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Optimal asset allocation for a general portfolio of life insurance policies

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ABSTRACT

Asset liability matching remains an important topic in life insurance research. The objective of this paper is to find an optimal asset allocation for a general portfolio of life insurance policies. Using a multi-asset model to investigate the optimal asset allocation of life insurance reserves, this study obtains formulae for the first two moments of the accumulated asset value. These formulae enable the analysis of portfolio problems and a first approximation of optimal investment strategies. This research provides a new perspective for solving both single-period and multiperiod asset allocation problems in application to life insurance policies. The authors obtain an efficient frontier in the case of single-period method; for the multiperiod method, the optimal asset allocation strategies can differ considerably for different portfolio structures.

1. Introduction

This article attempts to investigate optimal asset allocations in a stochastic investment environment for a general portfolio of life insurance policies. Asset liability matching remains an important issue for long-term liabilities, such as insurance policies or pension funds. Most insurance companies' assets consist of policyholders' premiums, such that each policy represents a specific liability for the insuring firm. Thus, maximizing investment returns may not be the primary goal of insurance companies, which may instead be more concerned with managing policyholders' premiums, such that returns adequately meet future benefits or guarantee profits to policyholders. Besides, insurance policies offered by insurers also vary in duration, so it is not realistic to discuss only the case of asset liability management for a single policy. Therefore, we consider asset liability management in a more general case in which random policy durations exist in a product's profile.

In this paper, we investigate the asset allocation issue on life insurance reserves. Previous research focuses on studying the distribution of reserves, from one policy (e.g. Panjer and Bellhouse, 1980; Bellhouse and Panjer, 1981; Dhaene, 1989) to portfolios (e.g. Waters, 1978; Parker, 1994a,b; Marceau and Gaillardetz, 1999). They examine the distribution of life insurance reserves in a specific interest rate model, for example, AR(1) model or ARCH(1) model. We introduce a multi-asset model and consider the asset allocation problem of life insurance companies.

For the asset allocation issue, widespread investigations consider the investment strategy for a single-period approach (e.g. Hurlimann, 2002; Sharpe and Tint, 1990; Sherris, 1992, 2006; Wilkie, 1985; Wise, 1984a,b, 1987a,b), whereas multiperiod asset allocations in discrete time rarely have been explored. Extensive research into the optimal multiperiod investment strategy concentrates mostly on continuous time models with dynamic controls (e.g. Chiu and Li, 2006; Emms and Haberman, 2007) or uses the martingale method (e.g. Wang et al., 2007). With these approaches, the optimal strategy takes the new information generated by filtration, but to solve the closed form solution in discrete time, they often suffer from mathematical complexity and intractability. Another approach is to get the numerical solution by stochastic programming (see Dempster, 1980; Carino et al., 1994). It overcomes the disadvantage of finding theoretical solution. The model can be constructed easily and realistically. However, it faces other problems. Stochastic programming constructs the possible asset return scenario by trees. A good description about the market depends on the number of nodes at each decision point. On the other hand, the time cost has an exponential growth as does the number of nodes. Thus, for the purpose of finding the solution in a tolerable time, the node number is often insufficient to describe the real market. Consequently, it is inevitable for the existence of large approximation error. Thus, to examine the appropriate investment strategy in discrete time, we must trade off between the convenience of the method and the accuracy of the solution.

Specifically, we examine two kinds of rebalancing methodologies: constant rebalancing and variable rebalancing. At the beginning of every year, the portfolio mix gets realigned according to one of these two methods. Constant rebalancing means that the portfolio mix should realign to a constant proportion of assets, or the single-period method. Variable rebalancing implies that the portfolio mix realigns with a different proportion of assets each time, or the multiperiod method.

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A continuing business line contains some mature polices and some new policies every year, so the single-period method is more suitable than a multiperiod method, because of the ongoing new policies. If the number of mature polices is close to the number of new policies, the structure of the policy portfolio remains similar between two neighboring decision dates. Therefore, it would be reasonable to adopt a constant rebalance strategy and retain the same weight of assets in a stable proportion every year. If the number of mature polices differs from the number of new policies, the durations of the policy portfolios should differ every year. Therefore, the proportion of constant rebalance requires recalculation, according to the updated durations of policy portfolios every year. Occasionally, a business line ceases to exist, so no new policies occur in the future. In this case, variable rebalancing with a multiperiod method is suitable for matching the rest of the periods of the liabilities.

Because insurance policies typically involve a long duration of more than five years, choosing the optimal investment strategies is crucial to ensure that insurance companies can maximize their profits while reducing their insolvency risk. We propose an optimization approach for analyzing the optimal portfolio problem for both single-period and multiperiod asset allocations.

In turn, we propose an optimization approach to generate the optimal investment strategy of an asset liability management model for long-term endowment policies. The proposed discrete time investment model includes both static, single-period and multiperiod optimal investment strategies for a time-dependent asset return process. We derive the formulae for the first and second moments of the accumulated asset value of the insurer based on a multi-asset return model. With these formulae, we can analyze the portfolio problems for both single-period and multiperiod methods. For the single-period method, we depict an efficient frontier under a constant rebalance strategy, which can be determined from arbitrary policy portfolios. In the case of the multiperiod method, we obtain a first approximation of the optimal asset allocation, as applied to a ceased life insurance product line. The numerical results show that the proportion of cash should increase when we compare a portfolio with uniform years before maturity with a portfolio comprised of new policies.

In Section 2, we formulate the explicit form of the first two moments of accumulated asset value, followed by the mean-variance analysis and an investigation of the optimal asset allocation strategy for various policy portfolios in Section 3. We then examine the parameter sensitivities and discuss the large sample problem in Section 4. Finally, we give the conclusion in Section 5.

2. Model setting

The liability reserve is the value of the difference between the present value of the future benefits and the future premiums received, which is often discounted by conservative rates. Reserves can be viewed as policyholders' credit. The investment manager of an insurance company attempts to make profits from these credits while also ensuring the solvency of the insurer.

The liability reserve of a policy portfolio of an insurer also can be expressed as the present value of the stochastic cash flows (Lai and Frees, 1995; Marceau and Gaillardetz, 1999). Let $CF(j)$ be the cash inflow of the insurer at time j, or the net difference between the premiums received and the benefits paid at time j $(j = 1, \ldots, n)$. In addition, $v(j)$ is the specified discount factor from time j to time 0. We define L as the liability reserve of a policy portfolio after the enforced premiums are paid at the valuation date $j = 0$, which can be expressed as

$$\begin{aligned} L &= PV\,(\text{future benefits}) - PV\,(\text{future premiums received}) \\ &= -\sum_{j=1}^{n} E\left[CF(j)\right] v(j), \end{aligned}$$

where n is the maximum remaining policy term for this portfolio. Thus, the reserve L is determined exogenously and equals the minimal asset value of the insurance company at the valuation date. Frequently, the life insurance authorities in various countries require that the liabilities are valued on a market basis. Therefore, the liabilities of a life insurance portfolio behave more like a portfolio of bonds. In addition, nowadays the life insurance authorities in various countries require that a certain amount of money needs to be set aside as capital at the end of each year. In this paper, we ignore these two issues and assume that the required asset at time 0 is L, consistent with Hurlimann (2002). In turn, this article proposes a feasible asset allocation model to manage L.

Let $I(j)$, $j = 0, \ldots, n$, be the accumulation factor from time j to time n, which depends on the asset allocation strategy of the insurance company, and $I(n) = 1$. We define $F(j)$ as the accumulated asset value after adding $CF(j)$ at time j $(j = 1, \ldots, n)$, and $F(0) = CF(0) = L$. Hence,

$$F(j+1) = F(j)\,\frac{I(j)}{I(j+1)} + CF(j+1),$$

and the accumulated asset value at time n can be written as

$$F(n) = \sum_{j=0}^{n} CF(j)\, I(j).$$

We can describe the first two moments of $F(n)$ with the following lemma.

