_t^s (s-u)b(u,s)\mathrm{d}u\Big]\mathrm{d}s$$
$$= 1_{\{\tau^R > t\}}\int_t^T C_s p(t,s)G_1(t,s)\mathrm{d}s$$

Following the similar pricing procedure, we have

$$I_{B2} = E\left[1_{\{\tau^R > T\}}\frac{B(t)}{B(T)}M\middle| F_t\right] = 1_{\{\tau^R > t\}}Mp(t,T)G_1(t,T)$$

To compute I_{B3}, note that conditionally on $F^r_{T^*} \vee F^I_{T^*}$, the density of the default time is given by

$$f_{\tau^R}(s) \equiv \frac{\partial}{\partial s} P\left(\tau^R \le s \middle| F^r_{T^*} \vee F^I_{T^*}\right) = \frac{\partial}{\partial s}\left[1 - \exp\left(-\int_0^s \lambda^R(u)\mathrm{d}u\right)\right] = \lambda^R(s)\exp\left[-\int_0^s \lambda^R(u)\mathrm{d}u\right]$$

Therefore, we obtain

$$\begin{aligned}
I_{B3} &= E\left[1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_R M | F_t\right] = \delta_R M E \\
&\quad \left(E\left[1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)} | F^r_{T^*} \vee F^I_{T^*} \vee H^A_t \vee H^R_t\right]\middle| F_t\right) \\
&= \delta_R M\, E\left(1_{\{\tau^R>t\}}\exp\left(\int_0^t \lambda^R(u)\mathrm{d}u\right)\right. \\
&\quad \left. E\left[1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)} | F^r_{T^*} \vee F^I_{T^*}\right]\middle| F_t\right) \\
&= 1_{\{\tau^R>t\}}\delta_R M E\left[\int_0^\infty 1_{\{t<s\le T\}}\frac{B(t)}{B(s)}\right. \\
&\quad \left.\times \exp\left(\int_0^t \lambda^R(u)\mathrm{d}u\right) f_{\tau^R}(s)\mathrm{d}s \middle| F_t\right] \\
&= 1_{\{\tau^R>t\}}\delta_R M E\left\{\int_t^T \lambda^R(s)\right. \\
&\quad \left.\times \exp\left[-\int_t^s [r(u)+\lambda^R(u)]\mathrm{d}u\right]\mathrm{d}s\middle| F_t\right\} \\
&= 1_{\{\tau^R>t\}}\delta_R M E\left\{\int_t^T \left[\lambda^R_0 + \lambda^R_1.\right.\right. \\
&\quad \left.\times r(s) + \beta_R \log\left[\frac{I(s)}{B(s)}\right]\right] \\
&\quad \times \exp\left(-\int_t^s \left[\lambda^R_0 + (1+\lambda^R_1) r(u).\right.\right. \\
&\quad \left.\left.\left. + \beta_R \log\left[\frac{I(s)}{B(s)}\right]\mathrm{d}u\right]\mathrm{d}s \middle| F_t\right)\right\}
\end{aligned} \quad \text{(B2)}$$

We divide (B2) into three parts as follows:

$$\begin{aligned}
J_{B1} &\equiv 1_{\{\tau^R>t\}}\delta_R M \lambda^R_0 \int_t^T \exp\left[-\lambda^R_0(s-t)\right] \\
&\quad E\left\{\exp\left(-\int_t^s (1+\lambda^R_1) r(u)\right.\right. \\
&\quad \left.\left. + \beta_R \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u \middle| F_t\right)\right\}\mathrm{d}s \\
J_{B2} &\equiv 1_{\{\tau^R>t\}}\delta_R M \lambda^R_1 \int_t^T \exp\left[-\lambda^R_0(s-t)\right] \\
&\quad E\left\{r(s)\exp\left(-\int_t^s (1+\lambda^R_1) r(u)\right.\right. \\
&\quad \left.\left. +\beta_R \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u \middle| F_t\right)\right\}\mathrm{d}s \\
J_{B3} &\equiv 1_{\{\tau^R>t\}}\delta_R M \beta_R \int_t^T \exp\left[-\lambda^R_0(s-t)\right] \\
&\quad E\left\{\log\left[\frac{I(s)}{B(s)}\right]\exp\left(-\int_t^s (1+\lambda^R_1) r(u)\right.\right. \\
&\quad \left.\left. +\beta_R \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u \middle| F_t\right)\right\}\mathrm{d}s
\end{aligned}$$

Similar to the pricing algorithm for I_{B1}, we have

$$J_{B1} = 1_{\{\tau^R>t\}}\lambda^R_0 \delta_R M \int_t^T p(t,s) G_1(t,s)\mathrm{d}s$$

To compute J_{B2}, assume that $X_1 = -\int_t^s (1+\lambda^R_1) r(u)\mathrm{d}u$, $X_2 = -\int_t^s \beta_R \log[I(u)/B(u)]\mathrm{d}u$ and $X_3 = r(s)$.

Given that (X_1, X_2, X_3) is a tri-normal distribution, we have

$$\begin{aligned}
m &= E\left(\exp(aX_1 + bX_2 + cX_3) \middle| F_t\right) \\
&= \exp\left(a\mu_{X_1} + b\mu_{X_2} + c\mu_{X_3} + \frac{2ab\sigma_{X_1X_2} + 2bc\sigma_{X_2X_3} + 2ac\sigma_{X_1X_3} + a^2\sigma^2_{X_1} + b^2\sigma^2_{X_2} + C^2\sigma^2_{X_3}}{2}\right)
\end{aligned}$$

where a, b and c are all real values.
Also, we have

$$\frac{\partial m}{\partial c}|_{c=0} = E_t[X_3 \exp(aX_1 + bX_2)] = \left(\mu_{X_3} + a\sigma_{X_1X_3} + b\sigma_{X_2X_3}\right)\exp\left(a\mu_{X_1} + b\mu_{X_2} + \frac{2ab\sigma_{X_1X_2} + a^2\sigma^2_{X_1} + b^2\sigma^2_{X_2}}{2}\right)$$

