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A Factor-Copula Based Valuation of Synthetic CDO-Squared under a Stochastic Intensity

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Abstract

This study extends the double student's t factor copula models developed by Hull and White (2004) for valuing CDO-Squared. First, the assumptions of non-homogeneous recovery rates are adopted to fit realistic aggregate loss of CDO collateral. Second, a stochastic hazard rate is proposed using the CIR intensity process to resolve the problem of inability of constant intensity rate to capture instantaneous credit spread dynamics. To construct the default probability distribution of CDO-Squared, the factor copula model is derived using the two-stage probability bucketing method to approximate loss distribution. Finally, the example of CDO-Squared issued by the Polaris Securities Group in Taiwan is presented to illustrate fair credit spread pricing for various tranches.

Keywords: Factor Copula, CIR Intensity Model, Index Tranche, CDO-Squared.

1. Introduction

The CDO represents default correlation product to migrate default risk from originators to investors. Presently, numerous originators have begun to consider various bespoke tranches of other CDOs (including standardized contract of CDS index) as underlying

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collateral for both increasing tranche return and diversifying underlying collateral.[1] The type of exotic CDOs is termed Synthetic CDO-Squared.[2]

For risk management and valuation of Synthetic CDO-Squared, default dependence is an important factor which can be influenced by overall economic, sectoral, and firm-specific conditions. Andersen, Sidenius, and Basu (2003) proposed the double Gaussian factor copula methodology to reduce the dimension of correlation parameters involving CDOs. The double student's t factor copula method proposed by Hull and White (2004) not only reduces computational inefficiency of non-homogeneous underlyings in the model of Andersen et al.(2003), but also has superior ability to capture the fat-tail properties of CDO underlyings than the double Gaussian copula model adopted by Li and Liao (2006).[3] Moreover, Hull and White (2004) utilized one-stage probability bucketing technique to scatter collateral loss distribution by selecting bucket numbers to construct conditional discrete loss distribution given the common factors. [4]

However, the approach of Hull and White (2004) can be modified as follows. First, homogeneous and constant recovery rate must be replaced by non-homogeneous random recovery rates to fit realistic recovery rates among different obligors. Second, because of constant intensity rate being inable to capture instantaneous credit spread dynamics of CDO obligors, constant intensity rate assumption needs to be replaced by stochastic intensity rate to capture the realistic time-varying default possibilities of obligors. Third, the one-stage probability bucketing technique proposed by Hull and White (2004) cannot be used directly to construct collateral loss distribution of CDO-Squared.

Recent investigations involving the estimation of random recovery rates of obligors include Hull and White (2004) and Andersen and Sidenius (2005), among others. However, the assumption of these studies that the recovery rates of all obligors of CDO are homogeneous does not fully accord with the market situation mentioned in Moody's report.[5] Additionally, these papers did not propose the calibration approach of parameters in their random recovery rate model. Hence, this study thus extends the homogeneous assumption of recovery rate to four types of random recovery rates to fit the realistic properties of all underlyings on CDS index. Second, this study utilizes recovery rates from the statistics of the Moody's and Credit Index proposed by Kim (1999) to obtain more accurate estimate of aggregate loss given default of CDO than the estimate used by Hull and White (2004).

Recent investigations involving the estimation of obligor default probabilities using stochastic intensity to value CDO include Duffie and Garleanu (2001), Schonbucher and Schubert (2001), Chen and Sopranzetti (2003), Voort (2004), and Li and Liao (2006).

[1] CDS index issued included CDX, iTraxx, and iBoxx indice.

[2] The tranche of CDO-Squared (called master CDO) is termed the "master tranche" and the bespoke tranche of other CDO (called inner CDO) is termed the "inner tranche" in this study.

[3] The non-homogeneous underlyings means large variation in principal between CDO underlyings.

[4] The common factors are referred to overall economic and sectoral conditions.

[5] The classification of recovery rates of corporate bonds is according to the Moody's report (2006). This report mentioned that obligor recovery rate is not only affected by macroeconomic factors, but also affected by whether obligor debt has been guaranteed.

However, there is no guarantee that the stochastic intensity processes mentioned above will be always positive. This study thus uses the CIR intensity process to describe the instantaneous credit spread of CDOs obligors.

Recent investigations estimating collateral loss distribution include Baheti, Mashal, Naldi, and Schloegl (2005), Hull and White (2006), and Zheng (2006). However, the computations involved in the model of Baheti et al. (2005) are inefficient when the underlyings of CDO-Squared collateral are more than five inner tranches. Regarding Hull and White (2006) and Zheng (2006), more stable numerical solutions are obtainable only when the simulation frequency approaches two hundred thousand times. Monte Carlo (MC) simulation involves a trade-off between accuracy and efficiency. This study thus proposes the factor copula model under the two-stage probability bucketing method as a more efficient and accurate method of approximating the collateral loss distribution of CDO-Squared than those mentioned above.

The Polaris Securities Group in Taiwan issued the first CDO-Squared using CDX 3∼7% tranche and 26 Taiwan inverse floating structured notes as collateral to resolve the problem of loss of inverse floating structured notes. For a hybrid portfolio containing these underlyings, the hybrid factor copula model is proposed, which involves the CIR stochastic intensity model and the KMV-Merton Model developed by Leland (2004) in a random recovery rate environment to price CDO-Squared. The remainder of this study is organized as follows: Section 2 presents the formulation of the hybrid factor copula model. Section 3 then employs the proposed model to value CDO-Squared issued by Polaris Securities Group in Taiwan. This section also presents the sensitivity analyses of the relevant model parameters. Finally, Section 4 presents conclusions.

2. The Valuation Framework of CDO-Squared

The following sections contain the following: (1) an introduction to the double student's t factor approach under a random recovery rate environment and relevant parameter calibration, (2) an illustration of the CIR stochastic intensity and the KMV model, (3) a presentation of the valuation model of CDO-Squared using the two-stage probability bucketing method.

2.1. Factor Copula Approach under Random Recovery Rate

Assuming N obligors the random vector of the default times is denoted as $(\tau_1, \ldots, \tau_N)$ and the joint distribution is denoted as F such that $F(t_1, t_2, \ldots, t_N) = Q(\tau_1 \leq t_1, \tau_2 \leq t_2, \ldots, \tau_N \leq t_N) = C(F_1(t_1), \ldots, F_N(t_N))$, where $Q(\cdot)$ represents a probability measure. $F_1(\cdot), \ldots, F_N(\cdot)$ represent the marginal distribution functions. Meanwhile, the $C(\cdot)$ function denotes the copula function of default times.