Lemma 2.1. *Assuming the asset returns and mortality processes are independent, the first two moments of the accumulated asset value at time n are given by*

$$E\left[F(n)\right] = \sum_{j=0}^{n} E\left[CF(j)\right] E\left[I(j)\right], \quad \textit{and}$$

$$Var\left[F(n)\right] = E\left[Var\left[F(n)\left|i^{*}\right.\right]\right] + Var\left[E\left[F(n)\left|i^{*}\right.\right]\right],$$

where

$$E\left[Var\left[F(n)\left|i^{*}\right.\right]\right] = \sum_{j=0}^{n}\sum_{k=0}^{n} E\left[I(j)\,I(k)\right] Cov\left[CF(j), CF(k)\right],$$

$$Var\left[E\left[F(n)\left|i^{*}\right.\right]\right] = \sum_{j=0}^{n}\sum_{k=0}^{n} E\left[CF(j)\right] E\left[CF(k)\right] Cov\left[I(j), I(k)\right],$$

and i^{*} represents the information set for the asset returns until time n.

Thus, to obtain the closed form of the first two moments of $F(n)$, we first must create the asset return and cash flow models, respectively. The first two moments of the accumulated function, $I(j)$, and the first two moments of the cash flow function, $CF(j)$, then can be calculated. We do not focus on the mortality model herein; we provide the cash flow model and calculations for the corresponding first two moments in Appendix A. Instead, we focus on calculating the accumulate function, for which we require an asset return model, and formulate an investment strategy to produce the accumulated value.

In the asset return model, we adopt the discrete model proposed by Huang and Cairns (2006), which includes three assets: a one-year bond (cash), a long-dated bond, and an equity asset. The log-return rates between time $t-1$ and t of these assets can be denoted $y(t-1)$, $\delta_b(t)$, and $\delta_e(t)$, respectively, with the following underlying processes[1]:

$$\begin{cases} y(t-1) = y + \phi(y(t-2) - y) + \sigma_y Z_y(t-1) \\ \delta_b(t) = y(t-1) + \Delta_b(t) \\ \quad = y(t-1) + \Delta_b + \sigma_{by} Z_y(t) + \sigma_b Z_b(t) \\ \delta_e(t) = y(t-1) + \Delta_e(t) \\ \quad = y(t-1) + \Delta_e + \sigma_{ey} Z_y(t) + \sigma_{eb} Z_b(t) + \sigma_e Z_e(t), \end{cases}$$

[1] For a general form of this asset model, please refer to chapter 2.1 of Campbell and Viceira (2002).

where $Z_y(t)$, $Z_b(t)$, and $Z_e(t)$ are $N(0,1)$ random variables that are independent of one another and i.i.d. through time t. The cash asset follows an AR(1) process and the excess returns on the long-dated bond (the equity asset) are i.i.d. and normally distributed. The time indicator of the parentheses represents the measurability of the random variables. For example, both $Z_y(t-1)$ and $y(t-1)$, are measurable at time $t-1$, whereas $Z_y(t)$, $Z_b(t)$, and $\delta_b(t)$ are not measurable until time t.

Let p_{1t} represent the proportion of the liability invested in equities and p_{2t} represent the proportion of the liability invested in long-dated bonds at time $t-1$. The remaining assets shall be held in the form of cash. Let $i(t)$ be the portfolio return from $t-1$ to t. Using a second-order Taylor approximation of the nonlinear function that relates the log-individual asset returns to the log-portfolio returns, we can write the log return on the fund from $t-1$ to t, $Z(t)$, as:

$$\begin{aligned}Z(t) &= \ln(1+i(t))\\ &= y(t-1) + (p_{1t}, p_{2t})\begin{pmatrix}\Delta_e(t)\\ \Delta_b(t)\end{pmatrix} + \frac{1}{2}(p_{1t}, p_{2t})\begin{pmatrix}v_{ee}\\ v_{bb}\end{pmatrix}\\ &\quad - \frac{1}{2}(p_{1t}, p_{2t})\,\Sigma\begin{pmatrix}p_{1t}\\ p_{2t}\end{pmatrix}\\ &= y(t-1) + (p_{1t}, p_{2t})\begin{pmatrix}\Delta_e & \sigma_{ey} & \sigma_{eb} & \sigma_e\\ \Delta_b & \sigma_{by} & \sigma_b & 0\end{pmatrix}\begin{pmatrix}1\\ Z_y(t)\\ Z_b(t)\\ Z_e(t)\end{pmatrix}\\ &\quad + \rho(p_{1t}, p_{2t})\end{aligned}$$

where

$$\rho(p_{1t}, p_{2t}) = \frac{1}{2}(p_{1t}, p_{2t})\begin{pmatrix}v_{ee}\\ v_{bb}\end{pmatrix} - \frac{1}{2}(p_{1t}, p_{2t})\,\Sigma\begin{pmatrix}p_{1t}\\ p_{2t}\end{pmatrix},$$

$$v_{ee} = Var\,\Delta_e(t), \qquad v_{bb} = Var\,\Delta_b(t),$$

$$\Sigma = \begin{pmatrix}v_{ee} & v_{eb}\\ v_{eb} & v_{bb}\end{pmatrix}, \quad \text{and} \quad v_{eb} = Cov(\Delta_e(t), \Delta_b(t)).$$

This discrete time approximation is an exact version within the limit of continuous time. Huang and Cairns (2006) show that the return $1+i(t)$ at time t can be described as a fair return at t on an initial investment of 1 at time $t-1$ under the continuous time model and the complete market assumption. The function $\rho(p_{1t}, p_{2t})$ is a second-order adjustment which ensures that the model is arbitrage-free.

Note that the return rate of each asset follows Log-normal distribution. Since the products of Log-normal random variables are themselves Log-normal, this means for an initial investment of 1 in a single asset at time 0 the final payoff at time t is Log-normal distributed. However, this advantage cannot be carried over directly from individual asset to portfolios. By means of the second-order Taylor approximation to define the portfolio return $Z(t)$, we can approximate the portfolio return with Log-normal distribution and so can take the advantage of the Log-normal assumption.

Theorem 2.2. *The first two moments of the accumulate functions $I(j)$ can be written as follows:*

(a) *The expected value, variance, and covariance of $Z(t)$ at the valuation date are:*

$$E[Z(t)] = y + (y(0)-y)\,\phi^{t-1} + (p_{1t}, p_{2t})\begin{pmatrix}\Delta_e\\ \Delta_b\end{pmatrix} + \rho(p_{1t}, p_{2t}),$$

$$Var[Z(t)] = \frac{\sigma_y^2}{1-\phi^2}\left(1-\phi^{2(t-1)}\right) + (p_{1t}, p_{2t})\,\Sigma\begin{pmatrix}p_{1t}\\ p_{2t}\end{pmatrix},$$

and

$$Cov[Z(t), Z(t+k)] = \frac{\sigma_y^2}{1-\phi^2}(1-\phi^{2(t-1)})\phi^k + \phi^{k-1}(p_{1t}, p_{2t})\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y; \quad \forall k \in N$$

