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On the optimal product mix in life insurance companies using conditional value at risk

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ABSTRACT

This paper proposes a Conditional Value-at-Risk Minimization (CVaRM) approach to optimize an insurer's product mix. By incorporating the natural hedging strategy of Cox and Lin (2007) and the two-factor stochastic mortality model of Cairns et al. (2006b), we calculate an optimize product mix for insurance companies to hedge against the systematic mortality risk under parameter uncertainty. To reflect the importance of required profit, we further integrate the premium loading of systematic risk. We compare the hedging results to those using the duration match method of Wang et al. (forthcoming), and show that the proposed CVaRM approach has a narrower quantile of loss distribution after hedging—thereby effectively reducing systematic mortality risk for life insurance companies.

1. Introduction

Over the past decade, a longevity shock has spread across human society. Benjamin and Soliman (1993), McDonald et al. (1998), Grundl et al. (2006) and Stallard (2006) confirm that unprecedented improvements in population longevity have occurred worldwide. The decreasing trend in the mortality rate has created a great risk for insurance companies. The existing literature has proposed a number of solutions to mitigate the threat of longevity risk to life insurance companies. These solutions can be classified into three categories. The *capital market solutions* include mortality securitization (see, for example Dowd, 2003; Lin and Cox, 2005; Cairns et al., 2006a; Blake et al., 2006a,b; Cox et al., 2006), survivor bonds (e.g. Blake and Burrows, 2001; Denuit et al., 2007), and survivor swaps (e.g. Dowd and Blake, 2006; Dowd et al., 2006). These studies suggest that insurance companies can transfer their exposures to the capital markets. Cowley and Cummins (2005) provide an excellent overview of the securitizations of life insurance assets and liabilities. The second set of solutions, the *industry self-insurance solutions,* include the natural hedging strategy of Cox and Lin (2007), the duration matching strategy of Wang et al. (forthcoming), and the reinsurance swap of Lin and Cox (2005). The advantages of these solutions are that the hedging does not require a liquid market and can be arranged at a lower transaction cost. Insurance companies can hedge longevity risk by themselves or with counterparties. The third kind of solution, known as *mortality projection improvement*, provides a more accurate estimation of mortality processes. As Blake et al. (2006b) propose, these studies fall into two areas: continuous-time frameworks (e.g. Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Schrager (2006), Dahl and Moller (2006)) and discrete-time frameworks, e.g., Brouhns et al. (2002), Renshaw and Haberman (2003) and Cairns et al. (2006b). Parameter uncertainty and model specification in relation to the mortality process have also attracted more attention in recent years.

Among the industry self-insurance solutions, the *natural hedging strategy* suggests that life insurance can serve as a hedging vehicle against longevity risk for annuity products. Wang et al. (forthcoming) employ duration as a measure of the product sensitivity to mortality change, and propose a *mortality duration matching* (MDM) *approach* to calculate the optimal product mix. Their work, however, is based on several restrictive assumptions. First, they assume that future mortality changes involve parallel shifts in the mean, and do not measure the higher-order moments of the mortality risk distribution. Second, the MDM approach applies to only two products. Third, the MDM approach is a pure risk-reduction method because the profit loading is not considered during the hedging procedure. Fourth, Cairns (2000), Melnikov and

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Romaniuk (2006) and Koissi et al. (2006) suggest that parameter risk is crucial when dealing with longevity risk. The parameter uncertainty does not play a role in the MDM approach, since Wang et al. (forthcoming) consider the mortality shift only in terms of its mean.

To overcome these problems, we employ the two-factor stochastic mortality model of Cairns et al. (2006b) and construct the Conditional Value-at-Risk Minimization (CVaRM) (Dowd and Blake, 2006) approach to control the possible loss. Managing products risk with parameter uncertainty is one feature of the CVaRM approach. The other feature is that we add the profit-loading constraint into the optimization. The premium-pricing principle suggested by Milevsky et al. (2006) is employed to estimate the required profit loadings, i.e., in order to compensate the stockholders bearing systematic mortality risk with the same Sharpe ratio as other asset classes in the economy.[1] Furthermore, the CVaRM approach could be easily implemented using linear programming (Rockafellar and Uryasev, 2000), and insurance companies could adopt it as their own internal risk-management tool.