Hence, using the expression (B3) with $c=0$, $a=b=1$, $\mu_i \equiv E_t(X_i)$, $\sigma^2_i \equiv \mathrm{Var}_t(X_i)$ and $\sigma_{ij} \equiv \mathrm{Cov}_t(X_i, X_j)$, for $i, j = 1, 2, 3$, we obtain

$$\begin{aligned}
J_{B2} &= 1_{\{\tau^R>t\}}\delta_R M \lambda^R_1 \int_t^T \exp\left[-\lambda^R_0(s-t)\right] \\
&\quad \exp\left[-(1+\lambda^R_1)\mu(t,s).\right. \\
&\quad + \frac{(s-t)^2}{4}\beta_R\sigma^2_I + \frac{(1+\lambda^R_1)^2}{2}\sigma^2(t,s) + \frac{(s-t)^3}{6}\beta^2_R\sigma^2_I \\
&\quad \left. -(1+\lambda^R_1)\beta_R\sigma_I\rho_{rI}\int_t^s (s-u)b(u,s)\mathrm{d}u\right] \\
&\quad \times\left[E_t[r(s)] - (1+\lambda^R_1)\mathrm{Cov}_t\left[r(s), \int_t^s r(u)\mathrm{d}u\right]\right. \\
&\quad \left. -\beta_R\sigma_I \mathrm{Cov}_t\left[r(s), \int_t^s (s-v)\mathrm{d}W^I(v)\right]\right]\mathrm{d}s
\end{aligned}$$

where

$$r(s) = f(t,s) - \int_t^s \Phi(u,s) b(u,s)\mathrm{d}u + \int_t^s \Phi(u,s)\mathrm{d}W^r(u)$$

$$\mu_0(t,s) \equiv E_t[r(s)] = f(t,s) - \int_t^s \Phi(u,s)b(u,s)\mathrm{d}u,$$

$$\mathrm{Var}_t[r(s)] = \int_t^s \Phi(u,s)\mathrm{d}u$$

$$\mathrm{Cov}_t\left[r(s), \int_t^s r(u)\mathrm{d}u\right] = -\int_t^s \Phi(u,s)b(u,s)\mathrm{d}u = -\int_t^s b(u,s)\mathrm{d}b(u,s) = \frac{b(t,s)^2}{2}$$

$$\mathrm{Cov}_t\left[r(s), \int_t^s (s-v)\mathrm{d}W^I(v)\right] = \mathrm{Cov}_t\left[\int_t^s \Phi(u,s)\mathrm{d}W^r(u), \int_t^s (s-v)\mathrm{d}W^I(v)\right] = \rho_{rI}\int_t^s \Phi(u,s)(s-u)\mathrm{d}u$$

Hence, we have

$$\begin{aligned}
J_{B2} = {} & 1_{\{\tau^R > t\}}\delta_R M\lambda_1^R \int_t^T \Bigg\{\exp\Bigg[-\lambda_0^R(s-t)..\\
& - (1+\lambda_1^R)\mu(t,s) + \frac{(s-t)^2}{4}\beta_R\sigma_I^2\\
& + \frac{(1+\lambda_1^R)^2}{2}\sigma^2(t,s)\Bigg\} + \frac{(s-t)^3}{6}\beta_R^2\sigma_I^2\\
& - (1+\lambda_1^R)\beta_R\sigma_I\rho_{rI}\int_t^s (s-u)b(u,s)\mathrm{d}u\Bigg]\\
& \times\Bigg[\mu_0(t,s) - (1+\lambda_1^R)\frac{b(t,s)^2}{2}\\
& - \beta_R\sigma_I\rho_{rI}\int_t^s \Phi(u,s)(s-u)\mathrm{d}u\Bigg]\mathrm{d}s\\
= {} & 1_{\{\tau^R > t\}}\delta_R M\lambda_1^R \int_t^T p(t,s)G_2(t,s)\mathrm{d}s
\end{aligned}$$

Similar to the pricing algorithm for J_{B2}, let $X_1 = -\int_t^s (1+\lambda_1^R)r(u)\mathrm{d}u$, $X_2 = -\int_t^s \beta_R \log[I(u)/B(u)]\mathrm{d}u$ and$X_4 = \log[I(s)/B(s)]$, and given that (X_1, X_2, X_4) is a tri-normal distribution, we have

$$\begin{aligned}
\left.\frac{\partial m}{\partial c}\right|_{c=0} & = E_t[X_4\exp(X_1+X_2)]\\
& = (\mu_{X_4} + \sigma_{X_1X_4} + \sigma_{X_2X_4})\\
& \quad \times\exp\left(\mu_{X_1} + \mu_{X_2} + \frac{\sigma_{X_1}^2 + \sigma_{X_2}^2 + 2\sigma_{X_1X_2}}{2}\right)
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
J_{B3} = {} & 1_{\{\tau^R > t\}}\delta_R M\beta_R \int_t^T \exp[-\lambda_0^R(s-t)]\\
& \times\exp\Bigg[-(1+\lambda_1^R)\mu(t,s). + \frac{(s-t)^2}{4}\beta_R\sigma_I^2\\
& + \frac{(1+\lambda_1^R)^2}{2}\sigma^2(t,s) + \frac{(s+t)^3}{6}\beta_R^2\sigma_I^2\\
& - (1+\lambda_1^R)\beta_R\sigma_I\rho_{rI}\int_t^s (s-u)b(u,s)\mathrm{d}u\Bigg]\\
& \times\Bigg[E_t\left(\log\left[\frac{I(s)}{B(s)}\right]\right) - (1+\lambda_1^R)\sigma_I\\
& \times\mathrm{Cov}_t\left[\int_t^s r(u)\mathrm{d}u, \int_t^s \mathrm{d}W^I(u)\right]\\
& - \beta_R\mathrm{Cov}_t\left(\int_t^s \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u, \log\left[\frac{I(s)}{B(s)}\right]\right)\Bigg]\mathrm{d}s
\end{aligned}$$