(b) *If we define $S(k) = \sum_{j=1}^{k} Z(j)$, then the accumulation factor $I(k)$, depending on the actuarial portfolio selection of the insurer, will be $I(k) = \exp\{S(n) - S(k)\}$. Moreover, the expected value, variance, and covariance of $S(k)$ are:*

$$E[S(k)] = ky + (y(0)-y)\frac{1-\phi^k}{1-\phi} + \sum_{j=1}^{k}\left[(p_{1j}, p_{2j})\begin{pmatrix}\Delta_e\\ \Delta_b\end{pmatrix} + \rho(p_{1j}, p_{2j})\right],$$

$$\begin{aligned}Var[S(k)] &= \frac{\sigma_y^2}{(1-\phi)^2}\left[(k) - \frac{1-\phi^{2k}}{1-\phi^2}\right] + \sum_{j=1}^{k}(p_{1j}, p_{2j})\,\Sigma\begin{pmatrix}p_{1j}\\ p_{2j}\end{pmatrix}\\ &\quad + \frac{2\sigma_y^2\phi}{(1-\phi^2)(1-\phi)}\left[(k-1) - \frac{\phi(1-\phi^{k-1})}{1-\phi} - \frac{1-\phi^{2(k-1)}}{1-\phi^2} + \frac{\phi^{k-1}(1-\phi^{k-1})}{1-\phi}\right]\\ &\quad + 2\sum_{j=1}^{k-1}\frac{1-\phi^{k-j}}{1-\phi}(p_{1j}, p_{2j})\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y, \quad \text{and}\end{aligned}$$

$$\begin{aligned}Cov[S(k), S(k+m)] &= Var[S(k)] + \frac{\sigma_y^2\phi(1-\phi^m)}{(1-\phi^2)(1-\phi)}\frac{(1-\phi^k)(1-\phi^{k-1})}{(1-\phi)}\\ &\quad + \sum_{j=1}^{k}\left[\frac{\phi^{k-j}(1-\phi^m)}{1-\phi}(p_{1j}, p_{2j})\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\right].\end{aligned}$$

(c) *The moments associated with the asset accumulation model are:*

$$E[I(j)] = \exp\left\{E[S(n)] - E[S(j)] + \frac{1}{2}[Var(S(n)) + Var(S(j)) - 2Cov(S(n), S(j))]\right\},$$

$$E[I(j)\,I(k)] = \exp\{C_{j,k} + D_{j,k}\}, \quad \text{and}$$

$$Cov[I(j), I(k)] = E[I(j)\,I(k)] - E[I(j)]\,E[I(k)],$$

where

$$C_{j,k} = 2E[S(n)] - \{E[S(j)] + E[S(k)]\}, \quad \text{and}$$

$$\begin{aligned}D_{j,k} &= \frac{1}{2}\{4Var[S(n)] - 4Cov[S(n), S(j)] - 4Cov[S(n), S(k)]\\ &\quad + Var[S(j)] + 2Cov[S(j), S(k)] + Var[S(k)]\}.\end{aligned}$$

We provide the proof of Theorem 2.2 in Appendix B.

By including the results of Theorem 2.2(c) and the Lemma in Appendix A into Lemma 2.1., we can obtain the explicit formulae of the first two moments of accumulated asset value.

3. Optimal asset allocation

In this section, we investigate the optimal asset allocation of three unique cases using both constant and variable rebalancing methodologies. These two methodologies are independent of the

Table 1
Number of policies maturing every year for each 10-policy portfolio.

Year(s) before maturity	1	2	3	4	5	6	7	8	9	10
Case A	0	0	0	0	0	0	0	0	0	10
Case B	1	1	1	1	1	1	1	1	1	1
Case C	1	2	1	1	0	3	0	1	0	1

investment model. Constant rebalancing means that the portfolio mix should realign to a constant proportion of assets. In contrast, the variable rebalancing method means that the portfolio mix could realign to a different proportion of assets at the beginning of the year. In other words, the realignment is limited to a constant proportion in constant rebalancing but is unconstrained in variable rebalancing. For being more realistic, short selling is forbidden in this study.

In an attempt to ensure realism, we consider three endowment policy portfolios that contain 10 policies each and use the same 10-year term. Case A contains 10 new policies at the valuation date, similar to the case of a single policy. Case B is a uniform case with 10 different maturity dates. Finally, Case C represents a realistic sample with maturity dates generated stochastically. The policyholders are all aged 30 at the policy issue date. For simplicity, we adopt an underlying mortality index based on the American man's life table[2] for 2002, set the death benefit and survival benefit to 1, and assume the yearly payment premium is 0.0845. Table 1 provides a detailed composition of the three portfolios.

In this study, the numerical results are based on the following parameters: $\phi = 0.8965$, $y = 0.0441$, $\Delta_b = 0.0265$, $\Delta_e = 0.0317$, $\sigma_y = 0.0170$, $\sigma_b = 0.0971$, $\sigma_{by} = -0.0294$, $\sigma_e = 0.1424$, $\sigma_{ey} = -0.0500$, $\sigma_{eb} = 0.0065$, $y(0) = 0.0192$, and $v(j) = e^{-y(0)j}$. They are calibrated by the US historical data from 1978 to 2007.[3] The low initial return rate of cash asset model ($y(0)$) reflects the low interest rate situation in the recent years. According to the historical data, the long-term mean of cash asset is 4.41%. The excess means of long-dated bond and stocks are 0.0265 and 0.0317, and the standard deviations of $\Delta_b(t)$ and $\Delta_e(t)$ are 10.14% and 15.1%, respectively.

3.1. Constant rebalancing

For constant rebalancing, we first look at the mean–standard deviation plot of $F(n)$, $n = 10$. According to Lemma 2.1 and Theorem 2.2, we depict Fig. 1 for Case B; those for Cases A and C are similar. Because they are all analogous and Case B is the most general case, we only depict it in this study. We see from Fig. 1 that an efficient frontier is obtained in the mean–standard deviation plot. At this efficient frontier, we can find an optimal investment strategy that either achieves the maximum accumulated value, given a certain level of risk, or obtains the minimum level of risk according to an identified, permissible level of accumulated value. The optimal investment strategy should be selected from this efficient frontier. For example, if the risk tolerance, or the standard deviation of accumulated value, is 1.5, the insurance company's portfolio mix should consist of 27% stock holdings, 42% long-dated bonds, and 30% cash,[4] which creates a maximum accumulated value of 1.74. If the insurance company's target accumulated value is 1, the insurance company should hold an asset mix comprised of 14% stocks, 19% long-dated bonds, and 67% cash, resulting in a

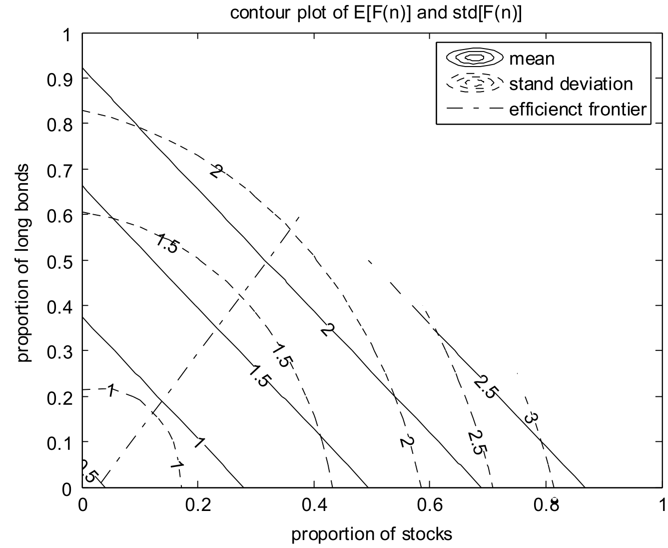

Fig. 1. Mean–standard deviation plot of Case B.

minimum standard deviation with the accumulated value of 1. As we can further note from Fig. 1, the efficient frontier moves down along the diagonal line if a short sales constraint exists. Therefore, we know that no cash should be invested if the target accumulated value is sufficiently large.

In the constant rebalancing case, insurance companies can find an optimal investment strategy from the efficient frontier to increase the expectation and/or to reduce the variance of accumulated asset value. For example, an insurance company may set its objective function as follows:

$$\max\left\{E[F(n)] - kVar[F(n)]^{0.5}\right\}, \tag{3.1}$$

where k is a specific, positive, real number. The candidates for the optimal strategy can be obtained from the efficient frontier. The value k represents how much risk an insurance company is willing to suffer. Furthermore, the numerical results of the following optimization problems are carried out by applying the statistical software Matlab7.0.