The results of our simulation reveal that the proposed CVaRM approach yields a less dispersed product distribution after hedging and so effectively reduces systematic mortality risk for life insurance companies. The MDM approach, on the other hand, has a limited effect on the dispersion of the product distribution. In addition, the CVaRM approach considers not only risk reduction but also the required profit constraint. We found that the required loading substantially changes the optimal product mix and so cannot be ignored.

The remainder of this article is organized as follows: Section 2 outlines the models, including the mortality model of Cairns et al. (2006b), the duration matching model of Wang et al. (forthcoming), the loading estimation of Milevsky et al. (2006) and the CVaRM approach. In Section 3 we introduce the mortality data, project future mortality and design the products. Section 4 presents the numerical examples in two scenarios: the two-product scenario without a required loading constraint and a multiple-product scenario with a required loading constraint. The hedging results of the MDM and CVaRM approaches are also compared in this section. Conclusions and implications are contained in Section 5.

2. The models

Before introducing the CVaRM approach, this section first briefly reviews the two-factor stochastic mortality model of Cairns et al. (2006b), the mortality duration matching model of Wang et al. (forthcoming), and the loading-estimation methods of Milevsky et al. (2006).

2.1. The two-factor stochastic mortality model

Several stochastic models proposed in the literature attempt to capture the mortality processes. We chose the two-factor mortality model, i.e., CBD model, as the underlying mortality process for two reasons. First, the CBD model characterizes not only a cohort effect but also a quadratic age effect. The two factors $A_1(t)$ and $A_2(t)$ in the CBD model represent all age general improvements in mortality over time and different improvements for different age groups. These two factors reflect both the trend effect and the age effect. Thus, the analysis will be economically or *biologically* meaningful when we consider the parameter changes of the factors over time. Second, the CBD model is a discrete-time model and can be more conveniently implemented in practice. This paper offers a brief description of the two-factor model; for a more detailed discussion, see Cairns et al. (2006b).

Let $q_{t,x}$ be the realized mortality rate for age x insured from time t to $t+1$. Assume that the mortality curve has a logistic functional form as follows:

$$q_{t,x} = \frac{e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}{1+e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}. \tag{1}$$

The two stochastic trends $A_1(t+1)$ and $A_2(t+1)$ follow a random-walk process with drift parameter μ and diffusion parameter C:

$$A(t+1) = A(t) + \mu + CZ(t+1), \tag{2}$$

where $A(t+1) = (A_1(t+1), A_2(t+1))^{\mathrm{T}}$ and $\mu = (\mu_1, \mu_2)^{\mathrm{T}}$ are 2×1 constant parameter vectors. C is a 2×2 constant upper-triangular Cholesky square-root matrix of the covariance matrix $V = CC^{\mathrm{T}}$ and $Z(t)$ is a two-dimensional standard normal random variable. To include the uncertainty of μ and C, Cairns et al. (2006b) invoke a normal-inverse-Wishart distribution from a non-informative prior distribution:

$$\begin{aligned}
&V^{-1}\,|D \sim Wishart(n-1, n^{-1}\hat{V}^{-1})\\
&\mu^{-1}\,|V, D \sim MVN(\hat{\mu}, n^{-1}V),\\
&\text{where } D(t) = A(t) - A(t-1),\\
&\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} D(t),\\
&\text{and } \hat{V} = \frac{1}{n}\sum_{t=1}^{n}\left(D(t)-\hat{\mu}\right)\left(D(t)-\hat{\mu}\right)^{\mathrm{T}}.
\end{aligned} \tag{3}$$

Thus, we can generate $A(t)$ from Eq. (2) with the parameters μ and C from Eq. (3). Then we get $q_{t,x}$, as Eq. (1) suggests.

