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Abstract

This paper proposes an asset liability management strategy to hedge the aggregate risk of annuity providers under the assumption that both the interest rate and mortality rate are stochastic. We assume that annuity providers can invest in longevity bonds, long-term coupon bonds, and short-term zero-coupon bonds to immunize themselves from the risks of the annuity for the equity holders subject to a required profit. We demonstrate that the optimal allocation strategy can lead to the lowest risk under different yield curves and mortality rate assumptions. The longevity bond can also be regarded as an effective hedging vehicle that significantly reduces the aggregate risk of the annuity providers.

1. Introduction

As the population ages and the deterioration of pension funds continue, hedging longevity risk is becoming increasingly important worldwide. Longevity risk, defined by Cairns et al. (2006b), is the uncertainty in the long-term trend in mortality rates. Specifically, it is the risk that the realized mortality rate is *lower* than the prediction in the long run. Many studies have suggested that an unprecedented improvement in population longevity has occurred globally over the course of the twentieth century (Stallard 2006). Longevity risk represents a critical threat to pension funds and private insurers because it increases the payout period and the liability costs of providing annuities. Hedging longevity risk has received serious attention because mispricing annuity products or misallocating investments could cause substantial deficits in financial institutions.

Longevity risk has motivated many studies in the last decade. The literature has contributed by proposing either better models to predict mortality rates or better strategies to hedge the longevity risk. Recently Renshaw and Haberman (2003) and Cairns et al. (2006a, b) have adapted discrete-time models to capture the randomness of mortality rates. At the same time, other authors (see, for example, Dahl 2004; Schrager 2006; Ballotta, et al. 2006; Hainaut and Devolder 2008) have proposed modeling mortality rates in a continuous-time framework. In addition, many hedging strategies have also been discussed. To hedge longevity risk, Blake et al. (2006a, b), Lin and Cox (2005), and Cox et al. (2006) have proposed using mortality securitization, while Blake and Burrows (2001), Denuit et al. (2007), and Dowd et al. (2006) have suggested using survivor bonds and survivor swaps. Cox and Lin (2007), Wang et al. (2010), and Tsai et al. (2010) analyzed natural hedging strategies. Following this line of research, our paper proposes a risk management approach to control the longevity risk of annuity providers. To the best of our knowledge, although the literature has provided many ingenious strategies to hedge longevity risk, most papers have analyzed the effects of their strategies under a common assumption that the insurance company faces longevity risk but has not considered the interest rate risk at the same time. Ignoring interest rate risk may result in underestimating the aggregate risk and

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misleading the hedging strategy. However, they do not consider risk management strategies providing annuities. Jalen and Mamon (2009) provide a model to price mortality-dependent contingent claims with stochastic mortality and interest rates.

In this paper we propose taking the analysis one step forward to develop a risk management approach that can deal with interest rate risk and longevity risk simultaneously. We assume that annuity providers can invest in longevity bonds, long-term coupon bonds, and short-term bonds to minimize the risks of annuities. Under the required rate of return, we derive an optimal solution that can be applied to different mortality and interest rate processes. This strategy leads to an optimal hedging strategy.

This article contributes to the literature in three ways. First, because our integrated approach deals with interest rate risk and longevity risk simultaneously, we propose a more realistic hedging strategy for the annuity providers. Second, we find that the optimal allocations are quite sensitive to the parameters in the stochastic interest rate process. On the other hand, the optimal allocations do not change much under different levels of parallel shifts in the interest rate. Third, we demonstrate that adding longevity bonds to the portfolio substantially improves the hedging efficiency. Notice that we assume that the change of mortality rate used in the annuity price is perfectly positively correlated to that used in the longevity bond. In fact, it may not be the case in reality. This means that there exists some basis risk when we use the longevity bond to hedge the annuity. The cost of the basis risk could greatly reduce the portfolio weight on the longevity bond.

The remainder of the paper is organized as follows. In Section 2 we derive the optimal solution for the annuity providers and review the stochastic interest and mortality rate model. In Section 3 we describe the data and design the assets and liabilities. The parameters for the stochastic interest and mortality risk are also specified. In Section 4 we present the numerical results and their analyses. Concluding remarks and a discussion are provided in the last section.

2. THE MODEL

We first introduce notation and the hedging scheme. We then offer a brief description of the CIR interest rate model of Cox et al. (1985), as well as of the two-factor stochastic mortality model of Cairns et al. (2006b).

