

THE TERM STRUCTURE OF RESERVE DURATIONS AND THE DURATION OF AGGREGATE RESERVES

Chenghsien Tsai

ABSTRACT

Estimating the duration gap of a life insurer demands the knowledge on the durations of liabilities and assets. The literature analyzed the durations of assets extensively but rendered limited analyses on the durations of insurance liabilities. This article calculated the reserve durations for individual policies and estimated the duration of the aggregate reserves. The results showed that the duration of the policy reserve might be negative and/or have a large figure. They further revealed an interesting pattern of the reserve duration with respect to the policy's time to maturity. A term structure with abnormal durations, however, does not result in an abnormal duration of the aggregate reserves.

INTRODUCTION

Managing the company's interest rate risk is vital to a life insurer. Life insurance policies are long-term contracts. Small changes in interest rates can therefore cause large changes in the policy reserve liability, which usually constitutes more than 90 percent of a company's total liabilities. To offset the resulting fluctuations in the value of the reserve liability, the company must look for an asset portfolio that produces matched changes in values. If the match is not perfect, the high liability-to-surplus ratio prevalent in the life insurance industry will make the mismatch large relative to the insurer's surplus. Movements in interest rates, therefore, can have a significant adverse impact on the solvency of a life insurance company.¹

Chenghsien Tsai is an associate professor in the Department of Risk Management and Insurance, National Chengchi University, No. 64, Section 2, Chihnan Road, Taipei, Taiwan 11623. The author can be contacted via e-mail: ctsai@nccu.edu.tw. The author is grateful to the two anonymous referees for their constructive comments, to Jia-Le Lin for his competent programming assistance, to Georgia State University and Santa Clara University for the kind support during the visit of the author, and to the National Science Council of Taiwan for its financial support (project numbers NSC 93-2416-H-004-041 and NSC 96-2918-I-004-006).

¹In other words, what is meant by the company's interest rate risk is the potential reduction in the surplus caused by interest rate movements. Movements in interest rates cause changes in the values of liabilities as well as assets. Using these value changes to measure the exposure of an insurer to the interest rate risk will exaggerate the exposure, however, because the values of liabilities and assets often change in the same direction and offset each other to a certain

A common measure of an institution's exposure to interest rate risk is the duration gap (DGAP). Samuelson (1945) was the first to introduce the concept of DGAP in analyzing how changes in interest rates may affect an institution. Redington (1952) invented the expression *immunization* and set up the fundamental equations for immunization strategies. One of the results of an immunization strategy is zero DGAP. Bierwag and Kaufman (1985) calculated four DGAPs to measure an institution's exposure to interest rate risk in accordance with alternative management goals. Further application and generalization of DGAP could be seen in Bierwag and Kaufman (1992, 1996). Fooladi and Roberts (2004) added the notion of convexity gap and default risk to DGAP to manage interest rate risk with better accuracy. The duration (gap) analysis has enjoyed widespread practitioner application in banks, insurance companies, and other financial institutions since the 1980s. Its usefulness as a measure of interest rate risk is undeniable and its use in finance markets today is extensive (Bierwag and Fooladi, 2006).

To calculate the DGAP of a life insurer, one usually has to calculate the durations of individual assets and liabilities.² The duration measure and the durations of the important assets held by life insurers such as bonds, mortgages, stocks, and real estate have been investigated extensively in the finance literature. As the review of Bierwag and Fooladi (2006) shows, the duration measure has been refined substantially with the considerations for stochastic interest rate processes, interest-rate-dependent cash flows, and default risk since the original work of Macaulay (1938). The durations of various asset classes, in addition to those of bonds, have also been explored substantially (e.g., Bierwag, Kaufman, and Toevs, 1983; Bierwag, 1987; Leibowitz et al., 1989; Bierwag, Corrado, and Kaufman, 1992; Babbel, Merrill, and Panning, 1997; Hayre and Chang, 1997; Hevert, McLaughlin, and Taggart, 1998; Cornell, 2000; Hamelink et al., 2002; Reilly, Wright, and Johnson, 2007) during the last quarter century.

In contrast, the durations of insurance liabilities have received limited attention. Babbel (1995) estimated the option-adjusted durations of the liabilities associated with a dozen life insurance products using a commercial software.³ Santomero and Babbel (1997) listed the effective durations of the liabilities of several life insurance products based on their on-site investigation on the risk management practices of insurance companies. Briys and Varenne (1997, 2001) calculated the effective duration of the liability associated with a single-premium participating contract having a minimum guaranteed rate of return.

The contribution of this article is that we discover interesting characteristics of the reserve durations of individual policies and explicitly estimate the duration of aggregate reserves. The former finding is new to the literature and has imperative implication

extent. Therefore, a more appropriate way to measure an insurer's interest rate risk is the net changes in the values of liabilities and assets, that is, the changes in the surplus values, with respect to interest rate changes.

²The alternative way is to aggregate expected cash inflows and outflows of all assets and liabilities and then calculate the duration of the net cash flow stream.

³Santomero and Babbel (1997) regard the measures of interest rate sensitivity that take into account the interest-sensitive cash flows of an asset or liability as "option-adjusted duration" or "effective duration." "Macaulay duration" and "modified duration" are the measures of interest rate sensitivity assuming that cash flows are insensitive to movements in interest rates.

to the life insurer's microhedging.⁴ The latter estimate fills in the missing details and analyses of the previous papers and is essential to the macrohedging of life insurers. First, we calculate the effective durations of the reserves for policies having different maturities.⁵ These durations are important because the policy reserve liability comprises the reserves of policies that are sold at different times and its duration is a weighted average of individual reserve durations. To date, no such calculations have yet been reported in the literature, and the results from this part have momentous implications to the microhedging of a life insurer. The interesting characteristics of the reserve durations of individual policies are discovered after these calculations.