2.2. The Mortality Duration Matching (MDM) method

Wang et al. (forthcoming) propose the MDM approach to calculate an optimal life insurance/annuity weight to immunize the value change from mortality risk. They propose the following product mix of life insurance:

$$w^D = \frac{D^a}{D^a + D^l}, \tag{4}$$

where D^a denotes the effective duration of the annuity and D^l denotes the effective duration of the life insurance. Formally, the effective duration can be calculated as follows:

$$D^a = -\frac{V^{a+} - V^{a-}}{2V^a \Delta q} \quad \text{and} \quad D^l = \frac{V^{l+} - V^{l-}}{2V^l \Delta q}.$$

The Δq refers to the change in the mortality rate, V^{a+} and V^{l+} represent the product values at higher mortality rate $(q+\Delta q)$ and V^{a-} and V^{l-} represent the values at the lower mortality rate $(q-\Delta q)$. If the change is small, this strategy leads to the product immunization as follows:

$$\Delta V = w^D D^l - (1-w^D)D^a = 0. \tag{5}$$

Wang et al. (forthcoming) also propose the mortality convexity adjustment for a large change as

$$C^a = \frac{V^{a-} + V^{a+} - 2V^a}{V^a(\Delta q)^2} \quad \text{and} \quad C^l = \frac{V^{l-} + V^{l+} - 2V^l}{V^l(\Delta q)^2}.$$

Then the product mix weight with convexity on life insurance is

$$w^C = \frac{D^a - \frac{\Delta q}{2}C^a}{D^a + D^l + \frac{\Delta q}{2}\left(C^l - C^a\right)}. \tag{6}$$

Here, the change is set as $\Delta q = \bar{q}(1+s) - \bar{q}$, where $\bar{q}$ is the mean of the mortality process and s is a shift proportion such as 1%. Thus, the change here involves a parallel shift in the mean.

[1] The non-systematic risk of products is not considered here. We assume that the non-systematic mortality risks are all diversified across policyholders via the law of large numbers. Shareholders bearing non-systematic mortality risk are not rewarded. We also assume that insurance companies will not suffer from insolvency risk.

2.3. Profit-loading estimation: The Sharpe ratio method

Milevsky et al. (2006) show that when the mortality rate is stochastic, the standard deviation per policy does not vanish despite the law of large numbers. Rather there exists systematic or market risk even in a large diversified product portfolio. The shareholders of an insurance company request a risk premium for bearing the systematic risk. Milevsky et al. (2006) propose that the risk premium π, which is used to compensate shareholders, be specified using the Sharpe ratio. The Sharpe ratio for the product premium is defined as

$$SR = \frac{E(V)(1+\pi) - E(V)}{\sigma(V)}, \tag{7}$$

where $E(V)$ is the expected or actuarially fair price of the product under the law of large numbers, and $\sigma(V)$ is the standard deviation of product values. When the capital market is in equilibrium, the SR in Eq. (7) may be set equal to the Sharpe ratio of some broadly diversified portfolio, such as the S&P500 index; then the risk premium π is implicitly specified by (7). For more details please see Section 4.2.

2.4. The Conditional Value-at-Risk Minimization (CVaRM) approach

Let the random variable v^i be the value of the ith product. Similarly let $E(v^i)$ be its present value or actuarially fair price. Since q is stochastic, v^i will generate deviations from $E(v^i)$. The loss proportions for each product are denoted as

$$r^i = \frac{v^i - E\left(v^i\right)}{E\left(v^i\right)}. \tag{8}$$

The total loss proportion is

$$r_p = \sum_i w^i r^i, \tag{9}$$

where w^i is the weight of the ith product in relation to the whole product. The ith product could refer to life insurance or an annuity. We engage in natural hedging to minimize the risk r_p by choosing different w^i. The Conditional VaR (CVaR) is proposed as a measure of the product risk. CVaR is chosen as a risk measure instead of VaR, because CVaR is a coherent measure, whereas VaR is not; this is shown by Artzner et al. (1997), Artzner et al. (1999) and Deprez and Gerber (1985). The CVaRM approach is expressed as

$$\underset{w^i}{\text{Min}} \quad E\left[r_p \left| r_p \geq r_p(\alpha)\right.\right] \tag{10}$$

$$s.t. \quad \sum_i w^i \cdot \pi^i \geq \bar{\pi}, \tag{11}$$

$$\sum_i w = 1, \quad \text{and} \quad 0 \leq w^i \leq 1 \tag{12}$$

where $E\left\{r_p \left| r_p \geq r_p(\alpha)\right.\right\}$ is the conditional expected loss that exceeds the threshold, $r_p(\alpha)$, under the specified probability α. In Eq. (11), π^i denotes the profit loading on the ith product charged by the insurance company and is estimated using the Sharpe ratio noted in Section 2.3. The weighted profit $\sum w^i \cdot \pi^i$ is constrained to be greater than or equal to $\bar{\pi}$. Here we let the target profit $\bar{\pi}$ be exogenously given. We ensure that the sum of the weights is equal to one and prohibit short selling via Eq. (12). Although CVaR is usually defined in terms of monetary value, here we represent it as a percentage loss; this avoids confusion over magnitude.