Table 2 shows the optimal asset allocation strategy for some k in the case in which constant rebalancing occurs in accordance with the objective function. The result of Case C depends on the random sample. When $k = 1.5$ or 2, the short sales constraint does not affect the optimal investment strategy. Thus, in Case B, in comparison with Case A, the insurance company holds more of the riskless cash asset (from 0.3250 to 0.4742 or from 0.4964 to 0.5964) and less of the risky assets, namely, stocks (from 0.2669 to 0.2116 or from 0.2033 to 0.1664) and long-dated bonds (from 0.4081 to 0.3143 or from 0.3003 to 0.2372). One policy matures in each of the ten years in Case B, and the insurance company needs to hold more cash to reduce its illiquidity risk for policies that mature at earlier dates. In comparison with Case B, an insurance company holds more of the riskless cash asset (from 4742 to 0.5437 or from 0.5964 to 0.6487) and less of the risky stock (from 0.2116 to 0.1841 or from 0.1664 to 0.1452) and long-dated bonds (from 0.3143 to 0.2723 or from 0.2372 to 0.2060) assets in Case C, because of a shorter duration (two policies will mature in two years and three policies will mature in six years). The insurance company therefore needs more cash in the earlier term for Case C than for Case B.

Also, we may observe the sensitivity of k from Table 2. As k increases, the weight of $Var[F(n)]$ increases, and the investment strategy should be more conservative. In contrast, as k decreases, the weight of $E[F(n)]$ increases, so the investment strategy should be more aggressive.

[2] Data source: The Human Mortality Database (http://www.mortality.org).

[3] The data for the calibration of one-year bonds and long-term bonds are on the Web address: http://federalreserve.gov/releases/h15/data.htm.

[4] The sum of the portfolio proportion (and hereafter) is not equal to 1 due to rounding error.

Table 2
Optimal asset allocation of the portfolio using constant rebalance.

k		Cash	Long-dated bond	Stock
1	Case A	0.0000	0.6034	0.3966
	Case B	0.1406	0.5246	0.3348
	Case C	0.2507	0.4570	0.2923
1.5	Case A	0.3250	0.4081	0.2669
	Case B	0.4742	0.3143	0.2116
	Case C	0.5437	0.2723	0.1841
2	Case A	0.4964	0.3003	0.2033
	Case B	0.5964	0.2372	0.1664
	Case C	0.6487	0.2060	0.1452

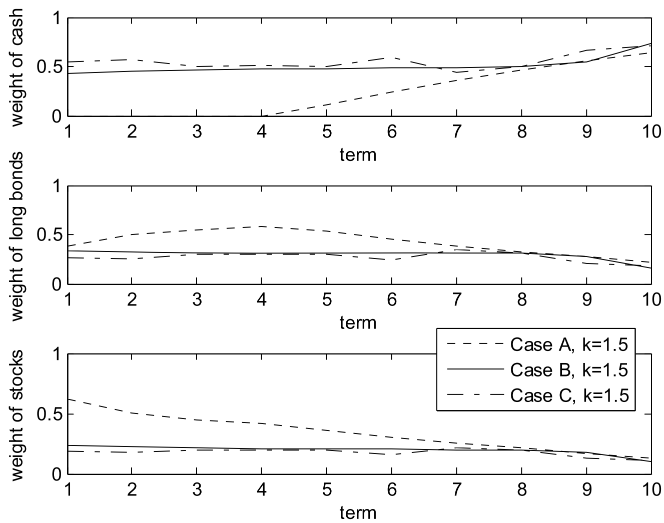

Fig. 2. Optimal variable asset allocation, $k = 1.5$.

3.2. Variable rebalancing

Fig. 2 depicts the optimal investment strategies for each case when $k = 1.5$, based on the application of the variable rebalancing methodology according to the objective function. We depict the results for all cases on a comparison plot. We first compare the optimal investment strategy of Cases A and B and find that the trend of optimal asset allocation varies greatly. In Case A, the proportion of cash increases while the proportion of stocks decreases, which matches a "top-down" investment strategy.[5] An insurance company will hold more risky assets at the beginning of the term to make more profits and then gradually switches to riskless assets to reduce its illiquidity risk and meet the liability payment in year 10 in Case A. In Case B, the top-down pattern is less significant to that of Case A, because one policy matures each year, so the insurance company needs to hold more cash upfront to meet the liability of its early maturing policies.

Fig. 2 also exhibits the optimal asset allocation for Case C. Due to the randomness of the remaining terms, the optimal investment strategy appears less smooth, though the pattern is similar to that for Case B (uniform case). Observing Fig. 2 and the remaining term of the policies (Case C, Table 1), we find that an increasing number of policies with coinciding maturity dates results in a larger proportion of cash held within the portfolio in the same period. For example, in Fig. 2, three policies will mature six years later, so the insurance company will hold a higher proportion of cash at the beginning of the sixth year.

5 This pattern will be more conspicuous when there is no short sales constraint.

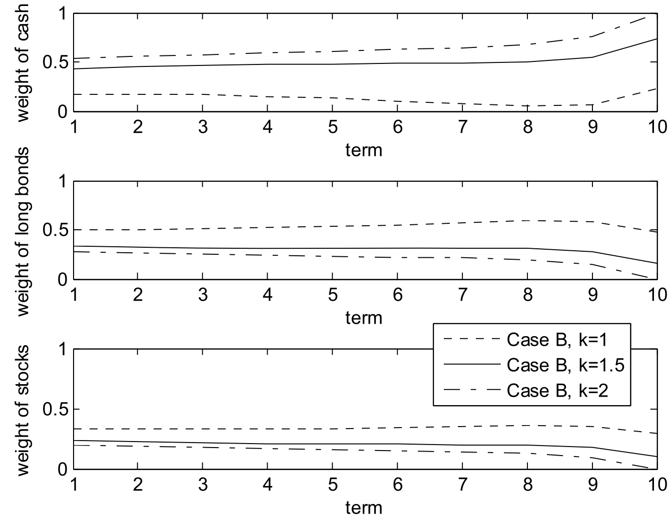

Fig. 3. Optimal asset allocation under variable rebalance, $k = 1$, 1.5, and 2.

4. Sensitivity analysis

The optimal asset allocation depends on the selection of a range of parameters. In this section, we examine the sensitivity of the optimal investment strategy to some parameters. We first investigate the sensitivity of the optimal multiperiod asset allocation to the parameter k, the weighted coefficient of the objective function, and then survey the sensitivity to the parameters of asset model. In terms of the sensitivity to the parameters of the asset model, we explore two situations: a simultaneous increase of the excess returns on risky assets (Δ_b and Δ_e), and a simultaneous increase of the volatility coefficients, which include σ_y, σ_b, σ_{by}, σ_e, σ_{ey} and σ_{eb}. Specifically, we explore the optimal asset allocation when the excess returns on risky assets or the volatility coefficients increase by 0.5 times. Finally, we discuss the impact of the portfolio size to optimal asset allocation and suggest a practice technique when there is a large sample problem. In this section, we only explore the uniform case (Case B) at $k = 1.5$.

4.1. Sensitivity of the optimal asset allocation to the parameter k

In Table 2, we show the optimal asset allocations for $k = 1$, 1.5, and 2 under constant rebalancing. Here, we display the corresponding results in Fig. 3 for the application of the variable rebalancing methodology. The coefficient k can be viewed as the risk tolerance measure of the insurance company, such that as k increases, the insurer's risk-averse attitude increases, and the investment strategy becomes more conservative.

4.2. Sensitivity of the optimal asset allocation to the parameters of asset model

We again explore two situations: (1) a 1.5 times increase in the excess return on risky assets, which we refer to as the high excess mean case, and (2) a 1.5 times increase in the variance on all three assets, which we refer to as the high variance case. Table 3 shows the results for the constant rebalance rule, and Figs. 4 and 5 show the results of the variable rebalance rule. The insurer should be more aggressive as the excess mean rises and more conservative as variance increases, in both two rebalancing rules.