2.1 The Hedging Approach

The hedging strategy is based on an asset-liability-management framework. We propose a single-period static hedging strategy, and the length of the hedge period is one year. We assume the annuity provider collects an A dollar annuity premium and invests the money in the long-term coupon bonds B, longevity bonds B^l, and short-term zero coupon bonds B^f defined as follows:

- $B(r_t)$: The present value of one default-free coupon bond depending on the interest rate r_t for $t = 0, 1, \ldots, T^B$, where T^B is the time to maturity of the coupon bond. In Section 3, the coupon bond is assumed to be issued at a 5% annual coupon rate, for $T^B = 30$ years, with a face value \$1.
- $B^l(r_t, m_{t,x})$: The present value of one longevity bond, depending on r_t and mortality rate $m_{t,x}$, for $t = 0, 1, \ldots, T^l$, where T^l is the time to maturity of the longevity bond; $m_{t,x}$ is the population mortality rate[1] of age x at time t determining the proportion of the longevity bond coupons that will be paid. The central death rate $m_{t,x}$ for individuals aged x in year t changes with time. In Section 3 the longevity bond we refer to is the EIB/BNP longevity bond that had a planned issue in November 2004 with a time to maturity of 25 years. The coupon payments were linked to a survivor index based

[1] The publicly available longevity bond structures are based on population mortality, not annuitant mortality. Thus, hedging mortality risks on annuities by longevity bonds could involve basis risk. It is conceivable that a large pension plan could get an investment bank or reinsurer to issue a longevity bond written on the pension plan's portfolio of annuities, but it would be an additional expense. We thank the referee who pointed out this issue.

on the realized mortality rate of males form England and Wales aged 65 in 2003. The EIB/BNP longevity bond was not ultimately issued.
- $B^f(r_t)$: The present value of a risk-free zero-coupon bond depends on the interest rate r_t for $t = 0, 1, \ldots, T^f$, where T^f is the time to maturity of the zero-coupon bond. We assume the zero-coupon bond is a $T^f = 1$ year pure discount bond with face value \$1 in Section 3.
- $A(r_t, m^A_{t,x})$: The total present value of the annuity depends on r_t and annuitant mortality rates $m^A_{t,x}$ for $t = 0, 1, \ldots, T^A$, where T^A, is the coverage period of the annuity. We assume that the annuitant mortality rate has some correlation with the population mortality rate at the same age x, that is, $\rho(m^A_{t,x}, m_{t,x}) > 0$. We do not require that they are perfectly positive correlated to each other. Thus, there exists some basis risk at the same age x if we use the longevity bonds to hedge the longevity risk of annuities. In Section 3 the annuity is whole life, issued for men aged 65, and pays the annuitant \$1 at the end of each year. We assume the annuity is single premium immediate annuity. There are no other guarantees or benefits in the annuity. We assume there are no transaction costs for buying or selling these bonds and annuities.

The initial equity of the annuity provider is the difference between assets (B, B^l, B^f) and liabilities (A). Thus, the equity denoted as E also depends on the interest rate r_t and mortality rate $m_{t,x}$ and can be expressed as

$$E(r_t, m_{0,x}) = x_1 B(r_t) + x_2 B^l(r_t, m_{0,x}) + x_3 B^f(r_t) - A(r_t, m^A_{0,x}), \tag{1}$$

where x_1, x_2, and x_3 are the amounts invested in the long-term coupon bonds, longevity bonds, and zero-coupon bonds, respectively; $x_1 B(r_t)$, $x_2 B^l(r_t, m_{t,x})$, and $x_3 B^f(r_t)$ are total investment values of the long-term coupon bonds, longevity bonds, and zero-coupon bonds, respectively.

To incorporate the risk of unexpected changes of current interest and mortality rate at the same time, we apply a two-variable Taylor expansion of E with respect to current interest rate r_0 and current mortality rate $m_{0,x}$ (for simplicity in notation, we later use m_0 to represent $m_{0,x}$):

$$\begin{aligned}\Delta E &= E(r_0 + \Delta r_0, m_{0,x} + \Delta m_0) - E(r_0, m_0)\\ &\approx E_{r_0}\Delta r_0 + E_{m_{0,x}}\Delta m_0 + \tfrac{1}{2} E_{r_0 r_0}\Delta r_0^2 + \tfrac{1}{2} E_{m_0 m_0}\Delta m_0^2 + E_{r_0 m_0}\Delta r_0 \Delta m_0,\end{aligned} \tag{2}$$