In the CVaRM approach, r_p is generated as follows. First, we apply the CBD model to simulate the mortality processes and corresponding distributions of v^i. We compute $E(v^i)$ and substitute it into Eq. (8) to obtain the distribution of r^i. We calculate r_p with Eq. (9). Also note that the CBD model allows parameter uncertainty to be considered and this approach makes it possible to incorporate longevity risk and parameter uncertainty simultaneously. To demonstrate the results of the optimization, we provide three examples in Section 4.

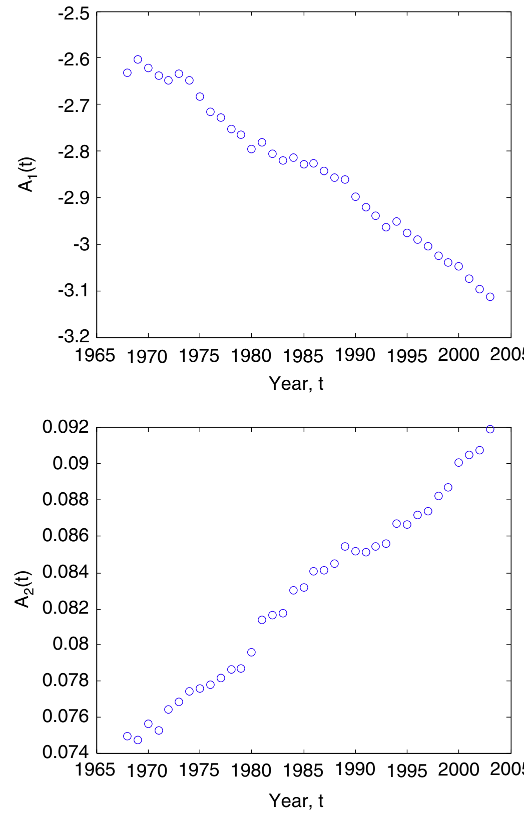

Fig. 1. The estimated parameters in the CBD model for men aged 60–84 from 1965 to 2003.

3. Mortality model estimation and product designs

This section describes the mortality data set and products. We employ the data from Cairns et al. (2007) and the JPMorgan LifeMetrics (2006); a sample of US men aged 60–84 from 1968 to 1979 and US men aged 60–89 from 1980 to 2003.[2] The estimated drift and diffusion parameters are

$$\hat{\mu} = \begin{pmatrix} -0.016289 \\ 0.0004769 \end{pmatrix} \quad \text{and}$$

$$\hat{V} = \hat{C}\hat{C}^{\mathrm{T}} = \begin{bmatrix} 0.00011695 & 0.00000334 \\ 0.00000334 & 0.00000031 \end{bmatrix}.$$

The hats indicate the estimated values. By substituting the coefficients $\hat{\mu}$ and $\hat{V}$ into Eq. (2), we obtain $A(t) = (A_1(t), A_2(t))^{\mathrm{T}}$. The paths of $A_1(t)$ and $A_2(t)$ are shown in Fig. 1. Then we convert the $q_{t,x}$ from (1) into a survival index S_t for a cohort of age x at time t.

There are three types of products in our numerical examples: whole-life annuity, whole-life insurance, and 20-year term-life insurance. The whole-life annuity is issued to men aged 60, and the cohort groups are paid \$1 at the end of each year. The whole-life insurance is issued to men aged 40 or 60, and the payout benefit

[2] The data set and CBD estimation model are available on the JPMorgan LifeMetrics website: http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/lifemetrics.

Table 1
Basic assumptions about products.