4.3. Sensitivity of the optimal asset allocation to the size of the portfolio

Table 4 shows the optimal asset allocation under constant rebalancing for different portfolio sizes, including 10,100,1000 and 10,000 policies in a single policy portfolio. Fig. 6 presents the

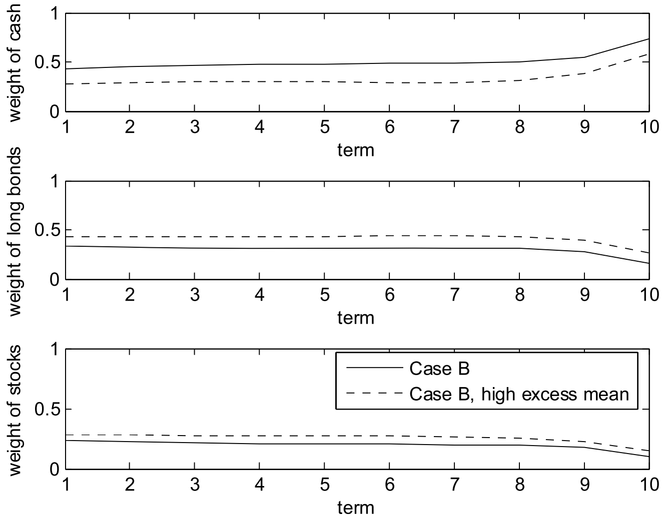

Fig. 4. Optimal asset allocation under the variable rebalance rule (high excess mean).

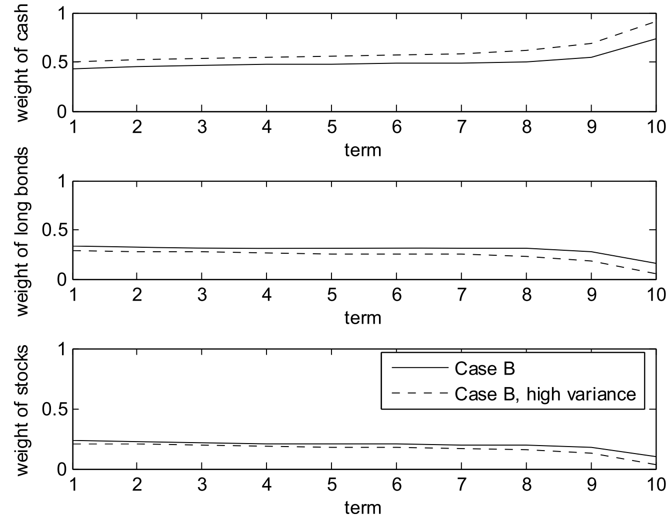

Fig. 5. Optimal asset allocation under the variable rebalance rule (high variance).

Table 3
Optimal asset allocation under the constant rebalance rule (varied parameters of asset model).

	Cash	Long-dated bonds	Stocks
Case B	0.4742	0.3143	0.2116
High excess mean	0.3089	0.4229	0.2682
High variance	0.5548	0.2605	0.1847

Table 4
Optimal asset allocation for different portfolio sizes under the constant rebalance rule.

Portfolio sizes	Cash	Long-dated bonds	Stocks
10	0.4742	0.3143	0.2116
100	0.4784	0.3116	0.2100
1,000	0.4788	0.3113	0.2099
10,000	0.4789	0.3113	0.2099

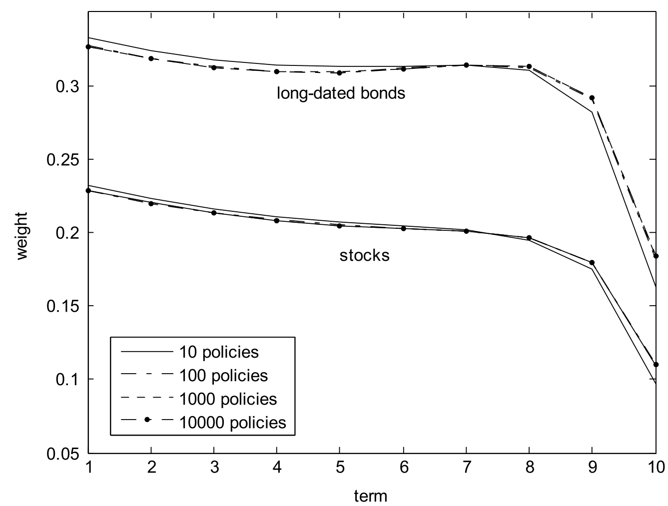

Fig. 6. Optimal asset allocation under the variable rebalance rule (different portfolio sizes).

impact of portfolio size to optimal asset allocation in the variable rebalancing strategy. The size of the portfolio indeed changes the optimal asset allocation with the objective function (3.1). We see from Table 4 and Fig. 6 that the insurance company is slightly getting towards being conservative when the portfolio size increases. However, the change of asset proportion is decreasing for both strategies of constant rebalancing and variable rebalancing. We find that the difference among a portfolio of 100 policies, 1000 policies and 10,000 policies is marginal. Hence we may conclude that the optimal asset proportion will converge to some value as the portfolio size approaches infinity.

4.4. Large sample problem — a practice technique

For a general policy portfolio with 10-year endowments, the years before maturity of each policy should be distributed stochastically across the integers 1 to 10. We set the uniform case as the base case and infer that if a portfolio is statistically indifferent from the base case, the optimal asset allocation strategy should be close to it. Following this idea, we compare this uniform case (100 policies) with the other four portfolios, each of which is composed of 100 policies, and the years before maturity for each policy is selected stochastically from 1, 2, …, 10. These four portfolios have the same statistical property, such that the years before maturity for these 100 policies of a single portfolio reach a p-value of 0.0487 according to the chi-square goodness-of-fit test. The null hypothesis states that the years before maturity for these 100 policies of a single portfolio may be selected from a discrete uniform distribution. Thus, these four portfolios are unlike those uniformly distributed in a statistical sense.

Table 5 provides the optimal asset allocation of the four special portfolios under the constant rebalance rule. Fig. 7 illustrates the optimal asset allocation of the four special portfolios under the variable rebalance rule. The results support our inference. As we show in Fig. 8, which depicts the distribution of the remaining term of the policies within each portfolio, most of the policies in Portfolio (a) mature in the second half of the 10 years, whereas in Portfolio (b), most of the policies will mature in the first half of the 10 years.

According to the constant rebalance rule, the proportion of cash for the base case is 47.84%, whereas the proportion of cash for the four special portfolios varies from 42.36% to 54.96%. As we demonstrated in Section 3.1, the insurance company should hold more cash to reduce its illiquidity risk for policies that mature earlier, so the highest proportion of cash should exist for Portfolio (b), whereas the smallest should occur for Portfolio (a). The optimal investment strategies for Portfolios (c) and (d) fall within the range marked by Portfolios (a) and (b).

With the variable rebalance rule, the patterns of the asset holding proportion of these four special cases are analogous to the base case, as depicted by the rough line (Fig. 7). As we observe in Case C (Section 3.2), in comparison with the base case, an increasing number of policies with coinciding maturity dates results in a larger

Table 5
Optimal asset allocation of the four special portfolios under the constant rebalance rule.

	Cash	Long-dated bonds	Stocks
Uniform case	0.4784	0.3116	0.2100
(a)	0.4236	0.3451	0.2314
(b)	0.5496	0.2685	0.1819
(c)	0.5010	0.2979	0.2011
(d)	0.4559	0.3253	0.2188

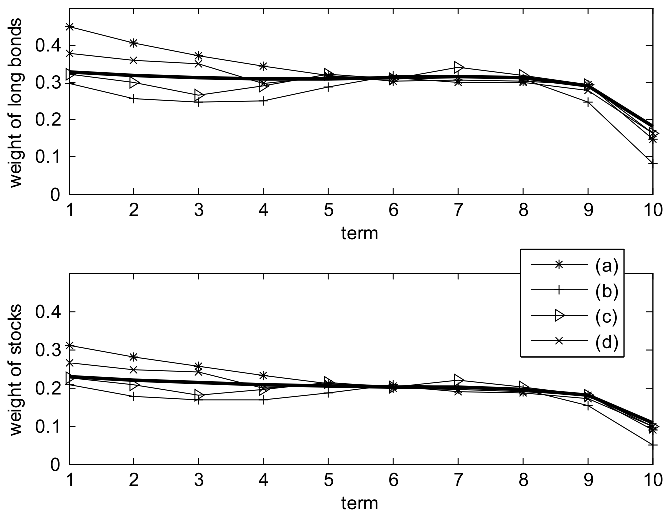

Fig. 7. Optimal asset allocation of the four special portfolios under the variable rebalance rule.