where ΔE is the change of the equity corresponding to Δr_0 and Δm_0, Δr_0, and Δm_0 represent the unexpected changes of the current mortality and interest rates, E_{r_0} and E_{m_0} are the partial derivatives of the equity with respect to r_0 and m_0, $E_{r_0 r_0}$, $E_{m_0 m_0}$ are the second partial derivatives, and $E_{r_0 m_0}$ is the cross-derivative with respect to r_0 and m_0. The higher-order terms are dropped in equation (2). We assume the unexpected change of annuitant mortality rate Δm^A_0 is positively correlated to Δm_0, and in the form $\Delta m^A_0 = \rho_0 \Delta m_0$, where ρ_0 is the correlation coefficient between the unexpected change of annuitant and population mortality rate. Assume $0 < \rho_0 < 1$, which means that they are not perfectly positive correlated to each other and there exists some basis risk at the same age x. Substituting equation (1) into equation (2), we obtain

$$\begin{aligned}\Delta E = {} & (x_1 B_{r_0} + x_2 B^l_{r_0} + x_3 B^f_{r_0} - A_{r_0})\Delta r_0 + (x_2 B^l_{m_0} - A_{m_0})\Delta m_0\\ & + (x_2 B^l_{r_0 m_0} - A_{r_0 m_0})\Delta r_0 \Delta m_0 + \tfrac{1}{2}(x_1 B_{r_0 r_0} + x_2 B^l_{r_0 r_0} + x_3 B^f_{r_0 r_0} - A_{r_0 r_0})\Delta r_0^2\\ & + \tfrac{1}{2}(x_2 B^l_{m_0 m_0} - A_{m_0 m_0})\Delta m_0^2,\end{aligned} \tag{3}$$

where B_{r_0}, $B^l_{r_0}$, $B^l_{m_0}$, $B^f_{r_0}$, A_{r_0}, and A_{m_0} are the partial derivatives for the assets and liabilities with respect to r_0 or m_0. Similarly, $B_{r_0 r_0}$, $B^l_{r_0 r_0}$, $B^f_{r_0 r_0}$, $A_{r_0 r_0}$, $B^l_{m_0 m_0}$, and $A_{m_0 m_0}$ are second partial derivatives. $B^l_{r_0 m_0}$ and $A_{r_0 m_0}$ are the cross-derivatives of r_0 and m_0.

If the future interest and mortality rate are correctly forecasted in the model, the unexpected change $\mathbb{E}(\Delta r_0)$ and $\mathbb{E}(\Delta m_0)$ will be zero.[2] We also assume that Δr_0 and Δm_0 are independent to each other. The

[2] We assume CIR and the Cairns et al. model has considered the trends of interest and mortality rate (and age effect) in the future. Based on their projection, the unexpected change are unbiased in mean, i.e., $\mathbb{E}(\Delta r_0)$ and $\mathbb{E}(\Delta m_0)$ are zero.

independence assumption provides a great simplification, especially on cross-terms, $\mathrm{E}(\Delta r_0 \Delta m_0) = 0$, $\mathrm{E}(\Delta r_0^2 \Delta m_0) = 0$, and $\mathrm{E}(\Delta r_0 \Delta m_0^2) = 3$.[3] Then the expected value of ΔE becomes

$$\mathbb{E}(\Delta E) = \frac{1}{2}\,[B_{r_0r_0}\sigma_{r_0}^2 \cdot x_1 + (B^l_{r_0r_0}\sigma_{r_0}^2 + B^l_{m_0m_0}\sigma_{m_0}^2) \cdot x_2 + B^f_{r_0r_0}\sigma_{r_0}^2 \cdot x_3 - A_{r_0r_0}\sigma_{r_0}^2 - A_{m_0m_0}\sigma_{m_0}^2],$$

where $\mathbb{E}(\Delta E)$ denotes the expected gains (or losses) of the equity with respect to the change of r_0 and m_0. $\mathbb{E}(\Delta r_0^2)$ and $\mathbb{E}(\Delta m_0^2)$ are rewritten as $\sigma_{r_0}^2$ and $\sigma_{m_0}^2$. The variance of ΔE is

$$\mathbb{V}(\Delta E) = \beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_1 x_2 + \beta_5 x_1 x_3 + \beta_6 x_2 x_3 + \beta_7 x_1 + \beta_8 x_2 + \beta_9 x_3 + \beta_{10}, \quad (4)$$

where $\beta_1, \beta_2, \ldots, \beta_{10}$ are coefficients of the partials derivatives and moments ($\sigma_{r_0}^2$, $\mathbb{E}(\Delta r_0^3)$, $\mathbb{E}(\Delta m_0^3)$) of r_0 and m_0. They are all independent of x_1, x_2, and x_3, and the full expressions are provided in the Appendix.