2.1.1. Factor Copula Model

Assuming the collateral portfolio of the k^{th} inner CDO contains N_k obligors, the loss amount generated by the i^{th} obligor from the k^{th} inner CDO are denoted as $l_{i,k}$ and the default time is denoted as $\tau_{i,k}$ in case of default before time T_j. The total portfolio loss of k^{th} the inner CDO experienced on $[0, T_j]$ is then

$$L_k(T_j) = \sum_{i=1}^{N_k} \ell_{i,k}(T_j) = \sum_{i=1}^{N_k} l_{i,k} \cdot 1_{\{\tau_{i,k}<T_j\}} = \sum_{i=1}^{N_k} l_{i,k}^{\max} \cdot (1 - R_{i,k}) \cdot 1_{\{\tau_{i,k}<T_j\}} \tag{1}$$

with expected value $E[L_k(T_j)] = \sum_{i=1}^{N_k} p_{i,k}(T_j) \cdot E(l_{i,k}|\tau_{i,k} < T_j)$, where $p_{i,k}(T_j) = E(l_{\{\tau_{i,k}<T_j\}})$ is the default probability before T_j, $\ell_{i,k(T_j)} = l_{i,k} \cdot 1_{\{\tau_{i,k}<T_j\}}$ is the default loss process before T_j, $l_{i,k} = l_{i,k}^{\max}(1 - R_{i,k})$ is the loss given default, $R_{i,k}$ is the recovery rate of i^{th} obligor in k^{th} CDO, and $l_{i,k}^{\max}$ presents the principal of i^{th} obligor on k^{th} CDO, $0 \le R_{i,k} \le 1$, $i = 1, \ldots, N_k$, $j = 1, \ldots, n$.

To efficiently capture fat-tail property of returns on assets, the return on asset and loss given default of the i^{th} obligor in the k^{th} inner CDO are defined as follows:[6]

$$r_{i,k} = a_{i,k} \cdot \sqrt{\left(\frac{v_1 - 2}{v_1}\right)} \cdot Z + \sqrt{(1 - a_{i,k}^2) \cdot \left(\frac{v_2 - 2}{v_2}\right)} \cdot \varepsilon_{i,k}, \tag{2}$$

$$l_{i,k}(z) = l_{i,k}^{\max} \cdot (1 - R_{i,k}(Z)) = l_{i,k}^{\max} \cdot (1 - \Phi(b_{i,k} + c_{i,k}Z + \xi_{i,k})). \tag{3}$$

where $r_{i,k}$ represents the return on asset of i^{th} obligor, Z represents common factor, $\varepsilon_{i,k}$ and $\xi_{i,k}$ represent the specific factors of the i^{th} obligor in the k^{th} inner CDO, $Z \sim t_{v_1}$, $\varepsilon_{i,k} \overset{iid}{\sim} t_{v_2}$, $\xi_{i,k} \overset{iid}{\sim} N(0, \sigma_{\xi_{i,k}}^2)$, $R_{i,k} \overset{iid}{\sim} N(0,1) \cdot \varepsilon_{i,k}$ and $\xi_{i,k}$ are both independent of Z, v_1 and v_2 are degrees of freedom in Z and $\varepsilon_{i,k}$ respectively, $\Phi(\cdot)$ is the Gaussian cumulative distribution function, and the default correlation $a_{i,k}$ can be estimated by the correlation between return on asset $r_{i,k}$ as in Equation (1) and common factor Z, $-1 \le a_{i,k} \le 1$, $i = 1, 2, \ldots, N_k$.[7]

Definition 1. The filtration $\Im_t^{i,k}$ denotes the information generated by the default intensity rates of the i^{th} obligor in the k^{th} inner CDO $\lambda_t^{i,k}$ up to time t.

Remark 2. Assuming Equation (2) with $Z \sim t_{v_1}$, $\varepsilon_{i,k} \overset{iid}{\sim} t_{v_2}$ and Z, $\varepsilon_{i,k}$ are independent. If $F_{\tau_{i,k}(0)} = \Phi(z_i) = \Pr(r_{i,k} \le \overline{r}_{i,k})$, where $F_{\tau_{i,k}(t)}(T_j) = 1 - E[e^{-\int_t^{T_j} \lambda^{i,k}(u)du} | \Im_t^{i,k}] = 1 - P_{\tau_{i,k}(t)}(t, T_j)$, $\overline{r}_{i,k}$ represents the barrier level of return of asset and $\Phi(z_i)$ is the Gaussian cumulative distribution of z_i, then the conditional default probability of the i^{th} obligor in the k^{th} CDO is

$$\begin{aligned} p_{\tau_{i,k}}(T_j, Z) &= \Pr(\tau_{i,k}(0) \le T_j | Z) \\ &= \Pr\left(\varepsilon_{i,k} \le \left(\frac{\Phi^{-1}(F_{\tau_{i,k}(0)}(T_j)) - a_{i,k} \cdot \sqrt{\left(\frac{v_1-2}{v_2}\right)} \cdot Z}{\sqrt{(1 - a_{i,k}^2)\left(\frac{v_2-2}{v_2}\right)}}\right) \middle| Z\right), \end{aligned} \tag{4}$$

[6] See Andersen et al. (2005).

[7] See, for example, Belkin et al. (1998).

where $i = 1, \ldots, N_k$, $j = 1, \ldots, n$.

Secondly, this study extends the approach of Andersen et al. (2005) to modeling four types of random recovery rates based on secured levels of obligor's liabilities. $\ell^m_{i,k}(T_j, Z) = l^m_{i,k}(Z) \cdot 1_{\{\tau_{i,k} < T_j\}}$ is thus defined as the discrete default process of the i^{th} obligor in the k^{th} inner CDO provided on Z before time T_j under a random recovery rate with m-possible outcomes. Moreover, $R^m_{i,k}$ $(= 1 - \ell^m_{i,k}(T_j, Z)/l^{\max}_{i,k})$ denotes the corresponding recovery rate of $\ell^m_{i,k}$. The realized recovery ratio of i^{th} obligor is $\overline{y}_{i,k} = 1 - (\overline{\ell}_{i,k}/l^{\max}_{i,k})$, and the realized loss $\overline{\ell}_{i,k}$ satisfies $0 \le \overline{\ell}_{i,k} \le l^{\max}_{i,k}$.

For numerical computation of factor copula given the random recovery rates, the first task is to establish a discrete collateral loss distribution via Proposition 3.

Proposition 3. *If the loss given default processes $l_{i,k}(Z)$ and $l^m_{i,k}(Z)$, and default process $1_{\{\tau_{i,k} < T_j\}}$ are independent conditional on Z, then the conditional default distributions given the random recovery rates are*

$$\Pr[\ell_{i,k}(T_j, Z) \le \overline{\ell}_{i,k}|Z] = 1 - \Pr[\tau_{i,k}(0) \le T_j|Z] \cdot \Pr[R_{i,k} \le \overline{y}_{i,k}|Z], \tag{5}$$

$$\Pr[\ell^m_{i,k}(T_j, Z) = \left(1 - \frac{s}{m}\right) \cdot l^{\max}_{i,k}|Z] \approx 1 - \Pr[\tau_{i,k}(0) \le T_j|Z] \cdot \Pr\left[R^m_{i,k} = \frac{s}{m}\middle|Z\right], \tag{6}$$

where $l_{i,k}$, $R_{i,k}$ are defined as Equation (3), $\Pr(R_{i,k} \le \overline{y}_{i,k}|Z) = \Phi[(\Phi^{-1}(\overline{y}_{i,k}) - u_{i,k} - b_{i,k} \cdot Z)/\sigma_{\xi_{i,k}}]$, $\Pr(R^m_{i,k} = \frac{s}{m}|Z) = \Pr(R_{i,k} \le (\frac{2s+1}{2m})|Z) - \Pr(R_{i,k} \le (\frac{2s-1}{2m})|Z)$, $s = 0, 1, \ldots, m-1$.