where

$$E_t\left(\log\left[\frac{I(s)}{B(s)}\right]\right) = \log\left[\frac{I(t)}{B(t)}\right] - \frac{1}{2}\sigma_I^2(s-t) = -\frac{1}{2}\sigma_I^2(s-t)$$

$$\begin{aligned}
\mathrm{Cov}_t\left(\int_t^s r(u)\mathrm{d}u, \int_t^s \mathrm{d}W^I(u)\right) & = \mathrm{Cov}_t\left[\int_t^s -b(u,s)\mathrm{d}W^r(u), \int_t^s \mathrm{d}W^I(u)\right]\\
& = \mathrm{Cov}\left[\int_t^s \rho_t(s-u)\mathrm{d}W^t(u)\right]\\
& = -\rho_{rI}\int_t^s b(u,s)\mathrm{d}u
\end{aligned}$$

$$\mathrm{Cov}_t\left(\int_t^s \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u, \log\left[\frac{I(s)}{B(s)}\right]\right) = \mathrm{Cov}_t\left[\int_t^s \sigma_I(s-u)\mathrm{d}W^I(u), \sigma_I\int_t^s \mathrm{d}W^I(v)\right] = \sigma_I^2\int_t^s (s-u)\mathrm{d}u = \frac{\sigma_I^2(s-t)^2}{2}$$

Hence, we have

$$\begin{aligned}
J_{B3} = {} & 1_{\{\tau^R > t\}}\delta_R M\beta_R \int_t^T \Bigg\{\exp\Bigg[-\lambda_0^R(s-t).\\
& - (1+\lambda_1^R)\mu(t,s) + \frac{(s-t)^2}{4}\beta_R\sigma_I^2\\
& + \frac{(1+\lambda_1^R)^2}{2}\sigma^2(t,s)\Bigg\} + \frac{(s-t)^3}{6}\\
& \times\beta_R^2\sigma_I^2 - (1+\lambda_1^R)\beta_R\sigma_I\rho_{rI}\\
& \int_t^s (s-u)b(u-s)\mathrm{d}u\Bigg]\times\Bigg[\frac{\sigma_I^2(s-t)}{2}\\
& [-1-\beta_R(s-t)] + (1+\lambda_1^R)\sigma_I\rho_{rI}\\
& \int_t^s b(u,s)\mathrm{d}u\Bigg]\Bigg\}\mathrm{d}s\\
= {} & 1_{\{\tau^R > t\}}\delta_R M\beta_R \int_t^T p(t,s)G_3(t,s)\mathrm{d}s
\end{aligned}$$

This completes the proof of the Theorem 2.

Appendix 3

Proof of Theorem 3: We first define that

$$\begin{aligned}C_{PB}(t) &= E\bigg[\int_t^T 1_{\{\tau^A > s\}}1_{\{\tau^R > s\}}C_s\frac{B(t)}{B(s)}\mathrm{d}s\\&+\int_t^T 1_{\{\tau^A \le s\}}1_{\{\tau^R > s\}}\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s\\&+1_{\{\tau^A > T\}}1_{\{\tau^R > T\}}\frac{B(t)}{B(T)}M\\&+1_{\{t<\tau^A\le T\}}1_{\{\tau^R > T\}}\frac{B(t)}{B(T)}\delta_A M\\&+1_{\{\tau^A > \tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_R M\\&+1_{\{\tau^A \le \tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_A\delta_R M\bigg|F_t\bigg]\\&= I_{C1}+I_{C2}+I_{C3}+I_{C4}+I_{C5}+I_{C6}\end{aligned}$$

To compute I_{C1}, by using the law of the iterated conditional expectations, we have

$$\begin{aligned}I_{C1} &= E\bigg[\int_t^T 1_{\{\tau^A > s\}}1_{\{\tau^R > s\}}C_s\frac{B(t)}{B(s)}\mathrm{d}s|F_t\bigg]\\&= E\bigg(\int_t^T C_s\frac{B(t)}{B(s)}E\big[1_{\{\tau^A > s\}}1_{\{\tau^R > s\}}\\&\quad\big|F^r_{T*}\vee F^I_{T*}\vee H^R_t\vee H^A_t\big]\mathrm{d}s|F_t\bigg)\\&= E\bigg(\int_t^T C_s 1_{\{\tau^A > t\}}\exp\bigg[\int_0^t \lambda^A(u)\mathrm{d}u\bigg]\frac{B(t)}{B(s)}\\&\quad E\big[1_{\{\tau^A > s\}}1_{\{\tau^R > s\}}\big|F^r_{T*}\vee F^I_{T*}\vee H^R_t\big]\mathrm{d}s|F_t\bigg)\\&= E\bigg(\int_t^T C_s 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\exp\bigg[\int_0^t[\lambda^A(u)\\&\quad+\lambda^R(u)]\mathrm{d}u\bigg]\frac{B(t)}{B(s)}E\big[1_{\{\tau^A > s\}}\\&\quad\times 1_{\{\tau^R > s\}}\big|F^r_{T*}\vee F^I_{T*}\big]\mathrm{d}s|F_t\bigg)\end{aligned}$$