Second, we take some weighted averages of individual policies' reserve durations to estimate the durations of aggregate reserves.⁶ Analyzing the aggregate reserve duration is essential to the macrohedging of a life insurer. Without the aggregate reserve duration, the insurer will not be able to know how to construct the asset portfolio to hedge its DGAP. Babbel (1995) and Santomero and Babbel (1997) estimated the effective durations of the reserves for some products, but they disclosed only the final results without the policy specification, pool composition, interest rate model, surrender behavior, or any other assumptions. Due to these missing details, the characteristics of the duration of the policy reserve liability remain obscure in the present literature.

We employ both effective duration and modified duration as the measures of interest rate sensitivity in this study. The effective duration is the better measure because the insurance literature pinpoints the importance of the interest rate sensitivity of cash flows when estimating the durations of life insurance liabilities. In this article, the cointegrated vector autoregression (VAR) model of the interest rate and the surrender rate established in Tsai, Kuo, and Chen (2002) are adopted to generate interest-dependent surrender rates and thus interest-sensitive cash flows. The modified duration is used to establish intuition. The differences between the modified duration and effective duration can further demonstrate how interest-sensitive cash flows affect the sensitivity of policy reserves to interest rate changes. A 20-year endowment product issued to 30-year-old males in different years serves as an illustrative example in this study.

After calculating the modified and effective durations of reserves for policies that have different years to maturity, we find that the duration of the policy reserve may

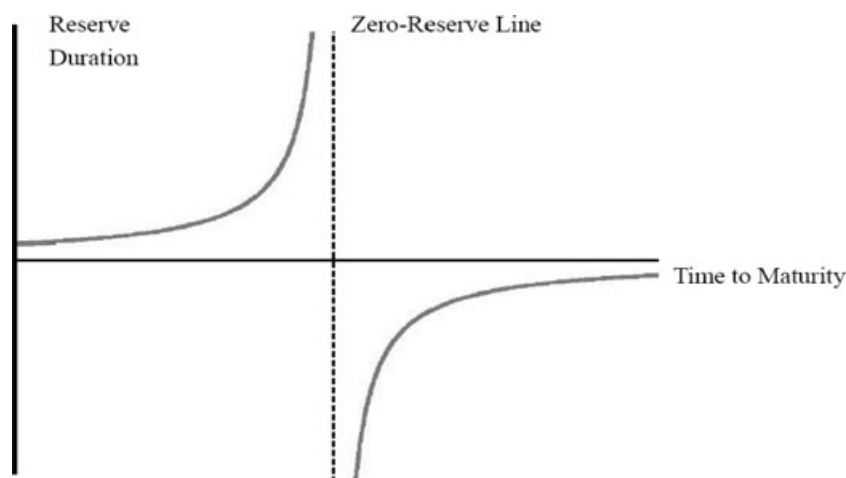
⁴A financial institution can hedge interest rate risk either at the micro level or at the macro level. Hedging a specific asset or liability is called microhedging while hedging the entire balance sheet DGAP is called macrohedging.

⁵To see clearer how this article is related to the finance literature, we can regard this part of analyses as having similar purposes to Hopewell and Kaufman (1973) and Haugen and Wichern (1974) who showed how duration changes with properties of bonds such as the maturity and the yield to maturity. It is similar to Morgan (1986) and Kalotay, Williams, and Fabozzi (1993) in considering the interest rate dependence of cash flows. Our treatment of mortality risk is similar to the adjustment for default risk done by Babbel, Merrill, and Panning (1997), Fooladi, Roberts, and Skinner (1997), and Jacoby (2003). In short, we applied some analyses of the aforementioned papers done on assets to the most important liabilities of life insurers, the policy reserves.

⁶This part of analyses on durations of policy reserve portfolios can be deemed similar to Bierwag, Corrado, and Kaufman (1990) who computed durations for bond portfolios.

FIGURE 1

General Pattern of the Reserve's Duration With Respect to the Policy's Time to Maturity



be negative and may have a figure far exceeding the policy's maturity.⁷ We further discover an interesting pattern of reserve durations with respect to maturities as shown in Figure 1.

Figure 1 is herein referred to as the term structure of reserve durations. It demonstrates that the reserve duration is a function of the policy's time to maturity, having a vertical asymptote at a break-even maturity. The break-even maturity is the time to maturity when the policy's reserve equals to zero. The vertical asymptote is therefore called the zero-reserve line. The duration increases with the maturity and approaches infinity when the maturity approaches the break-even maturity from the left. The duration turns into a negative infinity when the maturity crosses the break-even maturity and then increases with the maturity.

The immediate implication of the above results is that life insurers will encounter difficulties in performing microhedging for the reserves of individual policies against interest rate risk because reserves may have abnormal durations. A further inference from the above results is that life insurers may even be unable to perform macrohedging because the duration of the policy reserve liability on a life insurer's balance sheet is a weighted average of individual reserve durations and thus may have an abnormal value as well. This inference, nonetheless, ignores a feature of the reserve duration: abnormal duration values are coupled with small reserves. Because the underlying reserves of abnormal durations account for only a small percentage of the aggregate reserves, these abnormal values are immaterial in calculating the aggregate reserve duration. The duration of aggregate reserves therefore has a normal value even when the component reserves have abnormal values. For instance, the duration of the aggregate reserves resulting from equal numbers of endowment policies sold in different years can have a duration of less than 8 even though some policies have reserve

⁷Little (1984) and Kalotay (1984) also found that the duration might be longer than the bond's maturity when some of the cash flows are negative.

durations larger than 80. Further analyses show that growing/declining businesses lead to larger/smaller aggregate reserve durations because younger/older policies tend to have larger/smaller figures of reserve duration.

The remainder of this article is organized as follows. The next section specifies the expected cash flows of the endowment policies that are sold in different years. It also describes how the modified duration and effective duration of the policy reserve are calculated in this article. The section "The Term Structure of Modified Durations" reports the results of the modified duration, describes its term structure, and provides a rationale for the pattern of the term structure. The section "The Term Structure of Effective Durations" states the results for the effective duration and confirms the pattern of the term structure identified in the previous section. It further analyzes the differences between the modified durations and effective durations. The section "The Durations of Aggregate Reserves" calculates the effective duration of the aggregate reserves based on the results of individual reserve durations in the previous section. The durations of the aggregate reserves resulting from growing or declining underwriting businesses are also analyzed in this section. Finally, in the last section, the article is summarized and conclusions are drawn.