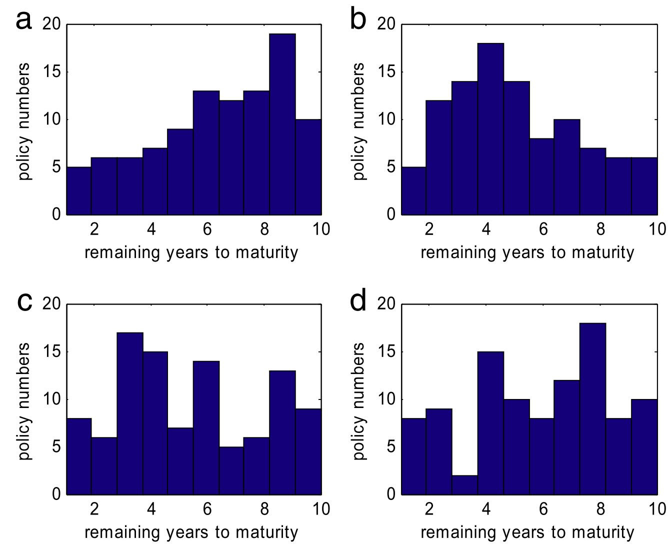

Fig. 8. Distribution of the number of policies within four different portfolios based on remaining years to maturity.

proportion of cash held within the portfolio during the same period. However, a smaller group of maturing policies results in a downward trend in the proportion of cash held. For example, in Portfolio (a), most of the policies mature in the second half of the 10 years; thus, in comparison with other portfolios, the insurance company does not have an immediate obligation to pay for its liability, which enables it to hold more risky assets at the beginning of the term to increase investment returns. In contrast, most of the policies mature in the first half of the 10 years in Portfolio (b); thus, the insurance company has an obligation to pay for the liability earlier. Therefore, it holds more riskless assets (and less risky assets) at the beginning of the term to reduce its illiquidity risk. Again, the optimal investment strategies for Portfolios (c) and (d) fall within the range represented by Portfolios (a) to (b). We display the complete and detailed plots of these four extremely skewed cases in Appendix C.

For the optimal asset allocation of a specific portfolio, the deviation from the base case decreases if the level of the skewness of the years before maturity for these 100 policies decreases, with both rebalancing rules. That is, if an insurance company knows the optimal asset allocation of the base case, following the properties mentioned above, it can gain an approximation of the investment strategy, based on that of the base case, without investigating the complicated combination of policies within a portfolio.

5. Conclusion

As Campbell and Viceira (2002) indicate, there are several important developments in the long-term portfolio choice area recently. These include computing power and numerical method, discovery of new closed form solution and access from approximate analytical solutions. In our study, we approximate the actual portfolio return by Log-normal random variable, which makes the moments tractable. We derive the formulae for the moments of the accumulated asset value of the insurer, and then get the numerical results by numerical method. Hence, we find the optimal asset allocation by numerical method, but also exploit the advantage from the approximate analytical method.

This paper successfully derives the formulae of the first and second moments of accumulated asset value based on a multi-asset return model. With these formulae, we can analyze portfolio problems and obtain optimal investment strategies. Therefore, this research provides a new perspective on solving both single-period and multiperiod asset allocation problems, as applied to life insurance policies.

We investigate the optimal asset allocation with both the constant and variable rebalancing methods. For constant rebalancing, we find an efficient frontier in the mean–standard deviation plot that occurs with arbitrary policy portfolios. The insurance company should hold more cash to reduce its illiquidity risk for portfolios in which policies will mature at earlier dates. In the case of variable rebalancing, we find that the optimal asset allocation strategy can differ considerably, given different portfolio structures. Thus, it is important for an insurance company to find a suitable investment strategy for different policy portfolios.

In the present model, we calculate the liability reserve exogenously for simplicity. In practice, the reserve should be valued on the market basis according to the requirement of various countries. In addition, nowadays the life insurance authorities in various countries require that a certain amount of money needs to be set aside as capital at the end of each year. These two issues are important and unavoidable in practice. It is somewhat difficult to take these two issues into account and we ignore these two issues in this paper. However, in practice if the market information can be acquired, the insurance company is able to value the liabilities on the market basis. It can obtain the reserve from the market information and then find the optimal asset allocation strategy by applying the method proposed in this paper. Moreover, an insurer can estimate the distribution of the accumulated asset value through simulation and so can know the probability of failing the authority's requirement. By adjusting the value of the parameter k (risk-tolerant parameter) in the objective function, one can find a conservative investment strategy that achieves an authority's requirement.

Appendix A. Cash flow model and its moments

We consider a general portfolio with m fully discrete policies. Let n_i be the term of the ith policy; r_i $(r_i < n_i)$ be the difference between the issue date and the valuation date; and $n = \max_{i\in\{1,2,\ldots,m\}}\{n_i - r_i\}$, which means the maximum years before the maturation of the policies within the portfolio. The age of the insured i at the date of issue is x_i, so the insured i is aged $x_i + r_i$ at the valuation date. A death benefit b_i exists if the insured i dies before the maturity and a survival benefit c_i otherwise. Both b_i and c_i are specific, and the benefit would be paid at the end of each policy year. A level premium π_i for the ith policy is payable at the beginning of each policy year.

We provide the moments related to the cash flow model in terms of the following lemma. A more detailed formula can be found in Marceau and Gaillardetz (1999).

Lemma. *Define*

$$D_{i,j} = \begin{cases} 1, & \text{if the insured } i \text{ dies during the time interval } (j-1, j] \\ 0, & \text{otherwise} \end{cases}$$

$$S_{i,j} = \begin{cases} 1, & \text{if the insured } i \text{ is alive at time } j \\ 0, & \text{otherwise.} \end{cases}$$

Then, $CF(j)$, time j's cash inflow, or the net difference between the premiums received and the benefits paid at time j, is

$$CF(j) = \sum_{i=1}^{m} S_{i,j}\pi_i 1_{(n_i-r_i>j)} - \sum_{i=1}^{m} D_{i,j} b_i 1_{(n_i-r_i\geq j)} - \sum_{i=1}^{m} S_{i,n_i-r_i} c_i 1_{(n_i-r_i=j)}, \quad j = 1,\ldots,n,$$

where 1_A is the indicator function with a value of 1 if condition A is satisfied and 0 otherwise.

Moreover, based on the moments of binomial distribution and assuming that the mortality processes of two different insured individuals are independent, the expected value, variance, and covariance of $CF(j)$ are:

$$E[CF(j)] = \sum_{i=1}^{m} \pi_{ij} p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i>j)} - \sum_{i=1}^{m} b_{ij-1|} q_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i\geq j)} - \sum_{i=1}^{m} c_{in_i-r_i} p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i=j)}, \quad j = 1,\ldots,n$$

$$\begin{aligned} Var[CF(j)] = {} & \sum_{i=1}^{m} b_i^2 \cdot {}_{j-1|}q_{x_i+r_i}^{(\tau_i)} \left(1 - {}_{j-1|}q_{x_i+r_i}^{(\tau_i)}\right) 1_{(n_i-r_i\geq j)} \\ & + \sum_{i=1}^{m} c_i^2 \cdot {}_j p_{x_i+r_i}^{(\tau_i)} \left(1 - {}_j p_{x_i+r_i}^{(\tau_i)}\right) 1_{(n_i-r_i=j)} \\ & + 2\sum_{i=1}^{m} b_i c_{ij-1|} q_{x_i+r_i}^{(\tau_i)} \cdot {}_j p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i=j)} \\ & + \sum_{i=1}^{m} \pi_i^2 \cdot {}_j p_{x_i+r_i}^{(\tau_i)} \left(1 - {}_j p_{x_i+r_i}^{(\tau_i)}\right) 1_{(n_i-r_i>j)} \\ & - 2\sum_{i=1}^{m} b_i \pi_i \cdot {}_{j-1|}q_{x_i+r_i}^{(\tau_i)} \cdot {}_j p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i>j)} \end{aligned}$$