	Whole-life annuity	Whole-life insurance	Term-life insurance
Age/Gender	60/Men	40 or 60/Men	40/Men
Payout benefit	\$1 per year	\$100	\$100
Coverage	Whole life	Whole life	20 years
Premium type	Single	Single	Single
Interest rate	3%	3%	3%
Deferred period	Immediately	Immediately	Immediately
Mortality process	CBD	CBD	CBD
Premium value	\$14.94	\$54.41/\$74.72	\$29.76

Table 2
The CVaRs and statistics of the loss distributions for Fig. 2.

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std. $\times 10^2$	$w^l \times 10^2$	Loading $\times 10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	0.0	1.438
All whole-life insurance	1.853	0.286	0.122	0.000	0.696	100.0	0.192
Duration match	3.324	0.519	0.222	−0.000	1.279	10.6	1.306
Duration with convexity	3.348	0.523	0.223	0.000	1.288	9.9	1.314
The CVaRM approach	1.725	0.270	0.116	0.000	0.662	67.3	0.599

The Std. and w^l are multiplied by 100. The loading values are multiplied by 10,000 (for example, 1.438 is 0.0001438).

is \$100. The term-life insurance is issued to men aged 40, and the payout benefit is also \$100. Both premiums are collected in a single premium today. For the sake of simplicity, the deferred periods are zero. The interest rate is 3%, and the mortality process follows the CBD two-factor model. The products' expected values for the whole-life annuity, whole-life insurance and 20-year term-life insurance are \$14.94, \$54.41/\$74.72, and \$29.76, respectively. The information is summarized in Table 1. We calculate the expected values of products on the basis of the mortality distributions generated by JPMorgan LifeMetrics (2006).[3]

4. Numerical analysis of the optimal product mix

To demonstrate the hedging effect, we construct three examples in two scenarios. In scenario one the insurer cares only about risk reduction and does not consider any profit loading. Here we choose a two-product framework and compare the hedging effects of the CVaRM and MDM approaches. We show that the CVaRM approach has a better hedging effect in terms of the aggregate distribution than the MDM approach does. The analysis is then extended to the multi-product framework in scenario two. We provide a three-product example with a required profit-loading constraint and find the optimal product mix. The results show that the CVaRM approach achieves a better hedging effect than the MDM approach under the required profit-loading constraint.

In the simulation, we first generate 10,000-times mortality processes to obtain the distributions of v^i. The loss distribution, r^i, is the product value minus its expected value, divided by the expected value as shown in Eq. (8). With these returns, we estimate the weights w^D (duration match), w^C (convexity adjustment) and w^{CVM} (CVaRM). The confidence intervals are chosen as $\alpha = 99\%$, 95% and 90%. To implement the CVaRM optimization, we follow the methodology of Rockafellar and Uryasev (2000). The algorithm is implemented in C++ and we used the CPLEX 7.0 Callable Library to solve the linear programming problem.

4.1. Scenario 1: Pure risk reduction and two-product hedging

We consider a two-product framework in this scenario. There are five cases. Case 1 is the distribution of the whole-life annuity; Case 2 is whole-life insurance; Case 3 is the mixed distribution of w^D; Case 4 is the mixed distribution of w^C and Case 5 is the mixed distribution of w_α^{CVM}. We solve w_α^{CVM} using the following approach:

$$\underset{w}{\text{Min}} \quad E\left[r_p \left| r_p \geq r_p(\alpha)\right.\right]$$

$$s.t. \quad w^a + w^l = 1, \quad \text{and } 0 \leq w^a,\ w^l \leq 1$$

where w^a and w^l are the weights for the annuity and life insurance, respectively.

The loss distributions are shown in Fig. 2. In Fig. 2, the whole-life annuities (Case 1) have the widest distribution among all cases and represent a high-risk product. The distribution of whole-life insurance (Case 2) has a more central distribution and lower risk than Case 1. The annuity issued for men aged 60 has a wider distribution than the life insurance issued for men aged 40 before hedging. We find a narrower distribution in Case 3, in which the annuity distribution is mixed with some life insurance, $w^D = 10.6\%$, as suggested by the MDM approach. We have a narrower and centered loss distribution in Case 4, but the effect of the convexity adjustment does not cause a significant improvement in the tail distributions. The postulation is that the convexity adjustment does not work well with distribution risk. The risk is significantly reduced in Case 5; the distribution reveals higher frequency in the center and lower frequency in the tails after hedging under the weight, $w_{99\%}^{\text{CVM}} = 67.3\%$. The hedging result for the CVaRM approach has the smallest tail risk among cases 3 through 5.