SPECIFICATIONS OF POLICY CASH FLOWS AND FORMULAS FOR RESERVE DURATIONS

Cash Flow Specifications

A 20-year endowment product is used as the analyzed policy. It is issued to 30-year-old males in different years. The death benefit and surrender value are assumed to be paid at the end of the year while the level premium and expenses are incurred at the beginning of the year. The expected net cash outflow at time t for the policy at the beginning of its policy year n (i.e., sold $n - 1$ years ago) after the n th net premium has been collected, where $t \in N$, $1 \leq t < 20 - n + 1$, and $1 \leq n < 20$, is then defined as⁸

$$E(CF_t | n) = [(t-1)p_{30+n-1}^{(\tau)} \times q_{30+n-1+t-1}^{(d)} \times F] + (t-1)p_{30+n-1}^{(\tau)} \times q_t^{(s)} \times S_{n-1+t})] - t p_{30+n-1}^{(\tau)} \times [P - (CMR_{n+t} \times P) - FExp_{n+t} - (VarCost \times P)], \quad (1)$$

where $t p_{30+n-1}^{(\tau)}$ is the probability that the policy for an insured, age of $30 + n - 1$, remains valid for t years,⁹ $q_{30+n-1+t-1}^{(d)}$ is the probability of the insured, age $30 + n - 1 + t - 1$, dying within 1 year, F denotes the death benefit paid at the end of the year in which the insured dies, $q_t^{(s)}$ is the probability that the policy is surrendered in year t ,¹⁰ S_{n-1+t} denotes the cash surrender value paid at the end of

⁸Note that the insured is at age $30 + n - 1$ when the policy is at the beginning of policy year n .

⁹Note that ${}_0 p_{30+n-1}^{(\tau)} = 1$. The upper script (τ) is used to indicate a function referring to all causes or total force of decrement. Two causes of decrement, death and surrender, are considered in this article and are denoted as the upper scripts (d) and (s) , respectively.

¹⁰Note that $1 - q_{30+n-1+t-1}^{(d)} - q_t^{(s)} = {}_1 p_{30+n-1+t-1}^{(\tau)}$ because a policy that is not terminated in 1 year due to either death or surrender is the policy that remains valid for 1 year. Furthermore, ${}_{t-1} p_{30+n-1}^{(\tau)} \times {}_1 p_{30+n-1+t-1}^{(\tau)} = t p_{30+n-1}^{(\tau)}$, that is, the probability of a policy with an insured age

policy year $n - 1 + t$,¹¹ P denotes the level premium received at the beginning of each surviving year, CMR_{n+t} represents the commission rate for the commission paid at the beginning of policy year $n + t$, $FExp_{n+t}$ represents the fixed expense paid at the beginning of policy year $n + t$, and $VarCost$ stands for the variable cost rate.

The first bracket term in Equation (1) represents the sum of the expected death benefit and surrender payment paid at the end of 1 year. The second term of Equation (1) denotes the expected net premium (net of the expected expenses) received at the beginning of the following year. The expected net premium received at time t equals the net premium $[P - (CMR_{n+t} \times P) - FExp_{n+t} - (VarCost \times P)]$ times the probability that the policy is valid at the time (${}_t p_{30+n-1}^{(\tau)}$). The expected death benefit equals the death benefit (F) times the probability that the policy remains valid for $t - 1$ year (${}_{t-1} p_{30+n-1}^{(\tau)}$) and then the insured dies within an year ($q_{30+n-1+t-1}^{(d)}$). Similarly, the expected surrender payment at time t equals the cash surrender value of policy year $n - 1 + t$ (S_{n-1+t}) times the probability that the policy remains valid for $t - 1$ year (${}_{t-1} p_{30+n-1}^{(\tau)}$) and then the insured surrenders the policy in year t ($q_t^{(s)}$). The time line regarding the above cash flows is plotted in the Appendix for further clarification.

At $t = 20 - n + 1$, where $1 \leq n \leq 20$, no more premiums will be collected because the insured has paid 20 premiums. Therefore, neither commissions nor variable costs are paid at $t = 20 - n + 1$. It is further assumed that no fixed expenses are incurred at maturity. The expected net cash outflow at the maturity of the policy then does not have the second term of Equation (1). Instead, it has a term for the surviving benefit. The expected cash outflow is therefore as follows:

$$E(CF_{20-n+1} | n) = [({}_{20-n} p_{30+n-1}^{(\tau)} \times q_{49}^{(d)} \times F) + ({}_{20-n} p_{30+n-1}^{(\tau)} \times q_{20-n+1}^{(s)} \times S_{20}^{ed})] + {}_{20-n+1} p_{30+n-1}^{(\tau)} \times F. \quad (2)$$

The actuarial assumptions about some of the above variables are shown in Table 1.

The present value of the expected cash flows from a policy right after its n th net premium is received, L_n , can then be expressed as

$$\sum_{t=1}^{20-n+1} [E(CF_t | n) \times v_t], \quad (3)$$

where $v_t = \frac{1}{(1+r_1)(1+r_2)\dots(1+r_t)}$ and r_t is the 1-year interest rate prevailing in year t . The random variable L_n represents the present value of the liability (i.e., the policy reserve) associated with the policy immediately after the first n net premiums are collected.

30 + $n - 1$ being valid for t years equals the probability of the policy being valid for $t - 1$ years times the probability of the policy with the insured age 30 + $n - 1 + t - 1$ remaining valid for 1 more year.

¹¹We can also say that the cash surrender value is paid at the end of year t .