and

$$\begin{aligned} Cov[CF(k), CF(j)] = {} & -\sum_{i=1}^{m} b_i^2 \cdot {}_{j-1|}q_{x_i+r_i}^{(\tau_i)} \cdot {}_{k-1|}q_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i\geq j)} \\ & - \sum_{i=1}^{m} b_i c_i \cdot {}_{k-1|}q_{x_i+r_i}^{(\tau_i)} \cdot {}_j p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i=j)} \\ & + \sum_{i=1}^{m} b_i \pi_i \cdot {}_{k-1|}q_{x_i+r_i}^{(\tau_i)} \cdot {}_j p_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i>j)} \\ & + \sum_{i=1}^{m} \pi_i^2 \cdot {}_j p_{x_i+r_i}^{(\tau_i)} \cdot \left(1 - {}_k p_{x_i+r_i}^{(\tau_i)}\right) 1_{(n_i-r_i>j)} \\ & - \sum_{i=1}^{m} b_i \pi_i \cdot \left(1 - {}_k p_{x_i+r_i}^{(\tau_i)}\right) \cdot {}_{j-1|}q_{x_i+r_i}^{(\tau_i)} 1_{(n_i-r_i\geq j)} \\ & - \sum_{i=1}^{m} \pi_i c_i \cdot {}_j p_{x_i+r_i}^{(\tau_i)} \cdot \left(1 - {}_k p_{x_i+r_i}^{(\tau_i)}\right) 1_{(n_i-r_i=j)} \end{aligned}$$

where $k < j$ and (τ_i) denote the specific mortality of the insured i.

Appendix B. Proof of Theorem 2.2

(a)

$$\begin{aligned} E[Z(t)] &= E[y(t-1)] + (p_{1t}, p_{2t}) \begin{pmatrix} E[\Delta_e(t)] \\ E[\Delta_b(t)] \end{pmatrix} + \rho(p_{1t}, p_{2t}) \\ &= y + (y(0) - y)\phi^{t-1} + (p_{1t}, p_{2t}) \begin{pmatrix} \Delta_e \\ \Delta_b \end{pmatrix} + \rho(p_{1t}, p_{2t}) \end{aligned}$$

$$\begin{aligned} Var[Z(t)] &= Var[y(t-1)] + (p_{1t}, p_{2t})\,\Sigma \begin{pmatrix} p_{1t} \\ p_{2t} \end{pmatrix} \\ &= \frac{\sigma_y^2}{1-\phi^2}\left(1 - \phi^{2(t-1)}\right) + (p_{1t}, p_{2t})\,\Sigma \begin{pmatrix} p_{1t} \\ p_{2t} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & Cov[Z(t), Z(t+k)] \\ &= Cov\left[y(t-1) + (p_{1t}, p_{2t}) \begin{pmatrix} \Delta_e(t) \\ \Delta_b(t) \end{pmatrix}, y(t+k-1)\right] \\ &= Cov\left[y(t-1) + (p_{1t}, p_{2t}) \begin{pmatrix} \Delta_e(t) \\ \Delta_b(t) \end{pmatrix}, \phi^k y(t-1) + \phi^{k-1}\sigma_y Z_y(t)\right] \\ &= \frac{\sigma_y^2}{1-\phi^2}(1 - \phi^{2(t-1)})\phi^k + \phi^{k-1}(p_{1t}, p_{2t}) \begin{pmatrix} \sigma_{ey} \\ \sigma_{by} \end{pmatrix} \sigma_y; \quad \forall k \in N \end{aligned}$$

(b)

$$\begin{aligned} E[S(k)] &= E\left[\sum_{j=1}^{k} Z(j)\right] \\ &= \sum_{j=1}^{k}\left[y + (y(0) - y)\phi^{j-1} + (p_{1j}, p_{2j}) \begin{pmatrix} \Delta_e \\ \Delta_b \end{pmatrix} + \rho(p_{1j}, p_{2j})\right] \\ &= ky + (y(0) - y)\frac{1-\phi^k}{1-\phi} \\ &\quad + \sum_{j=1}^{k}\left[(p_{1j}, p_{2j}) \begin{pmatrix} \Delta_e \\ \Delta_b \end{pmatrix} + \rho(p_{1j}, p_{2j})\right] \end{aligned}$$

$$\begin{aligned} Var[S(k)] &= Var\left[\sum_{j=1}^{k} Z(j)\right] \\ &= \sum_{j=1}^{k} Var[Z(j)] + 2\sum_{j=1}^{k-1}\sum_{m=j+1}^{k} Cov[Z(j), Z(m)] \\ &= (A) + (B), \end{aligned}$$

where

$$\begin{aligned} (A) &= \sum_{j=1}^{k} Var[Z(j)] \\ &= \sum_{j=1}^{k}\left[\frac{\sigma_y^2}{(1-\phi)^2}\left(1 - \phi^{2(j-1)}\right) + (p_{1j}, p_{2j})\,\Sigma \begin{pmatrix} p_{1j} \\ p_{2j} \end{pmatrix}\right] \\ &= \frac{\sigma_y^2}{(1-\phi)^2}\left[(k) - \frac{1-\phi^{2k}}{1-\phi^2}\right] + \sum_{j=1}^{k}(p_{1j}, p_{2j})\,\Sigma \begin{pmatrix} p_{1j} \\ p_{2j} \end{pmatrix} \end{aligned}$$