Since that the default times are conditional independent with respect to $F^r_{T*}\vee F^I_{T*}$, we have

$$\begin{aligned}I_{C1} &= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}E\bigg(\int_t^T C_s\exp\bigg[\int_0^t[\lambda^A(u)\\&\quad+\lambda^R(u)]\mathrm{d}u\bigg]\frac{B(t)}{B(s)}P\big[\tau^A > s\big|F^r_{T*}\vee F^I_{T*}\big]\\&\quad P\big[\tau^R > s\big|F^r_{T*}\vee F^I_{T*}\big]\mathrm{d}s|F_t\bigg)\\&= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}E\bigg[\int_t^T C_s\frac{B(t)}{B(s)}\\&\quad\exp\bigg(-\int_t^s[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\bigg)\mathrm{d}s|F_t\bigg]\end{aligned}$$

By virtue of (2), we obtain

$$\begin{aligned}I_{C1} &= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}E\bigg[\int_t^T C_s\frac{B(t)}{B(s)}\\&\quad\exp\bigg(-\int_t^s[\lambda^A(u)+\lambda^R(u)]\bigg)\mathrm{d}s|F_t\bigg]\\&= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\int_t^T C_s\exp\big[-(\lambda^A_0+\lambda^R_0)\\&\quad(s-t)\big]E\bigg(\exp\bigg[-\int_t^s\bigg[(1+\lambda^A_1+\lambda^R_1)\\&\quad r(u)+(\beta_A+\beta_B)\log\bigg[\frac{I(u)}{B(u)}\bigg]r(u)\bigg]\mathrm{d}u\bigg]|F_t\bigg)\mathrm{d}s\\&= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\int_t^T C_s p(t,s)G_4(t,s)\mathrm{d}s\end{aligned}$$

The proofs of I_{D3}, I_{C2}, I_{C3} also follow the same idea as I_{B1} and I_{C1}, thus we have

$$\begin{aligned}I_{C2} &= E\bigg[\int_t^T 1_{\{\tau^A\le s\}}1_{\{\tau^R > s\}}\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s|F_t\bigg]\\&= E\bigg[\int_t^T 1_{\{\tau^R > s\}}\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s|F_t\bigg]\\&\quad- E\bigg[\int_t^T 1_{\{\tau^R > s\}}1_{\{\tau^A > s\}}\delta_A C_s\frac{B(t)}{B(s)}\mathrm{d}s|F_t\bigg]\\&= 1_{\{\tau^R > t\}}\delta_A\int_t^T C_s p(t,s)G_1(t,s)\mathrm{d}s\\&\quad- 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\delta_A\int_t^T C_s p(t,s)G_4(t,s)\mathrm{d}s\end{aligned}$$

$$\begin{aligned}I_{C3} &= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}Mp(t,T)G_4(t,T),\\I_{C4} &= 1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\delta_A Mp(t,T)[G_1(t,T)-G_4(t,T)]\end{aligned}$$

For I_{C5}, we have

$$\begin{aligned}I_{C5} &= E\bigg[1_{\{\tau^A > \tau^R\}}1_{\{t<\tau^R\le T\}}\delta_R M\frac{B(t)}{B(\tau^R)}|F_t\bigg]\\&= \delta_R ME\bigg[E\bigg(1_{\{\tau^A > \tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\big|F^r_{T*}\vee F^I_{T*}\\&\quad\vee H^A_t\vee H^R_t\bigg]\bigg|F_t\bigg)\\&= \delta_R ME\bigg[1_{\{\tau^A > t\}}\exp\bigg(\int_0^t\lambda^A(u)\mathrm{d}u\bigg)\\&\quad E\bigg(1_{\{\tau^A > \tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\big|F^r_{T*}\vee F^I_{T*}\vee H^R_t\bigg)\bigg|_{F_t}\bigg]\\&= \delta_R ME\bigg[1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\exp\bigg(\int_0^t[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\bigg)\\&\quad E\bigg(1_{\{\tau^A > \tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\big|F^r_{T*}\vee F^I_{T*}\bigg)\bigg|_{F_t}\bigg]\\&= \delta_R ME\bigg[1_{\{\tau^A > t\}}1_{\{\tau^R > t\}}\exp\bigg(\int_0^t[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\bigg)\end{aligned}$$

$$\times\int_t^T\int_s^\infty \frac{B(t)}{B(s)}\lambda^A(v)\lambda^R(s)\exp\left(-\int_0^v \lambda^A(u)\mathrm{d}u\right)$$
$$\exp\left(-\int_0^s \lambda^R(u)\mathrm{d}u\right)\mathrm{d}v\mathrm{d}s|F_t\Bigg]=1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R ME$$
$$\times\left[\int_t^T \frac{B(t)}{B(s)}\lambda^R(s)\exp\left(-\int_t^s[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\right)\mathrm{d}s|F_t\right]$$
$$=1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R ME\left\{\int_t^T \frac{B(t)}{B(s)}[\lambda_0^R+\lambda_1^R r(s)\right.$$
$$+\beta_R\log\left[\frac{I(s)}{B(s)}\right]\Bigg]\exp-\left(\int_t^s[\lambda^A(u)\right.$$
$$\left.+\lambda^R(u)]\mathrm{d}u\right)\mathrm{d}s|F_t\Bigg\}$$
$$\equiv J_{C1}+J_{C2}+J_{C3}$$

where

$$J_{C1}\equiv 1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R ME$$
$$\left\{\int_t^T \frac{B(t)}{B(s)}\lambda_0^R\exp\left[-\int_t^s[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\right]\mathrm{d}s|F_t\right\}$$
$$=1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R M\lambda_0^R\int_t^T p(t,s)G_4(t,s)\mathrm{d}s$$
$$J_{C2}\equiv 1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R ME$$
$$\left\{\int_t^T \frac{B(t)}{B(s)}\lambda_1^R r(s)\exp\left[-\int_t^s[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\right]\mathrm{d}s|F_t\right\}$$
$$=1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R M\lambda_1^R\int_t^T p(t,s)G_5(t,s)\mathrm{d}s$$
$$J_{C3}\equiv 1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R ME\left\{\int_t^T \frac{B(t)}{B(s)}\beta_R\log\left[\frac{I(s)}{B(s)}\right]\right.$$
$$\exp\left[-\int_t^s[\lambda^A(u)+\lambda^R(u)]\mathrm{d}u\right]\mathrm{d}s|F_t\Bigg\}$$
$$=1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_R M\beta_R\int_t^T p(t,s)G_6(t,s)\mathrm{d}s$$