TABLE 1
Actuarial Assumptions About the Endowment Product^a

Insured's Age a	Mortality Rate of Age a $q_a^{(d)}$	At the Beginning of Policy Year n	Cash Surrender Value S_{n-1}	Commission Rate CMR_n	Fixed Expense $FExp_n$
30	0.0009790	1	N/A	62.40%	4,530
31	0.0010055	2	8,160.77	27.00%	1,359
32	0.0010481	3	39,789.39	20.60%	1,359
33	0.0011075	4	73,766.54	14.00%	1,359
34	0.0011826	5	110,191.76	13.00%	1,359
35	0.0012712	6	149,173.28	12.00%	1,359
36	0.0013711	7	190,830.52	10.00%	1,359
37	0.0014807	8	235,294.26	10.00%	1,359
38	0.0015989	9	282,706.96	10.00%	1,359
39	0.0017291	10	333,222.85	10.00%	1,359
40	0.0018749	11	387,003.56	7.00%	1,359
41	0.0020407	12	437,654.83	7.00%	1,359
42	0.0022297	13	490,341.50	7.00%	1,359
43	0.0024446	14	545,162.55	7.00%	1,359
44	0.0026795	15	602,227.49	7.00%	1,359
45	0.0029268	16	661,664.40	7.00%	1,359
46	0.0031784	17	723,620.17	7.00%	1,359
47	0.0034268	18	788,259.03	7.00%	1,359
48	0.0036671	19	855,759.96	7.00%	1,359
49	0.0039091	20	926,313.67	7.00%	1,359
50	N/A	20*	1,000,000	N/A	N/A

^aThese assumptions are made based on a life insurance policy once sold in Taiwan.

Notes: The death benefit is \$1,000,000 and the annual premium is \$45,300.

Variable cost rate is 0.1% that mainly consists of the guarantee fund contribution.

Mortality rates are 54% of those in the 1989 Taiwan Standard Ordinary Tables of Mortality (1989 TSO) and 54% is derived from the experiences of the life insurance industry in Taiwan.

We denote $q_{50}^{(d)}$, CMR_{20^*} , and $FExp_{20^*}$ (where 20* denote the end of policy year 20) as "N/A" because the policy matures at this time and neither mortality nor expenses apply any more. Here, S_0 is denoted as "N/A" because no cash value will be paid at the beginning of policy year 1.

The interest rate sensitivity of L_n (or the interest rate sensitivity of L_n 's mean) will be calculated for various levels of interest rates.¹²

In calculating the modified duration, r_t and $q_t^{(s)}$ are set as constants. To calculate the effective duration, the cointegrated VAR model established in Tsai, Kuo, and Chen

¹²Please note that what we will calculate is the duration of the policy reserve liability, not the surplus duration for a block of policies. Some readers may view expected premiums as assets and payments as liabilities and thus regard our calculation as surplus duration or DGAP. This view is inconsistent with the definition of the policy reserve that includes expected premiums to be a part of the reserve liability (see Black and Skipper, 2000, pp. 737, 741; Bowers et al., 1997, p. 205; SFAS 60, paragraph 21). Our calculations in the sections "The Term Structure of Modified Duration" and "The Term Structure of Effective Duration" are for the

(2002) is employed to simulate the surrender rates and the 1-year interest rates.¹³ Their VAR model is as follows:

$$\begin{aligned} \begin{bmatrix} \Delta q_t^{(s)} \\ \Delta r_t \end{bmatrix} &= \begin{bmatrix} -0.243^{***} \\ (-5.193) \\ -0.199 \\ (-0.890) \end{bmatrix} \begin{bmatrix} 1 & -1.053^{***} & -0.008 \\ & (-9.819) & (-1.148) \end{bmatrix} \begin{bmatrix} q_{t-1}^{(s)} \\ r_{t-1} \\ 1 \end{bmatrix} \\ &+ \begin{bmatrix} 0.240 & -0.046 \\ (1.650) & (-0.881) \\ -0.146 & 0.149 \\ (-0.210) & (0.597) \end{bmatrix} \begin{bmatrix} \Delta q_{t-1}^{(s)} \\ \Delta r_{t-1} \end{bmatrix} \\ &+ \begin{bmatrix} -0.012 & -0.151^{***} \\ (-0.094) & (-2.934) \\ -0.642 & -0.514^* \\ (-1.037) & (-2.085) \end{bmatrix} \begin{bmatrix} \Delta q_{t-2}^{(s)} \\ \Delta r_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^s \\ \varepsilon_t^r \end{bmatrix}, \quad (4) \end{aligned}$$

where $q_t^{(s)}$ denotes the surrender rate in year t , r_t denotes the 1-year interest rate in year t , Δ represents the first-order difference operator, $\mathbf{E} = [\varepsilon_t^s \ \varepsilon_t^r]' \sim N(\mathbf{0}, \hat{\Sigma})$, and $\hat{\Sigma} = \begin{bmatrix} 7.28 \times 10^{-6} & 8.09 \times 10^{-6} \\ 8.09 \times 10^{-6} & 1.67 \times 10^{-4} \end{bmatrix}$.¹⁴ The initial value of the 1-year rate is chosen among 0 percent, 2 percent, 4 percent, 6 percent, and 8 percent.¹⁵ The initial value for the

durations of individual policy reserves, that is, the durations of specific liabilities considered in microhedging. These durations are then used in the section "The Durations of Aggregate Reserves" as the building blocks to calculate the duration of aggregate reserves, the duration used along with the duration of total assets in obtaining the DGAP of a life insurer and in performing macrohedging.

¹³Cointegration modeling is designed to identify potential long-term relations between variables of interest. The intuition behind cointegration analysis is that even though a group of nonstationary variables might individually wander extensively, they may wander in such a way that they do not drift too far apart from one another. More specifically, a particular linear combination of variables may be stationary even though individually they are time series with unit roots. Such variables are said to be cointegrated. Cointegration analysis has become a popular method for economists to study the long-term relations between various economic variables, such as consumption and income, short- and long-term interest rates, and stock prices and dividends, since the seminal work of Engle and Granger (1987).