Proof. See Appendix 1-1.

2.1.2. Procedure for calibrating relevant parameters

The PROBIT model is used to estimate the default correlation and coefficient of the individual random recovery rate model.[8] The PROBIT equation is expressed as follows:

$$SDP_t = \Phi(X_{t-1}\beta + \varepsilon_t),$$

where SDP_t represents the default rate for investment grade corporate bonds, $\Phi(\cdot)$ represents Gaussian cumulative distribution, X_{t-1} represents macroeconomic variables at time $t-1$, and $\varepsilon_t \overset{iid}{\sim} N(0, 1)$.

Secondly, the credit index Z_t considered as the common factor Z in Equation (2) and (3) is defined as follows:

$$Z_t = \frac{\Phi^{-1}(SDP_t) - \mu_{\Phi^{-1}(SDP_t)}}{\sigma_{\Phi^{-1}(SDP_t)}}, \tag{7}$$

where $\mu_{\Phi^{-1}(SDP_t)}$ and $\sigma_{\Phi^{-1}(SDP_t)}$ represent the mean and standard deviation, respectively, of the inverse Gaussian of default rate.

[8] For detail, see Kim (1999) and Belkin, Suchower, and Forest (1998).

Finally, if the random recovery rate process, $R_{i,k}(Z)$ is expressed in Equation (3), then the default correlation $a_{i,k}$ can be estimated from the correlation between asset return $r_{i,k}$ in Equation (2) and credit index Z_t in Equation (7).[9] The coefficients $(b_{i,k}, c_{i,k}, \sigma_{\xi_{i,k}})$ of random recovery rate model in Equation (3) can also be estimated using the PROBIT method. For detail, see Brigo and Alfonsi (2005).

2.2. CIR Intensity and KMV-Merton Model

2.2.1. CIR Intensity Model

It is important for CDO-squared investors to capture the instantaneous credit spread dynamics of obligors in all inner CDO collaterals. Particularly, if the underlyings of CDO-squared collateral include bespoke tranches of the CDS index, then CDO-squared investors will benefit via the Greeks of CIR stochastic intensity process which describes their instantaneous credit spread trends of the underlying obligors in CDS index.[10] The intensity rate is thus set to be

$$\lambda_t^{i,k} = x_t^{\alpha_{i,k}} + \varphi_{i,k}(t;\alpha_{i,k}), \quad t \geq 0, \tag{8}$$

where $\varphi_{i,k}$ denotes a deterministic function that depends on $\alpha_{i,k} = (\mu_{i,k}, \theta_{i,k}, \sigma_{i,k}, x_0^{\alpha_{i,k}})$ and is integrable on closed intervals. $\mu_{i,k}$ denotes the adjustment speed of the intensity process $x_t^{\alpha_{i,k}}$, while $\theta_{i,k}$ represents the long-term average level of $x_t^{\alpha_{i,k}}$, σ is the standard deviation of $x_t^{\alpha_{i,k}}$, and $x_0^{\alpha_{i,k}}$ denotes an initial value of $x_t^{\alpha_{i,k}}$ and is selected from $x_0^{\alpha_{i,k}} = \lambda_0^{i,k} - \varphi_{i,k}(0;\alpha_{i,k})$.

The CIR intensity process is set as follows:

$$dx_t^{\alpha_{i,k}} = \mu_{i,k}\cdot(\theta_{i,k} - x_t^{\alpha_{i,k}})dt + \sigma_{i,k}\cdot\sqrt{x_t^{\alpha_{i,k}}}dW_t^{i,k}, \tag{9}$$

where $\mu_{i,k}$, $\theta_{i,k}$, $\sigma_{i,k}$, $x_0^{\alpha_{i,x}}$ are positive constants and $W_t^{i,k}$ is a standard Brownian motion.

To calibrate the relevant parameters $\alpha_{i,k} = (\mu_{i,k}, \theta_{i,k}, \sigma_{i,k}, x_0^{\alpha_{i,k}})$ of Equation (9), it is necessary to extract the implied intensity rate $\lambda_t^{i,k(implied)}$ corresponding to the CDS market quotes for increasing maturity T_j. First, the discounted value of the CDS contract is defined as follows:

$$\begin{aligned}
&V_{CDS}^{i,k}(t, \aleph, T_j, S_{i,k}, L_{GD}^{i,k}, \Gamma^{i,k(implied)}(\cdot)) \\
&\quad = 1_{\{\tau_{i,k}>t\}}\cdot\Bigg[S_{i,k}\cdot\int_t^{T_j} B(t,u)\cdot(T_{\beta_{i,k}(u)-1} - u)\cdot d_u\Big(e^{-(\Gamma^{i,k(implied)}(u)-\Gamma^{i,k(implied)}(t))}\Big) \\
&\qquad + \sum_{j=\beta_{i,k}(t)}^{n} B(t,T_j)\cdot S_{i,k}\cdot m\cdot e^{(\Gamma^{i,k(implied)}(t)-\Gamma^{i,k}(T_j))} \\
&\qquad + L_{GD}\cdot\int_t^{T_j} B(t,u)\cdot d_u\Big(e^{-(\Gamma^{i,k(implied)}(u)-\Gamma^{i,k(implied)}(t))}\Big)\Bigg],
\end{aligned} \tag{10}$$

[9] See, for example, Belkin et al.(1998).

[10] For detail, see Brigo and Alfonsi (2005).

where $t \in [T_{\beta_{i,k}(w)-1}, T_{\beta_{i,k}(w)}]$, $V^{i,k}_{CDS}(t, \aleph, T_j, S_{i,k}, L^{i,k}_{GD}, \Gamma^{i,k}(\cdot))$ represents the discounted value of the CDS contract at time t, $S_{i,k}$ denotes the fixed premium rate of the CDS contract where payments are received at times $\aleph = \{T_1, T_2, \ldots, T_n\}$, $m(\stackrel{\Delta}{=} T_j - T_{j-1})$ is the payment period, $L^{i,k}_{GD}$ is the loss given default, and $B(t, T_j)$ is the time t price of a zero-coupon bond maturing at T_j, and $\Gamma_t^{i,k(implied)}(t) \equiv \int_{-\infty}^{t} \lambda_t^{i,k(implied)}(s)ds$.[11]

Furthermore, because the intensity rate $\lambda_t^{i,k(implied)}$ shifts with changes in credit rating, this study assumes that $\lambda_t^{i,k(implied)}$ is a deterministic time-varying function. The implied default probabilities are then obtained from market quotes of CDS contract and the parameter $\alpha_{i,k} = (\mu_{i,k}, \theta_{i,k}, \sigma_{i,k}, x_0^{\alpha_{i,k}})$ is calibrated. Let $\lambda_t^{i,k(implied)}$ be a piecewise-constant and right-continuous function of time, defined as follows:

$$\lambda_t^{i,k(implied)} = \lambda_{T_j}^{i,k(implied)} \qquad \forall\, t \in [T_{j-1}, T_j), \tag{11}$$

where $\lambda_{T_j}^{i,k(implied)}$ is constant for $i = 1, \ldots, N_k$, $j = 1, 2, \ldots, n$, $k = 1, \ldots, M$.