Now we can choose the optimal investment variables x_1, x_2, and x_3 to minimize $\mathbb{V}(\Delta E)$ subject to the profit constraint:

$$\operatorname*{Min}_{x_1,x_2,x_3} \quad \mathbb{V}(\Delta E), \quad (5)$$

$$s.t. \quad x_1 I^B + x_2 I^l + x_3 I^f - A I^A + \mathbb{E}(\Delta E) = R_E E, \quad (6)$$

where I^B, I^l, and I^f are interest payments from the long-term coupon bonds, longevity bonds, and zero-coupon bonds at the end of the hedge period; the annuity payment is I^A if the annuitant survives at the end of the period. The total expected income earned from these contracts is $x_1 I^B + x_2 I^l + x_3 I^f - AI^A$, and it is independent of Δr_0 and Δm_0. The total expected change from the balance sheet is denoted by $\mathbb{E}(\Delta E)$ and depends on Δr_0 and Δm_0. Thus the left-hand side of equation (6) can be regarded as the total profit that consists of the expected changes from balance sheet and expected income earned from the contracts. On the right-hand side of equation (6), R_E is the required rate of return for the equityholders, and $R_E E$ is the total required profits. In Section 3 we assume R_E can be estimated by the Sharp ratio method following Milevsky et al. (2006). They propose that the shareholders of an insurance company request a risk premium for bearing systematic risk and will be compensated with the same Sharpe ratio as other asset classes in the capital market. Thus, we minimize the variance of the equity change, $\mathbb{V}(\Delta E)$, subject to the total profit (the profit from balance-sheet change plus certain interest earned from contracts) equaling the required profit. Then we solve for the optimal investment allocations, x_1, x_2, and x_3.

The Lagrange multiplier method is used to solve this constrained maximization problem. The solution to equation (6) is given by the following linear equations:

$$\Phi X = \Psi,$$

where

$$\Phi = \begin{bmatrix} 2\beta_1 & \beta_4 & \beta_5 \\ \beta_4 & 2\beta_2 & \beta_6 \\ \beta_5 & \beta_6 & 2\beta_3 \\ B_{r_0r_0}\sigma_{r_0}^2 + 2I^B & B^l_{r_0r_0}\sigma_{r_0}^2 + B^l_{m_0m_0}\sigma_{m_0}^2 + 2I^l & B^f_{r_0r_0}\sigma_{r_0}^2 + 2I^f \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (7)$$

and

$$\Psi = \begin{bmatrix} -\beta_7 \\ -\beta_8 \\ -\beta_9 \\ A_{r_0r_0}\sigma_{r_0}^2 + A_{m_0m_0}\sigma_{m_0}^2 + 2I^A + 2R_E E \end{bmatrix}. \quad (8)$$

[3] However, some studies identify that in the long run interest rates could be related to the size of population and mortality rates.

The solution of X in equation (7) is

$$x_j = \frac{|\Phi_j|}{|\Phi|}, \quad j = 1, 2, 3, \tag{9}$$

where the determinant $|\Phi_j|$ replaces the jth column of $|\Phi|$ whose components are the elements in Ψ. The solutions can be solved easily by widely available programming software.

2.2 The CIR Model

If the interest rate follows the stochastic process suggested by Cox et al. (1985), then the interest rate path can be expressed as

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\,dz, \tag{10}$$

where a, b, and σ are constants and dz follows a standard Brownian motion. The drift rate of the interest rate under above model is $a(b - r_t)$. The standard deviation of the interest rate is $\sigma\sqrt{r_t}$. Cox et al. (1985) solved equation (10) and showed that

$$B_t^f(r_t) = \alpha(t, T)e^{-\beta(t,T)r_t},$$

where B_t is the price of a zero-coupon bond at time t and

$$\alpha(t, T) = \left[\frac{2\sqrt{a^2 + 2\sigma^2}e^{(a+\sqrt{a^2+2\sigma^2})(T-t)/2}}{(\sqrt{a^2 + 2\sigma^2} + a)(e^{\sqrt{a^2+2\sigma^2}(T-t)} - 1) + 2\sqrt{a^2 + 2\sigma^2}}\right]^{2ab/\sigma^2},$$

$$\beta(t, T) = \frac{2(e^{\sqrt{a^2+2\sigma^2}(T-t)} - 1)}{(\sqrt{a^2 + 2\sigma^2} + a)(e^{\sqrt{a^2+2\sigma^2}(T-t)} - 1 + 2\sqrt{a^2 + 2\sigma^2}}. \tag{11}$$

2.3 The Two-Factor Stochastic Mortality Model

We choose the two-factor mortality model (the CBD model) as the underlying mortality process. We offer a brief description here; for a more detailed discussion, see Cairns et al. (2006b).