Finally, the process of derivation for I_{C6} is similar to I_{C5} as follows:

$$I_{C6}=E\left[1_{\{\tau^A\le\tau^R\}}1_{\{t<\tau^R\le T\}}\frac{B(t)}{B(\tau^R)}\delta_A\delta_R M|F_t\right]$$
$$=\delta_A\delta_R ME\Bigg[E\Bigg(1_{\{\tau^A\le\tau^R\}}1_{\{t<\tau^R\le T\}}$$
$$\frac{B(t)}{B(\tau^R)}\big|F_{T*}^r\vee F_{T*}^I\vee H_{T*}^A\vee H_t^R\Bigg)\Bigg|F_t\Bigg]$$
$$=1_{\{\tau^R>t\}}\delta_A\delta_R M\lambda_0^R\int_t^T p(t,s)G_1(t,s)\mathrm{d}s$$
$$+1_{\{\tau^R>t\}}\delta_A\delta_R M\lambda_1^R\int_t^T p(t,s)G_2(t,s)\mathrm{d}s$$
$$+1_{\{\tau^R>t\}}\delta_A\delta_R M\beta_R\int_t^T p(t,s)G_3(t,s)\mathrm{d}s$$
$$-1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_A\delta_R M\lambda_0^R\int_t^T p(t,s)G_4(t,s)\mathrm{d}s$$
$$-1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_A\delta_R M\lambda_1^R\int_t^T p(t,s)G_5(t,s)\mathrm{d}s$$
$$-1_{\{\tau^A>t\}}1_{\{\tau^R>t\}}\delta_A\delta_R M\beta_R\int_t^T p(t,s)G_6(t,s)\mathrm{d}s$$

This completes the proof of the Theorem 3.

Appendix 4

Proof of Theorem 4

$$\mathrm{TC_{SPV}}(t)=E\left[\int_t^T 1_{\{\tau^R>s\}}\mathrm{LC}_s\frac{B(t)}{B(s)}+1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\right.$$
$$\left[M\left(1+L\frac{v^R(T,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]+1_{\{t<\tau^R\le T\}}$$
$$\frac{B(t)}{B(\tau^R)}\left[M\left(1+L\frac{\delta_R p(\tau^R,U)-v^R(t_0,U)}{v^R(t_0,U)}\right)\right]|F_t\Bigg]$$
$$=I_{D1}+I_{D2}+I_{D3}$$

To compute I_{D1}, similar to the pricing procedure of I_{B1}, we have

$$E\left[\int_t^T 1_{\{\tau^R>s\}}\mathrm{LC}_s\frac{B(t)}{B(s)}|F_t\right]$$
$$=1_{\{\tau^R>t\}}\int_t^T \mathrm{LC}_s p(t,s)G_1(t,s)\mathrm{d}s$$

Similarly, for computing I_{D2}, we have

$$I_{D2}=E\left[1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\left[(1-L)M+LM\frac{v^R(T,U)}{v^R(t_0,U)}\right]|F_t\right] \tag{D1}$$

where

$$v^R(T,U)=E\left[\frac{B(T)}{B(U)}(\delta_R 1_{\{\tau^R\le U\}}+1_{\{\tau^R>U\}})|F_T\right]$$
$$=E\left[\frac{B(T)}{B(U)}[\delta_R+(1-\delta_R)\right.$$
$$E\big(1_{\{\tau^R>U\}}\big|F_{T*}^r\vee F_{T*}^I\vee H_T^R\big)\big]|F_T\Big]$$
$$=E\left[\frac{B(T)}{B(U)}\Bigg[\delta_R+(1-\delta_R)\right.$$
$$1_{\{\tau^R>T\}}\exp\left(-\int_T^U\lambda^R(s)\mathrm{d}s\right)\Bigg]\Bigg|F_T\Bigg]$$

We divide (D1) into two parts

$$J_{D1}=E\left[1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}(1-L)M|F_t\right]$$
$$=1_{\{\tau^R>t\}}(1-L)Mp(t,T)G_1(t,T)$$
$$J_{D2}=E\left\{1_{\{\tau^R>T\}}\frac{B(t)}{B(T)}\frac{LM}{v^R(t_0,U)}E\left[\frac{B(T)}{B(U)}\right.\right.$$
$$\left(\delta_R+(1-\delta_R)1_{\{\tau^R>T\}}\exp\left(-\int_T^U\lambda^R(s)\mathrm{d}s\right)\right)|F_T\Bigg]\Bigg|_{F_t}\Bigg\}$$
$$=E\left\{\frac{LM}{v^R(t_0,U)}\frac{B(t)}{B(U)}\Bigg[\delta_R+(1-\delta_R)\right.$$
$$\exp\left[-\left(\int_t^U-\int_t^T\right)\lambda^R(s)\mathrm{d}s\right]\Bigg]1_{\{\tau^R>T\}}|F_t\Bigg\}$$