The CVaRs and statistics are shown in Table 2. The 99% CVaRs of annuities and life insurance are 3.712 and 1.853, respectively. Hedged by the MDM and MDM with convexity, the 99% CVaRs decrease to 3.324 and 3.348. The CVaRM method decreases the 99% CVaR to 1.725. We can see similar effects for other confidence intervals, i.e., 95% and 90%. The CVaRM approach offers the smallest CVaR, which is even smaller than that for whole-life insurance. Column 5 shows that the CVaRM also has the smallest standard deviations (0.662%). This approach achieves a hedging effect in terms of the variance too. We do not add the required loading constraint in this scenario, but merely present the weighted loadings in the last column. The CVaRM approach has a lower weighted-loading profit (0.599 bps).

In Fig. 3 the whole-life insurance for the age-60 cohort is replaced by the age-40 cohort. In Fig. 3, the distribution for the whole-life insurance has a more centered distribution than its counterpart in Fig. 2. To reflect this change, the MDM approach holds more life insurance and increases w^D from 10.6% to 36.7%, i.e., see column 6 of Table 3. However, the CVaRM approach increases

[3] If the market prices of mortality-linked securities are available, then the mortality distribution could be transformed into a pricing distribution. For more details, see Cairns et al. (2006b).

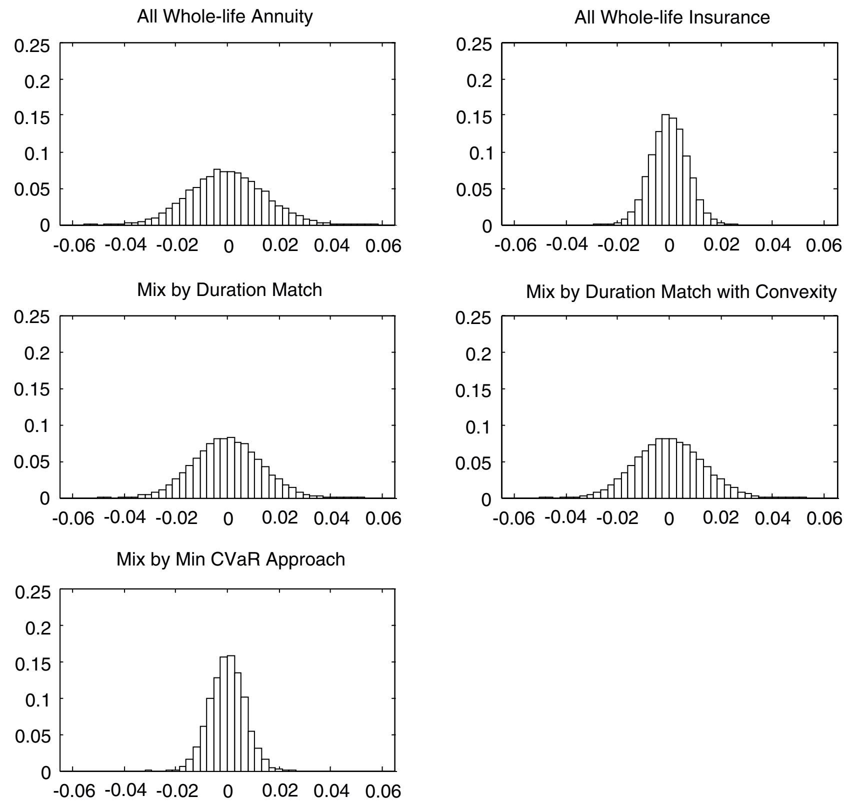

Fig. 2. The loss distributions for whole-life annuities at age 60 and whole-life insurance at age 40 (x-axis: values of r_p, y-axis: relative probability).