$q_{t,x}$ is the single-year mortality rate for age x insured from time t to $t + 1$. Cairns et al. (2006b) assume the mortality process as

$$q_{t,x} = \frac{e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}{1 + e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}, \tag{12}$$

where $A_1(t)$ and $A_2(t)$ can be regarded as age-general improvements in mortality over time and different improvements for different age groups. They reflect the "trend effect" and "age effect." The two stochastic trends follow a random walk process with drift parameter μ and diffusion parameter C:

$$A(t + 1) = A(t) + \mu + CZ(t + 1), \tag{13}$$

where $A(t + 1) = [A_1(t + 1), A_2(t + 1)]^T$, and $\mu = [\mu_1, \mu_2]^T$ are 2×1 constant parameter vectors. C is a 2×2 constant upper triangular. Their model can also include the parameter uncertainty of μ and C, invoking Bayesian methods and sampling from a noninformative prior distribution:

$$V^{-1}|D \sim Wishart(n - 1, n^{-1}\hat{V}^{-1}),$$
$$\mu^{-1}|V, D \sim MVN(\hat{\mu}, n^{-1}V), \tag{14}$$

where

$$D(t) = A(t) - A(t - 1),$$

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} D(t),$$

and

$$\hat{V} = \frac{1}{n} \sum_{t=1}^{n} (D(t) - \hat{\mu})(D(t) - \hat{\mu})^T.$$

$A(t)$ is generated from equation (13) with the parameters μ and C from equation (14). Then we obtain $q_{t,x}$ as equation (12) suggests. Cairns et al. (2006b) propose an approximation on the central death rate $m_{t,x}$ for individuals aged x in year t with respect to $q_{t,x}$:

$$m_{t,x} = \frac{q_{t,x}}{1 - \frac{1}{2}q_{t,x}}.$$

Convert the central death rate to the survival rates by $S_{t+1} = S_t(1 - m_{t,x})$.

Then Cairns et al. (2006b) apply the risk-neutral valuation approach to value the longevity bond price by summing the present value of the coupons (i.e., survival rate) over 25 years. The initial price of the longevity bond is

$$B^l(r_0, m_{0,x}) = \sum_{t=1}^{T^l} \alpha(0, t)e^{\beta(0,t)r_0}E^{Q(\lambda)}[S_t|M_0], \tag{15}$$

where r_0 is the current spot rate used in the CIR model and value of survival rate under the information set M_0 with respect to the risk-neutral probability $Q(\lambda)$. The probability that a person age x at time 0 will survive to time t, according to the model of the population underlying the longevity bond, is defined as

$${}_tp_x = E^{Q(\lambda)}[S_t|M_0].$$

Then the longevity bond price at time 0 will be

$$B^l(r_0, m_{0,x}) = \sum_{t=1}^{T^l} e^{-t\cdot i_t}{}_tp_x, \tag{16}$$

where i_t is the yield to maturity of the interest rate at time 0 to discount a payment at time t:

$$i_t = \frac{1}{t}\log(\alpha(0, t)e^{\beta(0,t)r_0}).$$

The values $i_1, i_2, \ldots$ give the term structure of interest rates at time 0, and we use them in calculating prices at time 0. The bond and annuity prices at time 0 are as follows:

$$B(r_0) = c \sum_{t=1}^{T^B-1} e^{-t\cdot i_t} + (1 + c)e_t^{-T_B\cdot i_T B}, \tag{17}$$

$$B^f(r_0) = e_t^{-T_B\cdot i_T B}, \tag{18}$$

$$A(r_0, m_0) = a_{x:\overline{T^A}|} = \sum_{t=1}^{T^A} e^{-t\cdot i_t}E^{Q(\lambda)}[S_t^A|M_0] = \sum_{t=1}^{T^A} e^{-t\cdot i_t}{}_tp_x^A, \tag{19}$$

where c is coupon payment of coupon bonds, and the superscript A indicates that the person is subject to annuitant mortality rather than population mortality.

3. NUMERICAL ANALYSIS

In this section to demonstrate the hedging strategy, we present a numerical example of an annuity provider. We assume that the annuity provider invests in longevity bonds, coupon bonds, and zero-coupon bonds to minimize the risks of equity holders subject to a targeted profit. The hedging effect of this approach is presented in Section 4.