¹⁴The lapse rate equation in Equation (4) suggests that changes in the lapse rate result from two sources: changes in the lagged variables and the levels of the lagged variables. The impact from the lagged variable levels can be represented by the cointegration vector $ECM_t = L_t - 1.053I_t - 0.008$. The cointegrated vector implies a long-term relation between the lapse rate and interest rate that can be expressed as $L_t = 0.008 + 1.053I_t$, and any deviation from the long-term equilibrium relation will cause the lapse rate to change. The lapse rate is also affected by changes in the interest rate two periods ago. The only significant coefficient in the interest rate equation is ΔI_{t-2} , which suggests that the interest rate process is like an AR(2).

¹⁵The initial of the interest rate is chosen to center around the 4 percent pricing rate of the policy.

surrender rate is at 7 percent, approximately the average termination rate in the United States during the sampling period of Tsai, Kuo, and Chen (2002).¹⁶

Duration Formulas

Modified duration (denoted as MD) has received a great deal of attention and is used quite often to measure and manage the interest rate risk (Tuckman, 1996, p. 139). It refers to the percentage price changes with respect to changes in the interest rate. More specifically,

$$MD = \frac{-\frac{\partial P}{\partial r}}{\frac{P}{\partial r}}, \quad (5)$$

where P is the value of an asset or liability and $\frac{\partial P}{\partial r}$ is the partial derivative of P with respect to the interest rate r . In calculating the modified duration of P , the differential of r is specified as one basis point (0.01 percent) and the difference between the resulting value and the starting value is used as the differential of P .

Equation (5) implies that the yield curve is flat and shifts in a parallel fashion, which is not consistent with the observations from the bond markets. In addition, modified duration is usually used when cash flows of the asset or liability are insensitive to interest rate fluctuations. Tsai, Kuo, and Chen (2002) and Kuo, Tsai, and Chen (2003), however, showed that the cash flows associated with the policy reserve correlate with the interest rates.¹⁷ Effective duration (ED) that takes into account the interest-sensitive cash flows and term structure behaviors is therefore a better measure for the interest rate risk of reserves. In this article, the effective duration of the policy reserve is defined as follows:

$$ED = \frac{-\frac{\partial \text{mean reserve}}{\partial r_0}}{\frac{\text{mean reserve}}{\partial r_0}}, \quad (6)$$

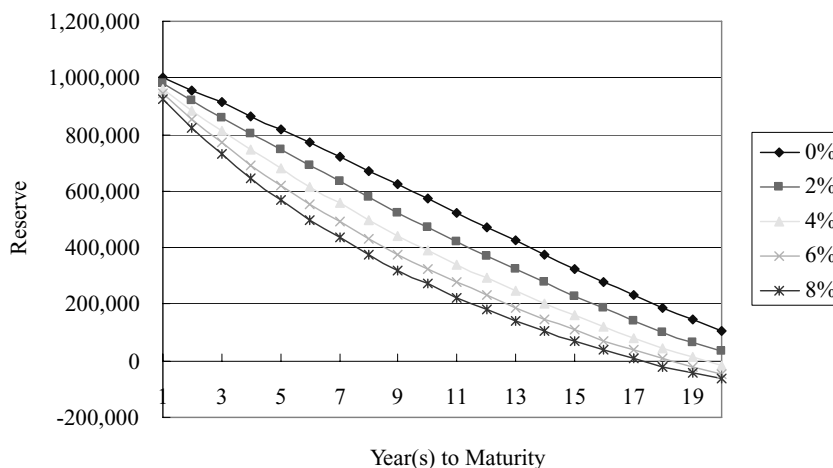
where r_0 denotes the initial value of the 1-year rate in the VAR model. The economic meaning of the effective duration is the percentage change of the mean reserve with respect to a change in the initial 1-year rate. In calculating the effective duration, the differential of r_0 is specified as 1 basis point and the difference between the resulted mean reserve and the starting mean reserve is used as the differential of *mean reserve* in simulation.

¹⁶The values of the 1-year rates and surrender rates in year -1 and year -2 are assumed to be the same as the initial values. In other words, the changes in the 1-year rates and surrender rates at time 0 and -1 are set as zero in simulation.

¹⁷This article does not adopt the model developed in Kuo, Tsai, and Chen (2003) because the average simulated interest rate rises from 6 percent to 11 percent in 20 years. The increasing mean of interest rates has significant impacts on the distribution of the policy reserve and will bring unnecessary complication to the analysis on reserve durations.

FIGURE 2

Term Structures of Policy Reserves Calculated Using Various Levels of Constant Interest Rates



THE TERM STRUCTURE OF MODIFIED DURATIONS

The Term Structure

To calculate the modified duration, L_n is first calculated using constant interest rates while assuming that the surrender rate remains at 7 percent for years to come. The policy reserves immediately after the first n net premiums are collected (L_n) are plotted in Figure 2.

Equation (5) is then applied, and the modified durations of the reserves for the endowment policies that are at the beginning of policy year 1 through 20 are obtained. The results are shown in Table 2.

Table 2 has two characteristics. First, the modified durations of several policy reserves are negative. For instance, the policies with maturities longer than 18 years have negative durations when the interest rate is 6 percent or 8 percent. Second, many policies have modified durations that are larger than their maturities. For example, the policies with maturities longer than 12 years have durations that are larger than the maturities when the interest rate is 0 percent or 2 percent. Some modified durations have very large figures such as 246.66 (the policy with 18 years to go and the interest rate is 6 percent) and 163.38 (the policy with 19 years to maturity at an interest rate of 4 percent). In Figure 3, the graph of the reserve's modified duration as a function of the policy's time to maturity reveals an interesting pattern.