By substituting $\lambda_t^{i,k(implied)}$ in Equation (11) into Equation (10), $\lambda_{T_j}^{i,k(implied)}$ for increasing maturity T_j can be bootstrapped from the term structure of the CDS market quotes.[12] By iteratively solving $\lambda_{T_j}^{i,k(implied)}$ from Equation (10), a set of equations for different T_j can be set as follows:

$$\begin{aligned}
&V^{i,k}_{CDS}\Big(0, \aleph, T_j, S^M_{1y}, L^{i,k}_{GD}, \lambda^{i,k}_{T_{1/m}} = \lambda^{i,k}_{T_{2/m}} = \cdots = \lambda^{i,k}_{T_1} =: \lambda^{i,k(implied)}_{T_1}\Big) = 0\\
&V^{i,k}_{CDS}\Big(0, \aleph, T_j, S^M_{2y}, L^{i,k}_{GD}, \lambda^{i,k(implied)}_{T_1}; \lambda^{i,k}_{T_{(m+1)/m}} = \lambda^{i,k}_{T_{(m+2)/m}} = \cdots = \lambda^{i,k}_{T_2} =: \lambda^{i,k(implied)}_{T_2}\Big) = 0\\
&\quad\vdots\\
&V^{i,k}_{CDS}\Big(0, \aleph, T_j, S^M_{ny}, L^{i,k}_{GD}, \lambda^{i,k(implied)}_{T_1}; \ldots; \lambda^{i,k(implied)}_{T_n-1}; \lambda^{i,k}_{T_{(m\cdot T_n - m+1)/m}} = \cdots\\
&\qquad = \lambda^{i,k}_{(m\cdot T_n - 1)/m} = \lambda^{i,k}_{T_n} =: \lambda^{i,k(implied)}_{T_n}\Big) = 0
\end{aligned} \tag{12}$$

Proposition 4. *Given the dynamics of $x_t^{\alpha_{i,k}}$ in Equation (9), the survival probability at time t given maturity at T_j is*

$$\begin{aligned}
P^{CIR}(t, T_j, x_t^{\alpha_{i,k}}; \alpha_{i,k}) &= E\Big(e^{-\int_t^{T_j} x^{\alpha_{i,k}}(u)du} \Big| \Im_t^{i,k}\Big)\\
&= A(t, T_j; \alpha_{i,k}) \exp\Big\{ - B(t, T_j; \alpha) x_0^{\alpha_{i,k}} \Big\}
\end{aligned} \tag{13}$$

where

$$A(t, T_j; \alpha_{i,k}) = \left[\frac{2\cdot v_{i,k}\cdot e^{(\mu_{i,k}+v_{i,k})(T_j-t)/2}}{2\cdot v_{i,k} + (\mu_{i,k}+\nu_{i,k})(e^{(T_j-t)\cdot\nu_{i,k}}-1)}\right]^{2\mu_{i,k}\cdot\theta_{i,k}/\sigma^2_{i,k}}, \quad \nu_{i,k} = \sqrt{k^2_{i,k} + 2\sigma^2_{i,k}},$$

[11] For detail see Brigo and Alfonsi (2005).

[12] Based on the market quote convention, CDS quotes on Bloomberg are limited to annual quotes while the CDS contract duration exceeds one year.

$$B(t,T_j;\alpha_{i,k}) = \frac{2(e^{(T_j-t)\cdot\nu_{i,k}}-1)}{2\nu_{i,k}+(\mu_{i,k}+\nu_{i,k})(e^{(T-t)\cdot\nu_{i,k}}-1)}.$$

Proof. See Cox, Ingersoll, and Ross (1985).

Proposition 5. *Defining* $\lambda_t^{i,k(implied)}$ *as Equation* (11), *then* $\varphi_{i,k}(t,\alpha_{i,k})$ *of Equation* (8) *can be derived as*

$$\varphi_{i,k}(t,\alpha_{i,k}) = \lambda_{T_j}^{i,k(implied)} + \frac{d}{ds}\ln\left(P^{CIR}(0,s,x_0^{\alpha_{i,k}})\right)\Big|_{s=t} \quad \forall\, T_{j-1}\le t<T_j, \tag{14}$$

where $P^{CIR}(0,u,x_0^{\alpha_{i,x}},\alpha_{i,k})$ *denotes the survival probability for CIR model in Equation* (13), *while* $\lambda_{T_j}^{i,k(implied)}$ *for increasing maturity* T_j *can be bootstrapped from the term structure of CDS market quotes in* $\lambda_{T_j}^{i,k(implied)}$ *of Equation* (11), $j=1,2,\ldots,n$.

Proof. See Appendix 1-2.

Because the markets of CDS contracts are not liquid for all maturities, the most liquid maturity $J^*_{i,k}$ of the i^{th} obligor in the k^{th}inner CDO is sifted from all maturities through the smallest difference between bid price $S^{Bid}_{i,k}(T_j)$ and ask price $S^{ask}_{i,k}(T_j)$.[13] Second, proposition 6 is used to find $\alpha^*_{i,k}$ such that the intensity rate $\lambda_t^{i,k}(\alpha^*_{i,k})$ is positive.

Proposition 6. *Assuming correlation between the interest and the intensity rates is equal to zero. If relevant parameters* $\alpha^*_{i,k}=(\mu^*_{i,k},\theta^*_{i,k},\sigma^*_{i,k},x_0^{\alpha_{i,k}})$ *of Equation* (9) *minimize* $\int_0^{J^*_{i,k}}\varphi^2_{i,k}(u,\alpha_{i,k})du$ *and satisfy the following constraints:* (1) $\int_0^{J^*_{i,k}}\varphi_{i,k}(s,\alpha_{i,k})ds>0$ *and* (2) $\varphi_{i,k}(s,\alpha_{i,k})>0$, $0\le s\le J^*_{i,k}$, *then the intensity rate* $\lambda_t^{i,k}(\alpha^*_{i,k})$ *is positive.*[14]

Proof. See Brigo and Mercurio (2006).

Finally, we substitute $1-P^{CIR}(t,T_j,\alpha^*_{i,k})$ in Equation (13) into $\Pr(\tau_{i,k}\le T_j|Z)$ of Equation (6) to calculate the conditional default probability of obligors in the CDS index.