Table 1
Parameters of CBD Mortality Model

λ_1	Market price of longevity risk associated with level shift in mortality	0.175
λ_2	Market price of longevity risk associated with tilt in mortality	0.175
x	Initial age of cohort	65
T	Bond maturity	25

Table 2
Parameters of CIR Interest Rate Model

a	Small speed of mean reversion	0.20
b	Long-run mean interest rate	0.05
σ	Volatility	0.08
r_0	Initial interest rate	0.03

3.1 The Asset Side

On the asset side, the long-term coupon bond is assumed to be issued at a 5% coupon rate, for 30 years, with a face value $1. The zero-coupon bond is a one-year pure discount bond with face value $1. For the longevity bond, we use the EBI/BNP longevity bond with a 25-year maturity and face value $1. Its cash flows are based on the actual mortality experience of the English and Welsh male population age 65. The coupons are equal to the fixed annuity multiplied by the percentage of the reference population still alive at each time point. We employed the model of it in Section 2.3 to project the mortality trend for future 25 years. The parameters are shown in Table 1.

Furthermore, we also assume the spot interest rate follows the CIR interest model. The assumed speed of mean reversion and volatility are shown in Table 2. The initial interest rate is 3%, and the long-run interest rate is 5%. The mean reversion and volatility parameter of CIR model are 0.2 and 0.08, respectively. We use this information to estimate the yield curve from 1 to 30 years.

3.2 The Liabilities Side

On the liabilities side, the annuity is assumed to be a 35-year term-life annuity, issued for men aged 65, and pays the annuitant $1 at the end of each year. It has no deferred period, and the premiums are collected as a single premium. The discount rate follows the CIR model as in Section 3.1, and the mortality process follows the mortality rate of the longevity bond produced by the correlation coeffi-

Table 3
Basic Assumptions on Assets and Liability

Asset Side	Coupon Rate	Maturity	Face Value	Price
Coupon bond	5%	30 years	$1	$1.101
Longevity bond	Base on $m_{t,x}$ of male age 65	25 years	$1	11.296
Zero-coupon bond	—	1 year	$1	0.9687
Liability Side	**Age/Gender**	**Coverage**	**Per Payment**	**Premium**
Term-life annuity	65/male Premium-type Single	35 years	$1 Deferred period Immediately	$10.697

cient. We assume the correlation coefficient is 0.75.[4] The value of the longevity bond on issuing day is \$11.296,[5] and the premium of the whole-life annuity is \$10.697. We summarize the information in Table 3.

4. THE RESULTS

The estimated partial derivatives for assets and liabilities are presented in Table 4. The coupon bond (30 years), longevity bond (25 years), and zero-coupon bond (1 year) have negative partial derivatives in regard to interest rates. The partial derivatives with respect to the interest rate are, respectively, −3.8545, −3.1279, and −1.6292 and decrease with time in absolute value. The second partial derivatives of coupon and zero-coupon bonds are positive, and coupon bonds have larger convexity than zero-coupon bonds. The longevity bonds and annuities both have negative sign when the interest rate changes. The second partial derivatives of interest are larger than the one for zero-coupon bonds but smaller than the one for the coupon bonds. For the longevity bond, the first partial derivatives with respect to the mortality rate are negative. The second partial derivatives with respect to the mortality rate are very small and can be neglected. By substituting these partial derivatives into equation (9), we can obtain the optimal allocation of x_1, x_2, and x_3.

The optimal solutions of x_1, x_2, and x_3 are shown in Table 5. The solutions under different interest parameters of the CIR model are also shown. We examine the effects of short-term interest rate r_0, long-term interest rate b, mean reversion speed a, and volatility σ on the optimal allocations. In the base case, case 1, the short-term interest rate r_0 is 3%, and parameters a, b, σ are 0.2, 3%, and 0.08, respectively. The optimal weight is 33.89% in long-term bonds and 12.64% in short-term bonds. In this base case, the yield curve is flat, and the annuity provider holds most assets in longevity bonds (53.47%).