To compute J_{D2}, we also divide into two parts

$$K_{D1} = E\Big[\frac{LM}{v^R(t_0,U)}\frac{B(t)}{B(U)}\delta_R E$$

$$(1_{\{\tau^R>T\}}\big|F^r_{T*}\vee F^I_{T*}\vee H^R_t)\big|F_t\Big]$$

$$= \frac{LM}{v^R(t_0,U)}\delta_R 1_{\{\tau^R>t\}}\exp\Big[-\lambda_0^R(T-t)-\mu(t,U)$$

$$-\lambda_1^R\mu(t,T)+\frac{(T-t)^2}{4}\beta_R\sigma_I^2+\frac{\sigma^2(t,U)}{2}+\frac{(\lambda_1^R)^2}{2}$$

$$\sigma^2(t,T)+\frac{(T-t)^3}{6}\beta_R^2\sigma_I^2+\lambda_1^R\rho_1(t,T,U)$$

$$+\beta_R\sigma_I\rho_2(t,T,U)+\lambda_1^R\beta_R\sigma_I\rho_3(t,T)\Big]$$

$$= \frac{LM}{v^R(t_0,U)}\delta_R 1_{\{\tau^R>t\}}p(t,T)$$

$$\exp\Big[-\lambda_0^R(T-t)-Y(t,U)(U-t)$$

$$-(\lambda_1^R-1)Y(t,T)(T-t)+\frac{(T-t)^2}{4}\beta_R\sigma_I^2$$

$$+\frac{(\lambda_1^R)^2-\lambda_1^R}{2}\sigma^2(t,T)+\frac{(T-t)^3}{6}\beta_R^2\sigma_I^2+\lambda_1^R\rho_1$$

$$\times(t,T,U)+\beta_R\sigma_I\rho_2(t,T,U)+\lambda_1^R\beta_R\sigma_I\rho_3(t,T)\Big]$$

$$= 1_{\{\tau^R>t\}}\frac{LM\delta_R}{v^R(t_0,U)}p(t,T)G_7(t,T,U)$$

$$K_{D2} = E\Big[\frac{LM}{v^R(t_0,U)}\frac{B(t)}{B(U)}(1-\delta_R)$$

$$E\Big(\exp\Big[-\Big(\int_t^U-\int_t^T\Big)\lambda^R(s)\mathrm{d}s\Big]$$

$$1_{\{\tau^R>T\}}\big|F^r_{T*}\vee F^I_{T*}\vee H^R_t\Big)\Big|F_t\Big]$$

$$= 1_{\{\tau^R>t\}}\frac{LM(1-\delta_R)}{v^R(t_0,U)}\exp\Big(-\lambda_0^R(U-t)$$

$$-\lambda_1^R Y(t,U)(U-t)+\frac{\lambda_1^R(1+\lambda_1^R)}{2}\sigma^2(t,U)$$

$$+\frac{(U-t)^2}{4}\beta_R\sigma_I^2+\frac{(U-t)^3}{6}\beta_R^2\sigma_I^2$$

$$-(1+\lambda_1^R)\beta_R\sigma_I\rho_{rI}\int_t^U(U-y)b(y,U)\mathrm{d}y\Big)$$

$$= 1_{\{\tau^R>t\}}\frac{LM(1-\delta_R)}{v^R(t_0,U)}p(t,U)G_1(t,U)$$

$$I_{D3} = E\Big\{1_{t<\tau^R\le T}\frac{B(t)}{B(\tau^R)}\Big[[(1-L)M].$$

$$+LM\frac{\delta_R P(\tau^R,U)}{v^R(t_0,U)}\Big]\Big|F_t.\Big\}$$

$$= E\Big\{E\Big(\frac{B(t)}{B(\tau^R)}\Big[(1-L)M+LM\frac{\delta_R p(\tau^R,U)}{v^R(t_0,U)}\Big]$$

$$1_{\{t<\tau^R\le T\}}\big|F^r_{T*}\vee F^I_{T*}\vee H^R_T\Big)\Big|_{F_t}\Big\}$$

$$= E\Big\{\int_t^T\frac{B(t)}{B(s)}\Big[(1-L)M+LM\frac{\delta_R p(s,U)}{v^R(t_0,U)}\Big]\lambda^R(s)$$

$$\exp\Big[-\int_t^s\lambda^R(u)\mathrm{d}u\Big]\mathrm{d}s|F_t\Big\} = J_{D3}+J_{D4}$$

$$J_{D3} = E\Big[\int_t^T(1-L)M\frac{B(t)}{B(s)}\Big(\lambda_0^R+\lambda_1^R r(s)$$

$$+\beta_R\log\Big[\frac{I(s)}{B(s)}\Big]\Big)\exp\Big[-\int_t^s\lambda^R(u)\mathrm{d}u\Big]\mathrm{d}s|F_t\Big]$$

$$= 1_{\{\tau^R>t\}}(1-L)M\lambda_0^R\int_t^T p(t,s)G_1(t,s)\mathrm{d}s$$

$$+1_{\{\tau^R>t\}}(1-L)M\lambda_1^R\int_t^T p(t,s)G_2(t,s)\mathrm{d}s$$

$$+1_{\{\tau^R>t\}}(1-L)M\beta_R\int_t^T p(t,s)G_3(t,s)\mathrm{d}s$$

$$J_{D4} = E\Big[\int_t^T\frac{LM\delta_R}{v^R(t_0,U)}\frac{B(t)}{B(s)}p(s,U)\Big(\lambda_0^R+\lambda_1^R r(s)$$

$$+\beta_R\log\Big[\frac{I(s)}{B(s)}\Big]\Big)\exp\Big[-\int_t^s\lambda^R(s)\mathrm{d}s\Big]\Big|F_t.\Big]$$

$$= K_{D3}+K_{D4}+K_{D5}$$

where

$$K_{D3} = E\Big[\int_t^T\frac{LM\delta_R}{v^R(t_0,U)}\frac{B(t)}{B(s)}p(s,U)\lambda_0^R$$

$$\exp\Big[-\int_t^s\lambda^R(s)\mathrm{d}s\Big]\Big|F_t.\Big]$$

$$= 1_{\{\tau^R>t\}}\int_t^T\frac{LM\delta_R}{v^R(t_0,U)}\lambda_0^R p(t,s)G_7(t,s,U)\mathrm{d}s$$

$$K_{D4} = \int_t^T\frac{LM\delta_R}{v^R(t_0,U)}\lambda_1^R\exp[-\lambda_0^R(s-t)]E\Big\{r(s)\exp$$

$$\Big(-\Big[\lambda_1^R\int_t^s r(u)\mathrm{d}u+\int_t^U r(u)\mathrm{d}u$$

$$+\int_t^s\beta_R\log\Big[\frac{I(u)}{B(u)}\Big]\mathrm{d}u\Big]\Big|F_t\Big)\Big\}\mathrm{d}s$$