Figure 3 contains three pairs of curves separated by implicit vertical asymptotes. These vertical asymptotes intersect the maturity axis at the times to maturities when the reserves are zero. This intercept is therefore called the break-even maturity in this article, and the vertical asymptote is called the zero-reserve line. The curve on the left-hand side of an asymptote is located in the positive domain whereas the right-hand-side curve is in the negative domain. More specifically, the modified duration is positive and increases without bound as the time to maturity approaches the

TABLE 2

The Term Structures of Modified Durations at Various Levels of Constant Interest Rates

Year(s) to Maturity	0%	2%	4%	6%	8%
1	1.00	0.98	0.96	0.94	0.93
2	1.97	1.93	1.89	1.85	1.82
3	2.91	2.85	2.79	2.74	2.69
4	3.84	3.76	3.68	3.61	3.53
5	4.77	4.66	4.56	4.46	4.37
6	5.70	5.57	5.44	5.32	5.20
7	6.64	6.49	6.34	6.19	6.05
8	7.62	7.44	7.27	7.10	6.94
9	8.63	8.44	8.25	8.07	7.89
10	9.71	9.51	9.32	9.13	8.96
11	10.88	10.69	10.52	10.36	10.21
12	12.14	12.00	11.88	11.79	11.74
13	13.58	13.53	13.54	13.64	13.83
14	15.26	15.41	15.71	16.22	17.03
15	17.28	17.83	18.75	20.26	22.86
16	19.77	21.08	23.35	27.60	37.13
17	23.02	25.88	31.64	46.64	147.13
18	27.53	33.97	52.04	246.66	-53.61
19	33.87	49.38	163.38	-63.79	-19.90
20	43.63	92.23	-121.25	-25.44	-10.78

break-even maturity from the left;¹⁸ it is negative and decreases without bound as the maturity approaches the break-even maturity from the right. In short, Figure 3 shows the term structure of modified durations and is new to the literature.

The Rationale

The pattern of Figure 3 originates from the change of the policy reserve over time and the change is driven by the evolutions of the two cash flow streams underlying the reserve. The reserve of a life insurance policy is the difference between two cash flow streams. Life insurance policy reserves, except for the reserves of single-premium policies, consist of not only expected cash outflows but also expected cash inflows. Death/survival benefits, surrender payments, policyholder dividends, commissions, and other expenses are the expected cash outflows; future premiums from policyholders are the expected cash inflows to life insurance companies. In contrast, a bond generates only cash outflows for the bond issuer after the bond is sold.¹⁹

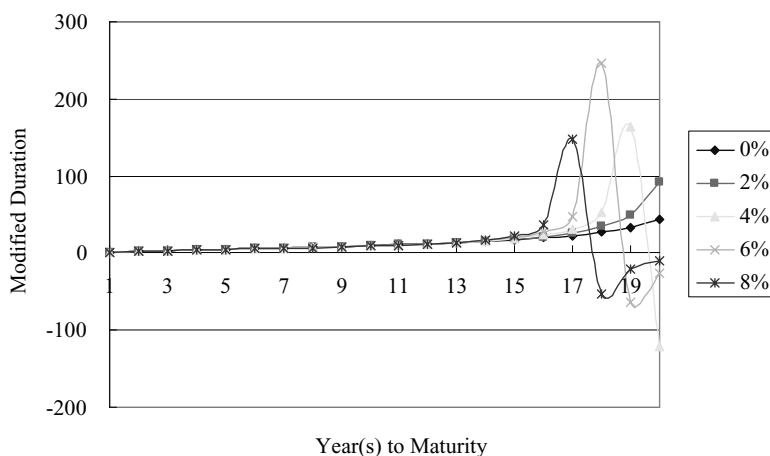
More specifically, the policy reserve equals the present value of expected cash outflows minus the present value of expected cash inflows. Furthermore, it has the opposite

¹⁸The reserves in the cases of 0 percent and 2 percent are positive for all maturities, and the implied break-even maturities are larger than 20 years. Therefore, the durations are all positive and increase with the time to maturity.

¹⁹A sold, single-premium life insurance policy produces only cash outflows in the future as well. Therefore, its reserve has regular duration features as a bond.

FIGURE 3

Term Structures of Modified Durations at Various Levels of Interest Rates



sign to the net present value (NPV) of a policy, which is the difference between the present value of future cash inflows and that of future outflows. A policy that is profitable to the life insurer should have a positive NPV and thus a negative reserve.²⁰ Newly sold multipremium policies should be profitable as long as they are priced correctly.²¹ The profit of a 2- or 3-year-old policy will be smaller than that of a brand new policy because fewer premiums are to be collected while the mortality rate increases. A correctly priced policy will become breakeven some time after it is sold. Old policies have positive reserves because there are few premiums left to be collected and the present value of future benefits is large due to the short maturities. The policy to be matured in a year has the largest reserve that equal to the policy's face amount divided by $(1 + \text{the 1-year interest rate})$. Therefore, the NPV/reserve of a policy is an increasing/decreasing function with respect to the policy's time to maturity as Figure 2 shows.²²

It is this pattern of the reserve that determines the pattern of the modified duration with respect to the policy's time to maturity. When the policy reserve is negative, its modified duration is negative as well. When the maturity is close to the breakeven maturity and thus the magnitude of the reserve is small, the modified duration has a large value. At the breakeven time to maturity, the policy reserve is zero and the modified duration has the magnitude of infinity.²³ In short, the newly identified

²⁰ Although negative reserves are not admissible in accounting and regulation, the valuation of an insurance company done by practitioners certainly takes the expected profits of in-force policies into account.

²¹ In contrast, the NPV of a single-premium policy after receiving the premium is always negative because no more premiums can be collected.

²² This feature holds for endowment and whole life insurance. It also holds for term life insurance except during the period close to the expiration date.

²³ Readers can observe that the maturities with zero reserves in Figure 2 correspond to the breakeven maturities in Figure 3.

pattern displayed in Figure 3 is driven by the pattern of the policy reserve that is in the denominator when calculating the modified duration.²⁴

For instance, the policy with 19 years to maturity has a reserve of \$12,837 at 4 percent interest rate whereas the present value of expected cash outflows and that of expected premiums are \$341,272 and \$328,435, respectively.²⁵ If the interest rate rises 1 percent, the present values of expected cash outflows and premiums will decrease by \$38,756 (\$341,272 – \$302,516) and \$19,893 (\$328,435 – \$308,542), respectively. The policy reserve will decrease by \$18,863 accordingly (\$38,756 – \$19,893). Because the net change from a 1 percent rise of the interest rate is larger than the policy reserves, the percentage change in values is more than 100 percent and results in a three-digit modified duration.²⁶ The modified duration of the policy reserve therefore may have a large figure when the reserve is small relative to the present values of individual cash flow streams.