Table 4
Partial Derivatives on Assets and Liabilities

	Coupon Bond	Zero-Coupon Bond	Longevity Bond	Annuity
First partial derivatives on r_0	$B_{r_0} = -3.8545$	$B^f_{r_0} = -1.6292$	$B^l_{r_0} = -3.1279$	$A_{r_0} = -3.0672$
Second partial derivatives on r_0	$B_{r_0r_0} = 16.614$	$B^f_{r_0r_0} = 2.6984$	$B^l_{r_0r_0} = 11.975$	$A_{r_0r_0} = 11.6$
First partial derivatives on m_0			$B^l_{m_0} = -1.0579$	$A_{m_0} = -0.78967$
Second partial derivatives on m_0			$B^l_{m_0m_0} = 1.9 \times 10^{-13}$	$A_{m_0m_0} = -1.7 \times 10^{-13}$
Cross-partial derivatives			$B_{r_0m_0} = 3.2409$	$A_{r_0m_0} = 2.8741$

Table 5
Optimal Allocations Value under Different CIR Parameters

Case	(r_0, a, b, σ)	Weight of B	Weight of B^l	Weight of B^f	$\mathbb{V}(\Delta E) \times 10^4$
1	(3%, 0.2, 3%, 0.08)	0.3389	0.5347	0.1264	0.003795
2	(1%, —, —, —)	0.2996	0.6061	0.0943	0.017469
3	(5%, —, —, —)	0.4221	0.3794	0.1985	0.000009
4	(3%, —, 5%, —)	0.3882	0.4140	0.1978	0.017200
5	(3%, —, 1%, —)	0.3062	0.6017	0.0921	0.000153
6	(—, 0.25, —, —)	0.3402	0.5342	0.1256	0.005477
7	(—, —, —, 0.15)	0.3466	0.5362	0.1172	0.001362

[4] The insurers can estimate their own mortality processes, which may have different correlation coefficients to the ones in the longevity bonds. We use $\rho_t = 0.75$ for all t, which is a simplified assumption. Actually, it need not to be so. We leave the study of time-variant correlation coefficients between annuitant and longevity bond for further research.

[5] The longevity bonds are priced as in Cairns et al. (2006b), but the discount rate is specified by the CIR model rather than the constant interest rate.

Table 6
Optimal Allocations with and without Longevity Bond

Case	Weight of B	Weight of B^l	Weight of B^f	$\mathbb{V}(\Delta E) \times 10^4$
With investment in B^l	0.3389	0.5347	0.1264	0.003795
Without investment in B^l	0.7654	0	0.2346	0.860060

This shows that the longevity bond is much more attractive than the coupon bond in this minimization-risk programming. We then consider the upward-sloping yield curve in case 2. Here the short-term interest rate is lower than long-term interest rate, and the rate of return on the zero-coupon bond is also lower. This effect causes the annuity providers to hold less in zero-coupon bonds (9.43%). If the short-term interest rate is higher than long-term rate, that is, a downward-sloping yield curve as in case 3, the annuity provider will tend to hold more zero-coupon bonds (19.85%). This is because the risks associated with the coupon bond are smallest (the first derivative plus second derivatives), and they are regarded as ‘‘a better thing’’ than long-term bonds and longevity bonds as the expected return increases. In case 4, if we have a upward-sloping yield curve caused by a higher long-term interest rate, the weights for B (38.82%) and B^f (19.78%) both increase while that for B^l decreases. In this case the long-term bonds become more attractive than the longevity bonds because the the increased returns in the future for longevity bonds are mitigated by mortality rate. However, the annuity provider is still mostly positioned in longevity bonds (41.40%). The short-term bonds also increase because of the risk-return trade-off effect between B and B^f. In the opposite case, case 5, the long-term interest decreases, and the annuity provider will tend to buy more longevity bonds because the yield on longevity bonds is higher than that on long-term bonds. In cases 6 and 7, the faster mean reversion speed $\alpha = 0.25$ and higher volatility $\sigma = 0.15$ result in rather small changes in B, B^l, and B^f. In these cases we find the allocations are less sensitive to the mean reversion and volatility parameters but more sensitive to the different shapes of the yield curves.

In Table 6 we consider the scenarios of the allocations with or without the longevity bond. The results show that when there is no longevity bond in the portfolio, the annuity provider tends to hold long-term bonds instead of short-term bonds. The variance of equity holders increases sharply from 0.003795 to 0.86006. We claim that, at the same required rate of return, the longevity bond indeed largely reduces the equity holders’ risk. The longevity bond effectively reduces the interest and mortality risk from the annuity.