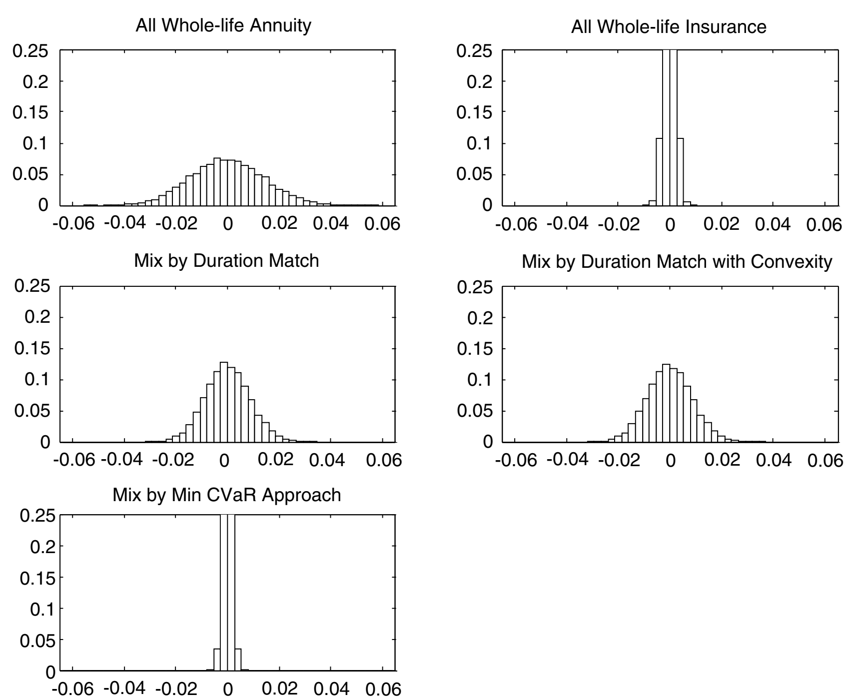

Fig. 3. The loss distributions for whole-life annuities and whole-life insurance both at age 60 (x-axis: values of r_p, y-axis: relative probability).

the weight $w_{99\%}^{\mathrm{CVM}}$ from 67.3% to 87.2%. The CVaRM approach recommends much more life insurance than the MDM approach.

Table 3 shows the CVaRs of these distributions. Consider the 99% CVaR in column 1 as an example: the whole-life annuity and whole-life insurance CVaR values are 3.712 and 0.583, respectively. After being hedged by the MDM and MDM with the convexity, the CVaRs decrease to 2.206 and 2.238, respectively. The CVaRM approach reduces CVaR to 0.379, the smallest value. These results show that the CVaRM approach offers a better hedging performance, in the scenario of the age-40 cohort.

The tendency in this risk-reduction scenario is to hold too much life insurance, as the CVaRM approach suggests. However, the weighted profit loadings of the CVaRM approach are the smallest (0.222 bps), as shown in the last column of Table 3. The loading constraints are included in the next scenario to fix this problem.

Table 3
The CVaRs and statistics of the loss distributions for Fig. 3.

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std. $\times 10^2$	$w^l \times 10^2$	Loading $\times 10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	0.0	1.438
All whole-life insurance	0.583	0.090	0.038	0.000	0.217	100.0	0.044
Duration match	2.206	0.345	0.147	0.000	0.851	36.7	0.926
Duration with convexity	2.238	0.350	0.149	0.000	0.863	35.9	0.938
The CVaRM approach	0.379	0.059	0.024	0.000	0.147	87.2	0.222

The Std. and w^l are multiplied by 100. The loading values are multiplied by 10,000 (for example, 1.438 is 0.0001438).

Table 4
The CVaRs and statistics of the loss distributions with $\bar{\pi} = 1$ bps.

Model	$\alpha = 0.99$	$\alpha = 0.95$	$\alpha = 0.90$	Average	Std. $\times 10^2$	w^a	w^{l1}	w^{l2}	Loading $\times 10^4$
All whole-life annuity	3.712	0.580	0.247	0.000	1.428	100.0	0.0	0.0	1.438
All whole-life insurance	1.853	0.286	0.122	0.000	0.696	0.0	100.0	0.0	0.192
All term-life insurance	6.532	1.017	0.435	0.000	2.517	0.0	0.0	100.0	1.268
The CVaRM approach	2.666	0.422	0.181	0.000	1.059	50.6	27.6	21.8	1.056

The Std. and w^l are multiplied by 100. The loading values are multiplied by 10,000 (for example, 1.438 is 0.0001438).