2.2.2. KMV-Merton Model

Owing to the lack of the CDS market quotes for certain structured notes in this hybrid portfolio, the KMV-Merton Model is used to calculate the default probability of obligors via financial statements. The probability can be estimated through the following steps:

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[13] $J^*_{i,k}=\arg\min_j\{|S^{Bid}_{i,k}(T_j)-S^{ask}_{i,k}(T_j)|\}$.

[14] see Brigo and Mercurio (2006), P.789-794.

(1) The distance to default $DD^{i,k}(t,T)$ can be calculated as follows:

$$DD^{i,k}(t,T_j) = \frac{\ln\left(\frac{V_A^{i,k}}{V_{B-KMV}^{i,k}}\right) + \left(\mu^{i,k} - \delta^{i,k} - (\sigma_A^{i,k})^2/2\right)\cdot(T_j - t)}{\sigma_A^{i,k}\cdot\sqrt{T_j - t}},$$

where $\mu_{i,k}$ $(= r + \lambda^{i,k})$ is an estimate of the expected annual return of firm assets, $\lambda^{i,k}$ denotes the asset risk premium, $\delta^{i,k}$ represents the fractional payout rate on assets (to both debt and equity), and T is the corporate bond maturity. Furthermore, V_{B-KMV} represents the default boundary of the obligor which equals book value of short term liabilities plus half of long term liabilities, r is the instantaneous risk-free rate, and $V_A^{i,k}$ and $\sigma_A^{i,k}$ denote the initial asset value and the volatility of the i^{th} obligor, respectively, calculated from KMV-Merton model.[15]

(2) $\Phi(-DD^{i,k}(t,T_j))$ is substituted into $\Pr(\tau_{i,k} \le T_j|Z)$ of Equation (6) to calculate the conditional default probability of obligors without CDS quotes.

2.3. The Valuation model of CDO-Squared under Probability Bucketing method

2.3.1. Contingent Payoff of inner CDO and master CDO (CDO-Squared)

The loss of the $p(k)^{\text{th}}$ bespoke tranche within $[L_{p(k)}, U_{p(k)}]$ for the k^{th} inner CDO until time T_j is described as $IL_{p(k)}(T_j)$:

$$\begin{aligned} IL_{p(k)}(T_j) &= (L_k(T_j) - L_{p(k)})^+ - (L_k(T_j) - U_{p(k)})^+ \\ &= \max\Big(\min(L_k(T_j), U_{p(k)}) - L_{p(k)}, 0\Big), \\ IR_{p(k)}(T_j) &= \Big(U_{p(k)} - L_{p(k)} - L_{p(k)}(T_j)\Big)^+, \end{aligned} \tag{15}$$

where $L_{p(k)}$, $U_{p(k)}$ and $IR_{p(k)}(T_j)$ represent the monetary values of lower, upper attachments, and the recovery value of the $p(k)^{\text{th}}$ tranche in the k^{th} inner CDO up to time T_j.

Assuming that the collateral of the master CDO is a portfolio comprising M inner tranches issued from various inner CDOs, then the loss of the q^{th} bespoke tranche within $[L_q, U_q]$ for the CDO-Squared or master CDO can be described as $ML_q(T_j)$:

$$\begin{aligned} ML_q(T_j) &= \max\left(\min\left(\sum_{k=1}^{M} IL_{p(k)}(T_j), U_q\right) - L_q, 0\right), \\ MR_q(T_j) &= \left(\sum_{k=1}^{M}\Big(U_{p(k)} - L_{p(k)}\Big) - ML_q(T_j)\right), \end{aligned} \tag{16}$$

[15] As leverage increases from zero, the equity risk premium will increase. An equity premium is about 6% when the average firm has about 35% leverage as assumed by Leland (2004). Payout rate assumption is 6% as assumed by Huang and Huang (2003). Besides, the calculating formulas of $V_A^{i,k}$ and $\sigma_A^{i,k}$ are taken from Rutkowski and Bielecki (2002), pp.51-57.

where L_q, U_q, and $MR_q(T_j)$ represent the monetary values of the lower and upper attachments, and the recovery value of the q^{th} tranche in the CDO-Squared until time T_j.

2.3.2. Loss distribution under two-stage probability bucketing method

The algorithm proposed by Hull and White (2004) accurately and efficiently approximates the distribution of collateral loss $L_k(T_j)$. However, this algorithm cannot deal with the distribution of collateral loss $\sum_{k=1}^{M} IL_{p(k)}(T_j)$. Therefore, this study proposes the two-stage probability bucketing method for establishing the probability distribution of CDO-Squared.[16] The procedure of the two-stage probability bucketing method is included in Appendix 2.

Finally, the unconditional default probability, $\left\{Q_M^{(h)}(T_j)\right\}_{h=1}^{v}$, and loss given default, $\left\{A_M^{(h)}(T_j)\right\}_{h=1}^{v}$, in the h^{th} bucket can be calculated by numerical integration of Gaussian Quadrature:[17]

$$\begin{aligned} Q_M^{(h)}(T_j) &= \int_{-\infty}^{\infty} Q_M^{(h)}(T_j, Z) \cdot g(Z) dZ, \\ A_M^{(h)}(T_j) &= \int_{-\infty}^{\infty} A_M^{(h)}(T_j, Z) \cdot g(Z) dZ, \end{aligned} \tag{17}$$

where the conditional default probability, $\left\{Q_M^{(h)}(Z, T_j)\right\}_{h=1}^{v}$, and loss given default, $\left\{A_m^{(h)}(Z, T_j)\right\}_{h=1}^{v}$, can be obtained from the algorithms of Appendix 2.

2.3.3. Fair Credit Spread of CDO-Squared

This study considers the q^{th} tranche of CDO-Squared that receives default payment only if the loss of the underlying asset pool lies between L_q and U_q ($L_q < U_q$), L_q represents the highest deductible loss of the q^{th} master tranche's issuer, and U_q is the highest issue of the q^{th} tranche, where $0 \le L_q \le U_q \le \sum_{k=1}^{M} (U_{p(k)} - L_{p(k)})$.

(1) Expected Loss ($EL_q(T_j)$)

$$EL_q(T_j) = \sum_{h=1}^{v} Q_M^{(h)}(T_j) \cdot \max(\min(A_M^{(h)}(T_j), U_q) - L_q, 0),$$

where v denotes the number of buckets in the loss distribution of master CDO and $A_M^{(h)}(T_j)$ represents the loss given default in the h^{th} bucket of the master CDO at payment dates T_j.

[16] The detailed algorithm can be requested from the authors.
[17] For detail, see Press et al. (2007).

(2) Fair Credit Spread (S_q)

$$S_q = \frac{\sum_{j=1}^{n} D(T_j) \cdot \Delta_j \cdot \{U_q - L_q - EL_q(T_j)\}}{\sum_{j=0}^{n} D(T_j) \cdot MR_q(T_j)},$$

where S_q denotes the fair credit spread of the q^{th} tranche within $[L_q, U_q]$, $D(T_j)$ represents the discount factor at time T_j, and $\Delta_j = T_j - T_{j-1}$ represents the duration between two payment dates.