5. CONCLUSION

Longevity risk has received serious attention both in the industry and in the research literature. Although the literature has provided many ingenious strategies to hedge the longevity risk, most papers analyzed the effects of the hedging strategies without considering the interest rate risk at the same time. However, ignoring interest rate risk may underestimate the aggregate risk and result in misleading hedging strategies. This paper proposes an asset liability management strategy to hedge the aggregate risk of equity for annuity providers by considering both independent stochastic interest rate risk and mortality rate risk. The numerical examples in our paper show that failing to integrate interest rate risk into the model for hedging longevity risk could cause a substantial risk for equity holders. We also find that the shape of the yield curve significantly affects the allocation weights. Moreover, the simulation results also suggest that the longevity bond plays a critical role in the integrated model. We demonstrate that our proposed approach can lead to optimal asset liability allocation and the longevity bond can serve as an effective vehicle to significantly reduce the aggregate risk for annuity providers.

Notice that our model proposes only a single-period, static hedging strategy to immunize the risk. Several issues should be considered if a model of dynamic hedging is considered. First, the insurer’s objective function should be the accumulated future value of the equity rather than the present value of it. Second, the static programming should be replaced by a dynamic programming. Third, the required returns should be subject to multiperiod constraints. The issues of the dynamic hedging are beyond the scope of this paper, but we believe it could provide fruitful results in a future study.

APPENDIX

The coefficients of equation (4) are the following:

$$\beta_1 = (B_{r_0})^2\sigma_{r_0}^2 + B_{r_0}B_{r_0r_0}\mathbb{E}(\Delta r_0^3),$$

$$\beta_2 = (B_{r_0}^l)^2\sigma_{r_0}^2 + (B_{m_0}^l)^2\sigma_{m_0}^2 + (B_{r_0m_0}^l)^2\sigma_{r_0}^2\sigma_{m_0}^2 + B_{r_0}^l B_{r_0r_0}^l\mathbb{E}(\Delta r_0^3) + B_{m_0}^l B_{m_0m_0}^l\mathbb{E}(\Delta m_0^3),$$

$$\beta_3 = (B_{r_0}^f)^2\sigma_{r_0}^2 + B_{r_0}^f B_{r_0r_0}^f\mathbb{E}(\Delta r_0^3),$$

$$\beta_4 = 2B_{r_0}B_{r_0}^l\sigma_{r_0}^2 + B_{r_0}B_{r_0r_0}^l\mathbb{E}(\Delta r_0^3) + B_{r_0}^l B_{r_0r_0}\mathbb{E}(\Delta r_0^3),$$

$$\beta_5 = 2B_{r_0}B_{r_0}^f\sigma_{r_0}^2 + B_{r_0}B_{r_0r_0}^f\mathbb{E}(\Delta r_0^3) + B_{r_0}^f B_{r_0r_0}\mathbb{E}(\Delta r_0^3),$$

$$\beta_6 = 2B_{r_0}^l B_{r_0}^f\sigma_{r_0}^2 + B_{r_0}^l B_{r_0r_0}^f\mathbb{E}(\Delta r_0^3) + B_{r_0}^f B_{r_0r_0}^l\mathbb{E}(\Delta r_0^3),$$

$$\beta_7 = -2A_{r_0}B_{r_0}\sigma_{r_0}^2 - A_{r_0}B_{r_0r_0}\mathbb{E}(\Delta r_0^3) - B_{r_0}A_{r_0r_0}\mathbb{E}(\Delta r_0^3),$$

$$\beta_8 = -2A_{r_0}B_{r_0}^l\sigma_{r_0}^2 - A_{r_0}B_{r_0r_0}^l\mathbb{E}(\Delta r_0^3) - B_{r_0}^l A_{r_0r_0}\mathbb{E}(\Delta r_0^3) - 2A_{m_0}B_{m_0}^l\sigma_{m_0}^2 - A_{m_0}B_{m_0m_0}^l\mathbb{E}(\Delta m_0^3)$$
$$- B_{m_0}^l A_{m_0m_0}\mathbb{E}({}_0\Delta r^3) - 2B_{r_0m_0}^l A\sigma_{r_0}^2\sigma_{m_0}^2,$$

$$\beta_9 = -2A_{r_0}B_{r_0}^f\sigma_{r_0}^2 - A_{r_0}B_{r_0r_0}^f\mathbb{E}(\Delta r_0^3) - B_{r_0}^f A_{r_0r_0}\mathbb{E}(\Delta r_0^3),$$

$$\beta_{10} = (A_{r_0})^2\sigma_{r_0}^2 + (A_{m_0})^2\sigma_{m_0}^2 + (A_{r_0m_0})^2\sigma_{r_0}^2\sigma_{m_0}^2 + A_{r_0}A_{r_0r_0}\mathbb{E}(\Delta r_0^3) + A_{m_0}A_{m_0m_0}\mathbb{E}(\Delta m_0^3).$$

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