A negative modified duration of the policy reserve can be justified from two other aspects. First, the implication about whether a rise in the interest rate increases or decreases the policy reserve is correct. A negative duration coupled with a negative reserve implies that increases in the interest rate will decrease the policy reserve, which is inferred from Equation (5):

$$\Delta P \approx -MD \times P \times \Delta r = (-) \times (-) \times (-) \times (+) < 0.$$

where ΔP and Δr are the increments of P and r . The downward shift of the reserve curve in Figure 2 caused by increases in the interest rate confirms this implication.

Second, it generates a good approximation of the change in policy reserve resulting from a change in the interest rate. For instance, the reserve of a policy with a maturity of 20 years decreases by \$186 to \$–15,514 from \$–15,328 when the interest rate increases from 4 percent to 4.01 percent.²⁷ The predicted change using the modified duration is

$$\Delta P \approx -(-121.25) \times (-15,328) \times 0.0001 = -186.$$

Indeed, the rationale for a negative reserve duration is obvious once we realize that a sold policy may be an asset rather than a liability to the insurance company. Imagine a situation in which an asset with a positive duration is treated as a negative liability. This negative liability then should have a negative duration so that the value changes with respect to changes in the interest rate are consistent, no matter how the asset is treated. Therefore, a policy with a negative reserve should have a negative duration because it is a *de facto* asset with a positive duration.

²⁴Whole life insurance has the same reserve pattern as endowment. The reserve duration of a whole life insurance policy therefore has the same pattern as Figure 1.

²⁵The amounts of \$341,272 and \$328,435 are obtained from the spreadsheet resulting in policy reserves (\$341,272 – \$328,435 = \$12,837).

²⁶A more accurate calculation for the modified duration is $\frac{-12,626.80 - 12,836.53}{12,836.53} \div 0.01\% = 163.38$ that is the number shown in Table 2.

²⁷Both amounts are taken from the spreadsheet used to calculate policy reserves.

TABLE 3

The Term Structures of Effective Durations With Various Initial Interest Rates

Year(s) to Maturity	0%	2%	4%	6%	8%
1	0.97	0.96	0.95	0.94	0.92
2	1.93	1.92	1.89	1.84	1.79
3	2.90	2.87	2.81	2.72	2.61
4	3.88	3.84	3.73	3.57	3.39
5	4.90	4.83	4.66	4.42	4.13
6	5.96	5.86	5.61	5.26	4.85
7	7.09	6.94	6.60	6.12	5.56
8	8.30	8.10	7.65	7.01	6.28
9	9.64	9.38	8.79	7.96	7.02
10	11.14	10.81	10.06	9.01	7.83
11	12.87	12.46	11.53	10.21	8.74
12	14.89	14.38	13.25	11.63	9.81
13	17.36	16.76	15.42	13.46	11.22
14	20.52	19.84	18.33	16.03	13.26
15	24.77	24.09	22.61	20.08	16.68
16	30.73	30.33	29.54	27.54	23.60
17	40.03	40.82	43.57	47.43	47.80
18	56.93	62.99	90.12	321.64	-150.05
19	94.23	134.10	-770.17	-57.47	-24.17
20	260.76	-897.89	-67.29	-23.74	-11.29

THE TERM STRUCTURE OF EFFECTIVE DURATIONS

The Term Structure

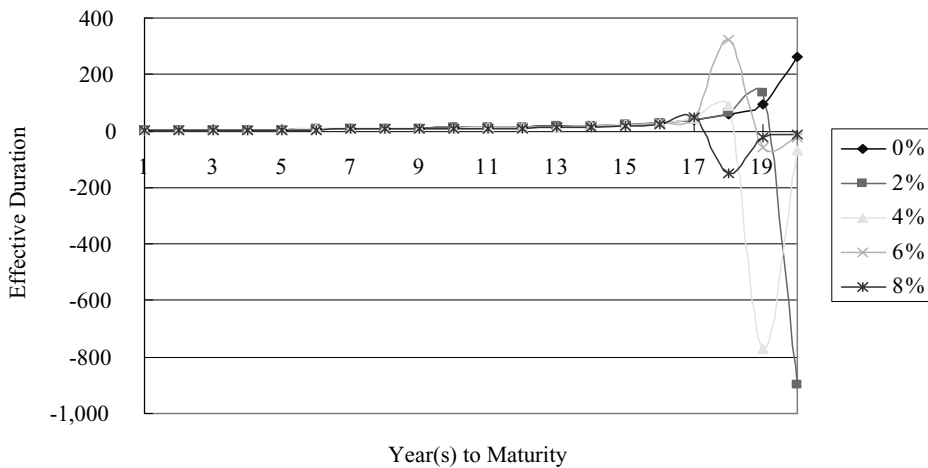
The calculations to obtain modified durations ignore two important facts: interest rates are stochastic and surrender rates depend upon interest rates. To incorporate interest-sensitive surrender rates and stochastic interest rates, this article draws on the cointegration model of Equation (4) to simulate surrender rates and the 1-year interest rates using various pairs of initial values. Then the simulated interest rates and surrender rates are substituted into Equations (1)–(3) to generate the reserve distributions for policies that are sold in different years. Equation (6) is then used to obtain the effective durations of reserves for the endowment policies with different times to maturity. The effective durations calculated using the initial surrender rate of 7 percent are reported in Table 3.²⁸

Table 3 confirms the findings from Table 2. First, several effective durations of policy reserves are negative. For instance, the policies with maturities longer than 18 years have negative durations when the interest rate is higher than 2 percent. Second, many effective durations are larger than the time to maturity of the policies. Some of them

²⁸We tried other initial values (e.g., 0 percent, 4 percent, 10 percent, and 14 percent) for the surrender rate to couple with the initial short rate of 4 percent. The results show that the initial value of the surrender rate has little impact on the effective durations and are thus omitted.