3. Empirical Results

3.1. The case of Polaris Security Group in Taiwan: CDO-Squared

Polaris Securities Group issued the first CDO-Squared in Taiwan on April 18, 2006. This issue totaled NT 12,225,000,000. The underlying assets include 26 Taiwan Corporate Bonds and 3∼7% tranche of USD 150,000,000 Dow Jones CDX NA Investment Grade Series 5 (D.J.CDX NA IG 5). The issue matures on February 18, 2011. Thus, there are a total of 20 periodic payment dates. Additionally, the asset pool includes 26 corporate bonds priced in NTD and issued in Taiwan and a 3%∼7% tranche of D.J.CDX NA IG 5 which is priced in USD.[18] The originator signed a forward contract to hedge against exchange rate risk.

3.2. Economic and credit model parameters calibrated

To build credit index and factor copula model, this study adopts relevant US economic variables to model the default rate of Moody's investment grade bonds using PROBIT.

Table 1. Coefficients of PROBIT model.

Economic variable	Coefficient	p-value
1. Intercept	3.2463*	2.53E-15*
2. Manufactory New Order- USA.	-3.5e^{-6*}	9.49E-05*
3. Producer Price Index-USA (2000 = 100)	-0.018*	6.87E-09*
4. Capacity Utility Rate.-U.S.A. (1997 = 100)	-0.02068*	3.43E-05*
5. American Composite Stock Index	-0.4799*	0.004944*

Note: $R^2 = 96.1\%$. "*" indicates statistical significance.

Table 1 shows that the R^2 value and the significant negative coefficient indicates that there is a negative correlation between the macroeconomic variable and the default

[18] Taiwan corporate bonds include Taiwan Cooperative Bank, Taiwan Cogen Power Corporation, Taiwan mobile, Walsin Lihwa Corporation, Formosa Chemical & Fiber Corporation, Bank SinoPac, Far Eastern International Bank, Taipei Fubon Bank, Chinatrust Financial Holding Co. Ltd, Taiwan High Speed Rall, Farmer Bank, Cathay United Bank, Shin Kong Financial Holding CO. LTD, and so on.

Table 2. Coefficients of random recovery rate model.

Debt type		Intercept (b_i)	slope (c_i)	Sigma ($\sigma_{\xi_{i,k}}$)
Senior-secured	(R_1)	-0.23223	-0.23319	0.29124
Senior-unsecured	(R_2)	0.304426*	-0.30065*	0.30696
Senior-subordinator	(R_3)	-0.51403	-0.02583	0.28475
Subordinator	(R_4)	0.079251*	-1.07934*	0.51891

Note: R^2 (in R_1)=31%, R^2 (in R_2)=40%, R^2 (in R_3)=32%, R^2 (in R_4)=53%. "*" are statistical significance.

probability of Moody's investment grade corporate bonds. That is, the decrease of credit index Z_t defined in Equation (7) represents an improvement in overall credit rating. Next, the coefficient a_i is estimated based on the correlation between stock S_t and credit index Z_t using Equation (7).[19] This study thus finds that the default correlations of 28 ticks exceed zero ($a_i > 0$) and is close to zero. This means that the returns on equities (ROE) of a few obligors are slightly stimulated by downgrade of the credit ratings. The correlations of the other 123 ticks range from -0.01 to -0.45. The phenomenon shows that most ROE's are significantly improved by upgrade of the credit ratings.

Table 2 reveals that the relationship between credit index and recovery rates defined by Equation (3) for different types of secured debts. The significant negative coefficients of slopes (c_i) indicate that a negative correlation between recovery rate and credit index, implying that as overall credit ratings gradually upgrade, the recovery rates will rise. Additionally, both degrees of freedom in double student't factor copula model are selected to be 4.[20] Finally, coefficients $(a_{i,k}, b_{i,k}, c_{i,k}, \sigma_{\xi_{i,k}})$ can be used to value CDO-Squared.

3.3. Sensitivity analyses

Sensitivity analyses of relevant model parameters help CDO-Squared investors make investment decisions relating to possible shifts in credit spread among various tranches.

3.3.1. Default correlation (a_i)

In Table 3, the results of scenarios I to II demonstrate that the credit spread of the. 0∼3% tranche of CDX index decreases with increased absolute default correlation while the credit spreads of other tranches increase. This occurs because higher correlation implies increased possibility of different companies all either defaulting or surviving. Thus, the possibility of loss suffered by investors holding 0∼3% tranche is reduced, while the loss possibility of other tranche is increased.

Scenario III compares the proposed method with the double student's t factor copula method of Hull and White (2004). The proposed model employing CIR intensity rate,

[19] The stock S_t includes 125 CDS issues contained in CDX index and 26 Taiwanese issues. The returns on equities and CDS quotes are obtained from Bloomberg. The returns on equities for Taiwanese issues are obtained from TEJ. Moreover, the data on corporate recovery rates are obtained from the Moody's report.

[20] The selection of degrees of freedom is based on Hull and White (2004).

Table 3. Sensitivity Analysis of Default Correlation (a_i).

Basis point	**0%∼3%**	**3%∼7%**	**7%∼10%**	**10%∼15%**	**15%∼30%**
1.Baseline model (CIR intensity rate, random recovery rate, and double-t copula model)	3,399	439	175	76	21
2.scenario I $a = 0.15$ for all cases	3,076	556	108	75	12
3.scenario II $a = 0.45$ for all cases	1,451	376	197	103	30
4.scenario III (Hull and White Model)[21]	3,404	455	196	89	18

random recovery rate, and double student's t factor copula methods is found to produce lower fair credit spreads for all tranches than the Hull and White (2004) model with the exception of the 15%∼30% tranche. The model assumption of constant hazard or recovery rates thus is unreasonable since daily market quotes with different maturities exist in CDS market, and thus credit spread information can be obtained from the market.

3.3.2. Credit mean-reverting speed ($\mu_{i,k}$) of the CIR intensity process

The results listed in Table 4 indicate that if $\mu_{i,k}$ of all obligors increases by 20% from the baseline, the credit spreads of all tranches will decrease with the credit spread of the 0∼3% tranche by up to 61 basis points. Conversely, if $\mu_{i,k}$ reduces by 20% relative to the baseline, then the credit spread of the 0∼3% tranche will increase up to 132 basis points. Consequently, represents the control ability of the obligor. Greater value of $\mu_{i,k}$ indicates larger improvement of obligor internal control. In contrast, lower value of $\mu_{i,k}$ means a lack of attention to obligor's internal control.