Given (X_0, X_1, X_2, $X3$), we have

$$m = E[\exp(dX_0 + eX_1 + fX_2 + gX_3)|F_t] = \exp\left(d\mu_{X_0} + e\mu_{X_1} + f\mu_{X_2} + g\mu_{X_3}. + \frac{2de\sigma_{X_0X_1} + 2df\sigma_{X_0X_2} + 2dg\sigma_{X_0X_3} + 2ef\sigma_{X_1X_2} + 2eg\sigma_{X_1X_3} + 2fg\sigma_{X_2X_3} + d^2\sigma_{X_0}^2 + e^2\sigma_{X_1}^2 + f^2\sigma_{X_2}^2 + g^2\sigma_{X_3}^2}{2}\right) \tag{D2}$$

where d, e, f, g are all real values.

Also, we have

$$\frac{\partial m}{\partial g}\Big|_{g=0} = E_t[X_3 \exp(dX_0 + eX_1 + fX_2)] = \big(\mu_{X_3} + d\sigma_{X_0X_3} + e\sigma_{X_1X_3} + f\sigma_{X_2X_3}\big)\exp\left[d\mu_{X_0} + e\mu_{X_1} + f\mu_{X_2} + \frac{d^2\sigma_{X_0}^2 + e^2\sigma_{X_1}^2 f^2\sigma_{X_2}^2 + 2de\sigma_{X_0X_1} + 2df\sigma_{X_0X_2} + 2ef\sigma_{X_1X_2}}{2}\right]$$

where $\mu_i \equiv E_t(X_i)$, $\sigma_i^2 \equiv \mathrm{Var}_t(X_i)$ and $\sigma_{ij} \equiv \mathrm{Cov}_t(X_i, X_j)$, for $i, j = 1, 2, 3$.

The pricing mechanism for moment generating function can be seen in Hogg and Craig (1970, p. 114) or the Appendix of Janosi *et al.* (2002). Hence, using the expression (D2) with $g = 0$, $d = e = f = 1$, $X_0 = -\int_t^U r(u)\mathrm{d}u$, $X_1 = -\int_t^s \lambda_1^R r(u)\mathrm{d}u$, $X_2 = -\int_t^s \beta_R \log[I(u)/B(u)]\mathrm{d}u$ and $X_3 = r(s)$, we obtain

$$\begin{aligned} K_{D4} = 1_{\{\tau^R > t\}} \int_t^T \frac{LM\delta_R}{v^R(t_0, U)} \lambda_1^R \exp\Big[&-\lambda_0^R(s-t) - \mu(t, U) - \lambda_1^R \mu(t, s) + \frac{(s-t)^2}{4}\beta_R \sigma_I^2 + \frac{\sigma^2(t, U)}{2} \\ &+ \frac{(\lambda_1^R)^2}{2}\sigma^2(t, s) + \frac{(s-t)^3}{6}\beta_R^2\sigma_I^2 + \lambda_1^R \rho_1(t, s, U) \\ &+ \beta_R \sigma_I \rho_2(t, s, U) + \lambda_1^R \beta_R \sigma_I \rho_3(t, s)\Big] \\ &\times \left[\mu_0(t, s) - \int_t^s \Phi(u, s) b(u, U)\mathrm{d}u - \frac{\lambda_1^R}{2} b(u, s)^2 - \beta_R \sigma_I \rho_{rI} \int_t^s \Phi(u, s)(s-u)\mathrm{d}u\right]\mathrm{d}s \\ = 1_{\{\tau^R > t\}} \int_t^T &\frac{LM\delta_R}{v^R(t_0, U)} \lambda_1^R p(t, s) G_8(t, s, U)\mathrm{d}s \end{aligned}$$

Similar to the pricing algorithm for K_{D4},

$$K_{D5} = \int_t^T \frac{LM\delta_R}{v^R(t_0, U)} \beta_R \exp\big[-\lambda_0^R(s-t)\big] \times E\left\{\log\left[\frac{I(s)}{B(s)}\right]\exp\left(-\left[\left(\lambda_1^R \int_t^s r(u)\mathrm{d}u + \int_t^U r(u)\mathrm{d}u + \int_t^s \beta_R \log\left[\frac{I(u)}{B(u)}\right]\mathrm{d}u\right]\Big|_{F_t}\right)\right\}\mathrm{d}s$$

Given (X_0, X_1, X_2, X_4), let $X_0 = -\int_t^U r(u)\mathrm{d}u$, $X_1 = -\int_t^s \lambda_1^R r(u)\mathrm{d}u$, $X_2 = -\int_t^s \beta_R \log[I(u)/B(u)]\mathrm{d}u$ and $X_4 = \log[I(s)/B(s)]$, we have

$$\begin{aligned} K_{D5} = 1_{\{\tau^R > t\}} \int_t^T \frac{LM\delta_R}{v^R(t_0, U)} \beta_R \exp\Big[&-\lambda_0^R(s-t) - \mu(t, U) - \lambda_1^R \mu(t, s) + \frac{(s-t)^2}{4}\beta_R \sigma_I^2 + \frac{\sigma^2(t, U)}{2} \\ &+ \frac{(\lambda_1^R)^2}{2}\sigma^2(t, s) + \frac{(s-t)^3}{6}\beta_R^2\sigma_I^2 + \lambda_1^R \rho_1(t, s, U) \\ &+ \beta_R \sigma_I \rho_2(t, s, U) + \lambda_1^R \beta_R \sigma_I \rho_3(t, s)\Big] \\ &\times \left[\frac{\sigma_I^2(s-t)}{2}[-1 - \beta_R(s-t)] + \sigma_I \rho_{rI} \int_t^s b(y, U)\mathrm{d}y + \lambda_1^R \sigma_I \rho_{rI} \int_t^s b(y, s)\mathrm{d}y\right]\mathrm{d}s \\ = 1_{\{\tau^R > t\}} \int_t^T &\frac{LM\delta_R}{v^R(t_0, U)} \beta_R p(t, s) G_9(t, s, U)\mathrm{d}s \end{aligned}$$

This completes the proof of the Theorem 4.