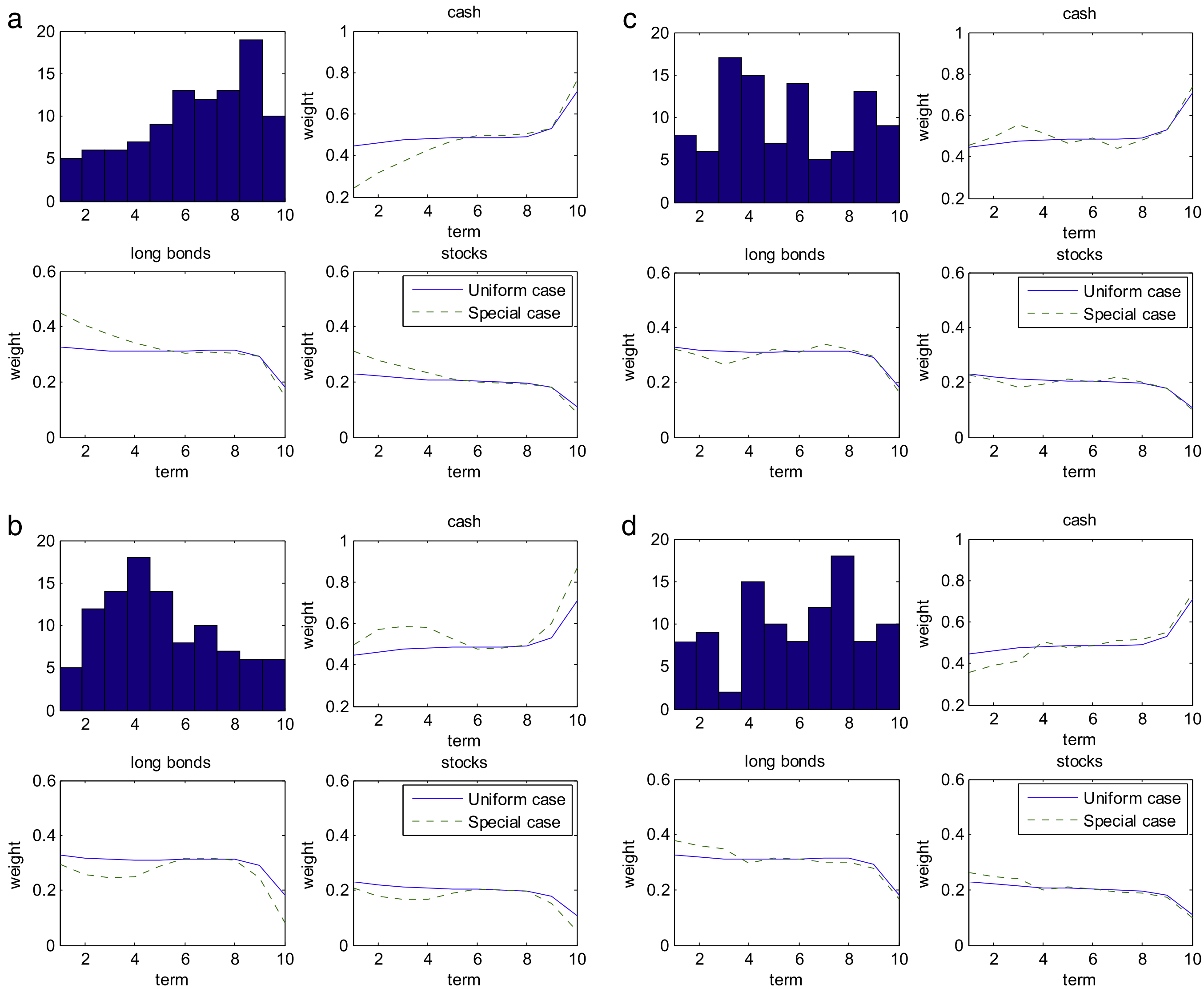

Fig. C.1.

$$
\begin{aligned}
(B) &= 2\sum_{j=1}^{k-1}\sum_{m=j+1}^{k}\left[\frac{\sigma_y^2}{1-\phi^2}(1-\phi^{2(j-1)})\phi^{m-j}\right. \\
&\quad \left.+\phi^{m-j-1}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\right] \\
&= 2\sum_{j=1}^{k-1}\frac{\sigma_y^2}{1-\phi^2}(1-\phi^{2(j-1)})\frac{\phi\left(1-\phi^{k-j}\right)}{1-\phi} \\
&\quad +2\sum_{j=1}^{k-1}\frac{1-\phi^{k-j}}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y \\
&= \frac{2\sigma_y^2\phi}{\left(1-\phi^2\right)\left(1-\phi\right)}\sum_{j=1}^{k-1}\left(1-\phi^{k-j}-\phi^{2(j-1)}+\phi^{k+j-2}\right) \\
&\quad +2\sum_{j=1}^{k-1}\frac{1-\phi^{k-j}}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y \\
&= \frac{2\sigma_y^2\phi}{\left(1-\phi^2\right)\left(1-\phi\right)}\left[(k-1)-\frac{\phi\left(1-\phi^{k-1}\right)}{1-\phi}\right. \\
&\quad \left.-\frac{1-\phi^{2(k-1)}}{1-\phi^2}+\frac{\phi^{k-1}\left(1-\phi^{k-1}\right)}{1-\phi}\right] \\
&\quad +2\sum_{j=1}^{k-1}\frac{1-\phi^{k-j}}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y
\end{aligned}
$$

$$
\begin{aligned}
Cov\left[S\left(k\right),S\left(k+m\right)\right] &= Cov\left[S\left(k\right),S\left(k\right)+\sum_{j=k+1}^{k+m}Z\left(j\right)\right] \\
&= Var\left[S\left(k\right)\right]+Cov\left[\sum_{j=1}^{k}Z\left(j\right),\sum_{j=k+1}^{k+m}Z\left(j\right)\right] \\
&= Var\left[S\left(k\right)\right]+\left(C\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\left(C\right) &= \sum_{j=1}^{k}\sum_{l=k+1}^{k+m}Cov\left[Z\left(j\right),Z\left(l\right)\right] \\
&= \sum_{j=1}^{k}\sum_{l=k+1}^{k+m}\left[\frac{\sigma_y^2}{1-\phi^2}(1-\phi^{2(j-1)})\phi^{l-j}\right.
\end{aligned}
$$

$$+\phi^{l-j-1}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\Bigg]$$

$$=\sum_{j=1}^{k}\left[\frac{\sigma_y^2}{1-\phi^2}(1-\phi^{2(j-1)})\frac{\phi^{k+1-j}\left(1-\phi^m\right)}{1-\phi}\right]$$

$$+\sum_{j=1}^{k}\left[\frac{\phi^{k-j}\left(1-\phi^m\right)}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\right]$$

$$=\frac{\sigma_y^2\phi\left(1-\phi^m\right)}{\left(1-\phi^2\right)\left(1-\phi\right)}\sum_{j=1}^{k}(1-\phi^{2(j-1)})\phi^{k-j}$$

$$+\sum_{j=1}^{k}\left[\frac{\phi^{k-j}\left(1-\phi^m\right)}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\right]$$

$$=\frac{\sigma_y^2\phi\left(1-\phi^m\right)}{\left(1-\phi^2\right)\left(1-\phi\right)}\frac{\left(1-\phi^k\right)\left(1-\phi^{k-1}\right)}{\left(1-\phi\right)}$$

$$+\sum_{j=1}^{k}\left[\frac{\phi^{k-j}\left(1-\phi^m\right)}{1-\phi}\left(p_{1j},p_{2j}\right)\begin{pmatrix}\sigma_{ey}\\ \sigma_{by}\end{pmatrix}\sigma_y\right].$$

(c) Because $I\left(j\right)=\exp\left\{Z\left(j+1\right)+Z\left(j+2\right)+\cdots+Z\left(n\right)\right\}$ and $Z\left(t\right)$ follows a normal distribution for all t conditional on time 0, we have

$$\begin{aligned}E\left[I\left(j\right)\right]&=E\left[\exp\left(Z\left(j+1\right)+Z\left(j+2\right)+\cdots+Z\left(n\right)\right)\right]\\&=E\left[\exp\left(S\left(n\right)-S\left(j\right)\right)\right]\\&=\exp\left\{E\left[S\left(n\right)-S\left(j\right)\right]+\frac{1}{2}Var\left[S\left(n\right)-S\left(j\right)\right]\right\}\\&=\exp\left\{E\left[S\left(n\right)\right]-E\left[S\left(j\right)\right]+\frac{1}{2}\left[Var\left(S\left(n\right)\right)\right.\right.\\&\qquad\left.\left.+Var\left(S\left(j\right)\right)-2Cov\left(S\left(n\right),S\left(j\right)\right)\right]\right\}\end{aligned}$$

$$\begin{aligned}E\left[I\left(j\right)I\left(k\right)\right]&=E\left[\exp\left[S\left(n\right)-S\left(j\right)\right]\exp\left[S\left(n\right)-S\left(k\right)\right]\right]\\&=E\left[\exp\left[2S\left(n\right)-\left(S\left(j\right)+S\left(k\right)\right)\right]\right]\\&=\exp\left\{2E\left[S\left(n\right)\right]-E\left[S\left(j\right)\right]-E\left[S\left(k\right)\right]\right.\\&\qquad\left.+\frac{1}{2}Var\left[2S\left(n\right)-S\left(j\right)-S\left(k\right)\right]\right\},\end{aligned}$$

where

$$\begin{aligned}&Var\left[2S\left(n\right)-S\left(j\right)-S\left(k\right)\right]\\&\quad=\left[4Var\left(S\left(n\right)\right)+Var\left(S\left(j\right)\right)+Var\left(S\left(k\right)\right)\right.\\&\qquad-4Cov\left(S\left(n\right),S\left(j\right)\right)-4Cov\left(S\left(n\right),S\left(k\right)\right)\\&\qquad\left.+2Cov\left(S\left(j\right),S\left(k\right)\right)\right].\end{aligned}$$

Appendix C. Distribution of the remaining term and the optimal asset allocation of each special portfolio under the variable rebalance rule

See Fig. C.1.

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