Table 4. Sensitivity Analysis of $\mu_{i,k}$ parameter of index trnache.

basis point	0∼3%	3∼7%	7∼10%	10∼15%	15-30%
-20%	3,521	473	184	91	24
-10%	3,450	459	179	86	23
Baseline	3,399	439	175	76	21
+10%	3,359	430	154	66	19
+20%	3,338	421	140	61	18

3.3.3. CDO-Squared valuation and analysis

Investor losses on tranches result from the collateral losses of the master CDO, which in turn are dependent on the underlying losses of the inner CDOs. Investors in CDO-Squared thus must track the feature of the credit obligors of inner CDOs to avoid unexpected losses. To achieve tranche sensitivity of 3∼7%, the results reveal that credit spread does not change significantly with changes in $\mu_{i,k}$. In Table 5, the individual

[21] The parameters of double student's t factor copula method that Hull and White (2004) adopted are: default correlations of all underlying equal 0.3, all intensity rates equal 0.15, all recovery rates equal 0.4, and the number of buckets is 500 (equal-length).

Table 5. Sensitivity analysis of parameter of CDO-Squared.

basis point	0∼4.4%	4.4∼9%	9∼18%	18∼33%	33∼100%
-20%	3,953.5	676.4	203.6	131	27
-10%	3,953	676.3	203.4	131	27
Baseline	3,952	676	203	131	27
+10%	3,950	674	202.5	131	27
+20%	3,949	673.6	202.4	131	27

credit spread of the tranches of CDO-Squared remains almost unchanged with changes in $\mu_{i,k}$. Investors in CDO-Squared thus do not consider internal control improvement of the original underlying obligors of individual CDO if the tranche pool of CDO-Squared contains no equity tranche of CDO.

4. Conclusion

CDO issuance is associated with numerous benefits. For issuing institutions, CDO issuance not only achieves regulatory capital relief or higher asset returns, but also improves liquidity via capital redeployment. Meanwhile, investors can use the CDOs to diversify investment risk. From the perspective of capital markets, CDOs can activate bond and loan markets as a result of transferring default risk of bonds and loans to investors from the CDOs issuer. An example of a CDO-Squared issued by Polaris Securities Group in Taiwan is presented and valued using the proposed models.

Therefore, this study thus obtains the following results. Default dependence is found to be important in managing credit risk and valuing credit derivatives. This work finds that the credit spread of 0∼3% tranche decreases with increasing absolute default correlation. However, the credit spreads of other tranches increase. This phenomenon occurs mainly because the higher increment of correlation implies that the possibilities that all companies either default or survive increase simultaneously. Consequently, the possibility of investors holding 0∼3% tranche suffering losses is reduced. In contrast, the probability of losses for other tranches is increased. Subsequently, larger increase in $\mu_{i,k}$ is associated with improved internal control of the obligor. Smaller $\mu_{i,k}$ indicates inattentive internal control of the obligor.

Compared with the double student's t factor copula method developed by Hull and White (2004), we find that the proposed model using CIR intensity rate, random recovery rate of various secured-level brackets, and double student's t copula achieves fairer credit spreads of tranches than the Hull and White (2004) model. The stochastic assumption of positive mean-reverting hazard rate and recovery rates of various classifications are more realistic than the assumption of constant ones since daily market quotes with different maturities exist in CDS market to expose obligor credit spread information via market trading. Finally, investors in CDO-Squared do not consider internal control improvement of original underlying obligors of individual CDO if the pool of tranches of CDO-Squared does not contain equity tranche of CDO.

Appendix 1.

1. Proof of Proposition 3

$$\begin{aligned}\Pr(\ell_{i,k} \leq \overline{\ell}_{i,k}|Z) &= \Pr(l_{i,k} \leq \overline{\ell}_{i,k}|Z) \cdot \Pr(\tau_{i,k} \leq T_j|Z) + \Pr(l_{i,k} = 0|Z) \cdot \Pr(\tau_{i,k} > T_j|Z),\\ &= 1 - \Pr(\tau_{i,k} \leq T_j|Z) \cdot \Phi\left[\frac{\Phi^{-1}(\overline{y}_{i,k}) - \mu_{i,k} - b_{i,k} \cdot Z}{\sigma_{\xi_{i,k}}}\right].\end{aligned}$$

$$\begin{aligned}\Pr(\ell^m_{i,k} = (1 - s/m) \cdot l^{\max}_{i,k}|Z) &\triangleq 1 - \Pr(\tau_{i,k} \leq T_j|Z) \cdot \left[\Pr\Big(R_{i,k}(Z) \leq \frac{2s+1}{2m}\Big)\right.\\ &\left.-\Pr\Big(R_{i,k}(Z) \leq \frac{2s-1}{2m}\Big)\right],\\ &= 1 - \Pr(\tau_{i,k} \leq T_j|Z) \cdot \Pr\Big(R^m_{i,k} = \frac{s}{m}|Z\Big)\end{aligned}$$

where $\Pr(R^m_{i,k} = \frac{s}{m}|Z) \triangleq \Phi[\Phi^{-1}(\frac{2s+1}{2m}) - b_i - c_i Z)/\sigma_{\xi_i}] - \Phi[\Phi^{-1}(\frac{2s-1}{2m}) - b_i - c_i Z)/\sigma_{\xi_i}]$.

2. Proof of Proposition 4

The risk-neutral survival probability of i^{th} obligor under deterministic intensity rate is

$$\begin{aligned}P(\tau_{i,k} > t) = e^{-\int_0^t \lambda^{i,k}_{T_s} ds} &= e^{-\sum\limits_{j=1}^{n}[1_{\{T_{j-1} \leq t < T_j\}} \cdot [\lambda^{i,k(implied)}_{T_{j-1}} + \lambda^{i,k}_{T_j} \cdot (t-j+1)]]}\\ &= \prod_{j=1}^{n} e^{-1_{\{T_{j-1} \leq t < T_j\}} \cdot \Gamma^{i,k(implied)}_{T_j}(t)}.\end{aligned} \tag{A.1}$$

Also, we can derive the risk-neutral survival probability under CIR intensity rate:

$$P(\tau_{i,k} > t) \triangleq E(e^{-\int_0^t \lambda^{i,k}(s) ds}|\Im^{i,k}_t) = E(e^{-\Gamma^{i,k}(t)}|\Im^{i,k}_t). \tag{A.2}$$

In addition, if the correlation between interest rate and default rate processes is zero, the price of CDS under stochastic intensity model is the same as in deterministic intensity $\lambda^{i,k(implied)}$ model.[22] By substituting $\lambda^{i,k}_t$ of Equation (8) into $\Gamma^{i,k}(t)$ of Equation (A.2), $e^{-\Gamma^{i,k(implied)(t)}_{T_j}}$ of Equation (A.1) can also be expressed as:

$$e^{-\sum\limits_{j=1}^{n}[1_{\{T_{j-1} \leq t < T_j\}} \cdot [\lambda^{i,k(implied)}_{T_{j-1}} + \lambda^{i,k}_{T_j} \cdot (t-j+1)]]} = E(e^{-\int_0^t x_s^{a_{i,k}} ds}|\Im_t) \cdot e^{-\int_0^t \varphi_{i,k}(s,\alpha_{i,k}) ds} \tag{A.3}$$

Next, by substituting $P^{CIR}(0, t, x^{\alpha_{i,k}}_t, \alpha_{i,k})$ of Equation (13) into $E(e^{-\int_0^t x_s^{\alpha_{i,k}} ds})$ in Equation (A.3), we can obtain the equality of the parameters $\varphi_{i,k}(t, \alpha)$:

$$\varphi_{i,k}(t, \alpha_{i,k}) = \lambda^{i,k(implied)}_{T_j} + \frac{d}{ds} \ln\Big(P^{CIR}(0, s, x^{\alpha_{i,k}}_0, \alpha_{i,k})\Big)\Big|_{s=t} \quad \forall\ T_{j-1} \leq t < T_j.$$

[22] For detail, see Brigo and Alfonsi (2005).