4.2. Scenario 2: Multi-product mix with loading constraint

Assume that the insurance company sells three life insurance products in the market. The three products are whole-life annuities for the age-60 cohort, whole-life insurance and 20-year term-life insurance for the age 40 cohort. The CVaRM approach with the required profit-loading constraint is

$$\begin{aligned} &\underset{w}{\text{Min}} \quad E\left[r_p \,\middle|\, r_p \geq r_p(\alpha)\right] \\ &s.t. \quad w^a \cdot \pi^a + w^{l_1} \cdot \pi^{l_1} + w^{l_2} \cdot \pi^{l_2} \geq \bar{\pi}, \\ &\qquad\; w^a + w^{l_1} + w^{l_2} = 1, \quad \text{and } 0 \leq w^a,\ w^{l_1},\ w^{l_2} \leq 1 \end{aligned}$$

where w^a is the weight of the whole-life annuity, w^{l_1} is the weight of the whole-life insurance, and w^{l_2} is the weight of the term-life insurance. π^a, π^{l_1} and π^{l_2} are profit loadings on the annuity, whole-life and term-life insurance, respectively, and are 1.438, 0.192, and 1.268 basis points, as implied by the premium-pricing principle, with the Sharpe ratio being equal to 15%. Consider the example of whole-life insurance: $E(v^{l1}) = 54.41$, $\sigma(r^{l1}) = 0.00696$ (in Table 4), and $SR = 15\%$ indicates that $\pi^{l_1} = 0.192$ basis points.[4] These loadings reflect the profit requested by the shareholders bearing the systematic mortality risk in the capital market. The higher the standard deviation, the higher the profit loading. The target return, $\bar{\pi}$, is set equal to 1 basis point. The CVaRs of the three-product distributions and their product mix are shown in Table 4.

In Table 4, the all whole-life insurance has a small CVaR value, 1.853 under $\alpha = 0.99$. However, the CVaRM approach cannot hold this product too much under the inequality constraint. The CVaRM approach proceeds with the trade-off between r_p and π^i, and then gives $w^a = 50.6\%$, $w^{l_1} = 27.6\%$ and $w^{l_2} = 21.8\%$ under the 99% confidence interval. In row 4 of Table 4, we have mild CVaRs, e.g., CVaR = 2.666 for $\alpha = 0.99$, and the weighted loading is 1.056 basis points, which is very close to 1 basis point which means that the constraint is active. The other two mortality confidence intervals, 95% and 90%, lead to similar hedging results.

5. Conclusion and discussion

This article proposes a new approach to optimize the insurer's product mix under systematic mortality risk. By incorporating the natural hedging strategy of Cox and Lin (2007), the two-factor stochastic mortality model of Cairns et al. (2006b), and the Sharpe ratio-loading price of Milevsky et al. (2006), we construct a CVaRM approach to evaluate the product mix. We consider two numerical scenarios: the two-product case without a loading constraint and the multi-product case with a loading constraint. In the first scenario, the CVaRM approach exerts a better risk-reduction effect than the MDM approach. In the second scenario, the three-product example reveals a trade-off between the CVaR and the required loadings. The results show that the proposed CVaRM approach leads to an optimal product mix and effectively reduces the mortality risks associated with forecasting longevity patterns for life insurance companies.

Some important issues for future research and practice clearly deserve further investigation. First, this paper deals with the parameter risk, but ignores the misspecification or modeling risk. For example, the real mortality process may not follow the CBD model. Second, this paper omits the basis risk of the mortality rate between life insurance and annuities because of the data limitations. Our numerical example assumes that the mortality processes for life insurance and annuities are the same. In fact, the mortality experiences may differ for these products. Third, in this study, the premium loadings for each product are decided individually by means of the Sharpe ratio. To maintain rigidity, they should be priced according to their contributions to the aggregated risk, in a way similar to the beta concept of the Capital Asset Pricing Model (CAPM). This work is beyond the scope of this paper, and so we leave it for future study. Finally, we illustrate the hedging strategy with a mortality term structure, but a flat interest-rate yield curve. An analysis of the combined effects of stochastic mortality and stochastic interest rate would offer more realistic results.

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[4] Here we assume that the premium loadings are given, i.e., the firm with a natural hedging strategy can take a free ride on others without natural hedging.

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