Appendix 5

Proof of Theorem 5

$$
\begin{aligned}
\mathrm{TC}_{\mathrm{PB}}(t) = E\Bigg[&\int_t^T 1_{\{\tau^A > s\}} 1_{\{\tau^R > s\}} \mathrm{LC}_s \frac{B(t)}{B(s)} \\
&+ \int_t^T 1_{\{\tau^A \le s\}} 1_{\{\tau^R > s\}} \delta_A \mathrm{LC}_s \frac{B(t)}{B(s)} \\
&+ 1_{\{\tau^A > T\}} 1_{\{\tau^R > T\}} \frac{B(t)}{B(T)} \\
&\left[M\left(1 + L\frac{v^R(T,U) - v^R(t_0,U)}{v^R(t_0,U)}\right)\right] \\
&+ 1_{\{t < \tau^A < T\}} 1_{\{\tau^R > T\}} \\
&\frac{B(t)}{B(T)} \delta_A \left[M\left(1 + L\frac{v^R(T,U) - v^R(t_0,U)}{v^R(t_0,U)}\right)\right] \\
&+ 1_{\{\tau^A > \tau^R\}} 1_{\{t < \tau^R \le T\}} \frac{B(t)}{B(\tau^R)} \\
&\left[M\left(1 + L\frac{\delta_R P(\tau^R,U) - v^R(t_0,U)}{v^R(t_0,U)}\right)\right] \\
&+ 1_{\{\tau^A \le \tau^R\}} 1_{\{t < \tau^R \le T\}} \frac{B(t)}{B(\tau^R)} \\
&\delta_A \left[M\left(1 + L\frac{\delta_R P(\tau^R,U) - v^R(t_0,U)}{v^R(t_0,U)}\right)\right] \Bigg| F_t. \Bigg] \\
= {} & I_{E1} + I_{E2} + I_{E3} + I_{E4} + I_{E5} + I_{E6}
\end{aligned}
$$

Using the same pricing procedure of Appendices 3 and 4, we have

$$
I_{E1} = 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} \int_t^T \mathrm{LC}_s p(t,s) G_4(t,s) \mathrm{d}s
$$

$$
\begin{aligned}
I_{E2} = {} & 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} \int_t^T \delta_A \mathrm{LC}_s p(t,s) G_1(t,s) \mathrm{d}s \\
& - 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} \int_t^T \delta_A \mathrm{LC}_s p(t,s) G_4(t,s) \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
I_{E3} = {} & E\left[1_{\{\tau^A > T\}} 1_{\{\tau^R > T\}} \frac{B(t)}{B(T)} \cdot \right. \\
& \left. \left[M + (1-L)\frac{v^R(T,U) - v^R(t_0,U)}{v^R(t_0,U)}\right] \middle| F_t \right] \\
= {} & J_{E1} + J_{E2}
\end{aligned}
$$

where

$$
\begin{aligned}
J_{E1} &= E\left[1_{\{\tau^A > T\}} 1_{\{\tau^R > T\}} (1-L) M \frac{B(t)}{B(T)} \middle| F_t. \right] \\
&= 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} (1-L) M p(t,T) G_4(t,T)
\end{aligned}
$$

$$
\begin{aligned}
J_{E2} = E\Bigg\{ & 1_{\{\tau^A > T\}} \frac{B(t)}{B(T)} \frac{LM}{v^R(t_0,U)} E\Bigg[\frac{B(T)}{B(U)} \\
& \left(\delta_R + (1-\delta_R) 1_{\{\tau^R > T\}} \exp\left(-\int_T^U \lambda^R(s)\mathrm{d}s\right) \middle| F_T. \right) \\
& 1_{\{\tau^R > T\}} | F_t \Bigg]\Bigg\}
\end{aligned}
$$

We divide J_{E2} into two parts and obtain

$$
\begin{aligned}
K_{E1} &= E\left\{ 1_{\{\tau^A > T\}} \frac{B(t)}{B(T)} \frac{LM}{v^R(t_0,U)} E\left[\frac{B(T)}{B(U)} \delta_R 1_{\{\tau^R > T\}} \middle| F_t. \right]\right\} \\
&= 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} \frac{LM\delta_R}{v^R(t_0,U)} p(t,T) G_{10}(t,T,U)
\end{aligned}
$$

$$
\begin{aligned}
K_{E2} = E\Bigg\{ & 1_{\{\tau^A > T\}} \frac{B(t)}{B(T)} \frac{LM}{v^R(t_0,U)} E\Bigg[\frac{B(T)}{B(U)} \Bigg((1-\delta_R) \\
& \times 1_{\{\tau^R > T\}} \exp\left[-\int_T^U \lambda^R(s)\mathrm{d}s\right] \Bigg| F_T. \Bigg) 1_{\{\tau^R > T\}} \Bigg| F_t. \Bigg]\Bigg\} \\
= {} & 1_{\{\tau^A > t\}} 1_{\{\tau^R > t\}} \frac{LM}{v^R(t_0,U)} (1-\delta_R) p(t,U) \\
& \times G_{13}(t,T,U)
\end{aligned}
$$

Finally, the proofs of I_{E4}, I_{E5}, I_{E6} also follow the same idea as I_{D3} and is omitted.