Appendix 2.

This study proposes the two-stage probability bucketing method for establishing the probability distribution of CDO-Squared collateral loss.[23]

(1) First stage

To establish the discrete default probability distribution in the k^{th} inner CDO, potential losses of the k^{th} inner CDO are divided into the following individual ranges: $\{0, b_{k,0}\}, \{b_{k,0}, b_{k,1}\}, \ldots, \{b_{k,w_k-1}, \infty\}$. $\{0, b_{k,0}\}$ is defined as the 0^{th} bucket of k^{th} inner CDO, $\{b_{k,h-1}, b_{k,h}\}$ as the h^{th} bucket $(1 \le h \le w_k - 1)$, and $\{b_{k,w_k-1}, \infty\}$ as the w_k^{th} bucket. That is, the first stage aims to estimate the default probability that the total loss lies in the h^{th} bucket of the k^{th} inner CDO for all h. Consequently, the following variables are defined to establish the default probability distribution: $IP_{i,k}^{(h)}(T_j, Z)$ denotes the probability that the collateral cumulative loss of k^{th} inner CDO at time T_j lies in the h^{th} bucket as the credit event of the i^{th} obligor occurs given Z; $IA_{i,k}^{(h)}(T_j, Z)$ represents the average cumulative loss conditional on the cumulative loss of the k^{th} inner CDO being in the h^{th} bucket as the credit event of the i^{th} obligor occurs given Z.

$IP_{i,k}^{(h)}(T_j, Z)$ and $IA_{i,k}^{(h)}(T_j, Z)$ are calculated iteratively by considering an additional obligor in the collateral pool from nothing until there are N_k obligors in the collateral of the k^{th} CDO.

After considering all N_k obligors of the k^{th} CDO, the conditional total loss distribution on w_k buckets is scattered over the following points $\{IA_{N_k}^{(h)}(T_j, Z)\}_{h=1}^{w_k}$ and its corresponding conditional default probabilities are $\{IP_{N_k}^{(h)}(T_j, Z)\}_{h=1}^{w_k}$. Therefore, the default loss of the h^{th} bucket in the k^{th} obligor of master CDO thus is defined as $D_{N_k}^{(h)}(T_j, Z)$ by Equation (1.4). The $InB_k^*(T_j, Z)$ bucket that lies in $[L_{p(k)}, U_{p(k)}]$ and the $OvB_k^*(T_j, Z)$ bucket that lies in $(U_{p(k)}, \sum_{i=1}^{N_k} l_{i,k}^{\max}]$ can be sifted from $D_{N_k}^{(h)}(T_j, Z)$ in the Equation (A.5).

$$D_{N_k}^{(h)}(T_j, Z) = \max\left(\min\left(IA_{N_k}^{(h)}(T_j, Z), U_{p(k)}\right) - L_{p(k)}, 0\right), \tag{A.4}$$

$$InB_k^*(T_j, Z) = \{h | 0 < D_{N_k}^{(h)}(T_j, Z) < (U_{p(k)} - L_{p(k)}),\ h = 1, \ldots, w_k\}, \tag{A.5.1}$$

$$OvB_k^*(T_j, Z) = \{h | D_{N_k}^{(h)}(T_j, Z) = (U_{p(k)} - L_{p(k)}),\ h = 1, \ldots, w_k\}. \tag{A.5.2}$$

Using the above procedure, the average cumulative loss, $IL_k^{m_k+1}(T_j, Z)$, and its corresponding as default probability, $IP_k^{m_k+1}(T_j, Z)$, in the k^{th} CDO tranche can be considered the k^{th} obligor of master CDO using Equation (1.6):[24]

$$IL^{m_k+1}(T_j, Z) = \{D_{N_k}^{(h)}(T_j, Z) | h \in InB_k^*(T_j, Z)\} \cup \{D_{N_k}^*(T_j, Z) = (U_{p(k)} - Lp(k))\},$$

[23] The detailed algorithm can be requested from the authors.

[24] m_k is defined as $m_k = \sum_{h=1}^{w_k} 1_{\{0 < D_{N_k}^{(h)}(T_j, Z) < (U_{p(k)} - L_{p(k)})\}}$.

$$IP^{m_k+1}(T_j,Z)=\{P_{N_k}^{(h)}(T_j,Z)|h\in InB_k^*(T_j,Z)\} \bigcup\left\{P_{N_k}^*(T_j,Z)=\sum_{h\in OvB_k^*(T_j,Z)} P_{N_k,k}^{(h)}(T_j,Z)\right\}. \tag{A.6}$$

Furthermore, the average cumulative default loss of all inner CDO tranches, $\{IL^{m_k+1}(T_j,Z)\}_{k=1}^M$, and the corresponding conditional default probability, $\{IP^{m_k+1}(T_j,Z)\}_{k=1}^M$, are used to implement the following stage of probability bucketing in the master CDO.

(2) Second stage

The following ranges of buckets $\{0,B_0\},\{B_0,B_1\},\ldots,\{B_{v-1},\infty\}$ are selected for loss distribution of the master CDO. Next, the conditional probability of cumulative loss being in the h^{th} bucket $Q_k^{(h)}(T_j,Z)$ and its corresponding average cumulative loss is $A_k^{(h)}(T_j,Z)$ which is defined as the loss of the $p(k)^{\text{th}}$ inner tranche by time T_j occurring given Z. $Q_k^{(h)}(T_j,Z)$ and $A_k^{(h)}(T_j,Z)$ are calculated iteratively when the k^{th} obligor of the master CDO (namely the $p(k)^{\text{th}}$ tranche of the k^{th} inner CDO) with m_k+1 pairs of average cumulative loss, $IL_k^{m_k+1}(T_j,Z)$, and the corresponding default probability, $IP_k^{m_k+1}(T_j,Z)$, are simultaneously included in the above different buckets of master CDO. After all obligors of master CDO have been considered via the above procedure, the conditional total loss distribution on v buckets is scattered over the following points $\{A_m^{(h)}(T_j,Z)\}_{h=1}^v$ and its corresponding conditional default probabilities are $\{Q_M^{(h)}(T_j,Z)\}_{h=1}^v$.

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