FIGURE 4

Term Structures of Effective Durations With Various Initial Interest Rates



have very large figures, such as 321.64 (the policy with 18 years to maturity when the initial interest rate is 6 percent) and 260.76 (the policy with 20 years to maturity when the initial interest rate is 0 percent).

More important, the term structure of effective durations exhibits the pattern as shown in Figure 1. The term structure for a given initial interest rate consists of two curves separated by a vertical asymptote as can be detected from Figure 4. The effective duration is positive, increases with the time to maturity of the policy, and approaches infinity when the policy's maturity approaches the break-even maturity. The other curve is in the negative domain and decreases without bound as the maturity of the policy decreases to the break-even maturity. In other words, the findings from the deterministic cases in the section "The Term Structure" hold in the cases of stochastic interest rates and interest-dependent surrenders. This is reasonable because the rationale stated in the section "The Rationale" is entirely independent from the behaviors of interest rates and surrenders and consequently should apply to effective durations as well.

Effective Dollar Duration

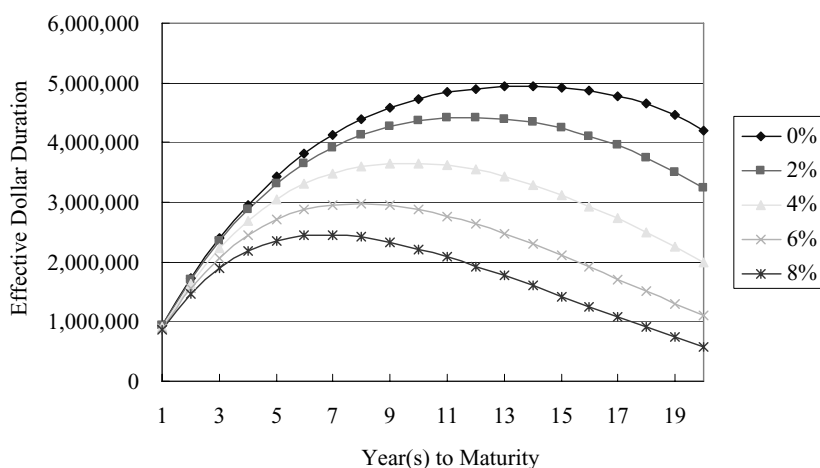
To further demonstrate the validity of the above results, the effective dollar duration (*EDD*) is calculated. The *EDD* measures changes in the mean reserves with respect to the initial 1-year rate changes. Specifically,

$$EDD = -\frac{\partial \text{mean reserves}}{\partial r_0}. \quad (7)$$

The *EDD* can be deemed as the slope of the policy reserve—interest rate curve with the opposite sign. Its magnitude should decrease with the initial interest rate due to

FIGURE 5

Effective Dollar Durations Corresponding to the Effective Durations in Figure 4



the convexity of the present value function.²⁹ More specifically, the absolute value of the function's slope (i.e., the value of *EDD*) decreases with the interest rate because the present value function is convex with respect to the interest rate. The *EDDs* of reserves for policies with different times to maturity under different initial interest rates are plotted in Figure 5.³⁰

Figure 5 shows a clean pattern where the *EDD* decreases with the initial interest rate. The term structure of *EDDs* shifts downwards as the initial interest rate rises. This check further confirms the robustness of the calculations in this study.

The *EDD* of reserves is not monotonic with respect to the time to maturity because the two factors determining the *EDD* interfere with each other. All other things being equal, a longer maturity generates a larger *EDD* and so does a larger policy reserve. The policy reserve however decreases with the time to maturity. The policy with the longest time to maturity has the smallest reserve whereas the one with the shortest time to maturity has the largest reserve. Therefore, the function of the *EDD* with respect to the time to maturity is not monotonic.³¹

Effective Duration Versus Modified Duration

The comparison between Table 3 and Table 2 shows that many effective durations are larger than the corresponding modified durations. This is contrary to the arguments of Babbel (1995) and Santomero and Babbel (1997). Through Equation (11.3) in his paper, Babbel demonstrates that the positive correlation between cash flows and

²⁹Note that the VAR model of Equation (4) has the property of partial adjustments in both interest rate and surrender rate. A higher initial interest rate/surrender rate will therefore result in a higher average interest rate/surrender rate.

³⁰We calculated *EDD* using the equation that $EDD = ED \times \text{the starting mean reserve}$.

³¹We also calculated the modified dollar durations of reserves and found term structures similar to Figure 5.

interest rates generates a positive second term of the partial derivative of P with respect to interest rate r . The generated second term will result in a smaller duration.

Babbel's equation cannot be directly applied here because the reserves calculated using the VAR model differ from the reserves calculated by the deterministic way described in the section "The Term Structure." The deterministic method using a low interest rate (e.g., 0 percent, 2 percent, or 4 percent) overestimates the reserve because it assumes that the interest rate will remain at this low level for 20 years. On the other hand, the average interest rates generated by the VAR model with low initial interest rates increases gradually with time and thus generate smaller reserves. The deterministic method using a high interest rate (e.g., 8 percent) underestimates the reserve due to the convexity of the present value function with respect to the interest rate. In particular, the decrease in the present value due to an increase in the interest rate is smaller than the increase in the present value for an equivalent amount of decrease in the interest rate. The reserve calculated using the deterministic method with a high interest rate is hence smaller than the stochastic reserve even though the average interest rates are similar. Because the reserves calculated in the section "The Term Structure of Modified Duration" and the section "The Term Structure of Effective Duration" are different, one cannot simply apply Babbel's equation to assert that the effective duration of the policy reserve will be smaller than the modified duration.

The newly identified pattern of Figure 1 provides a graphical explanation of why the effective duration of the policy reserve is larger than the modified duration in some cases yet is smaller in other cases. It is apparent that the break-even maturities in Figure 3 are larger than those in Figure 4. For instance, the break-even maturity in Figure 3 when the interest rate is 4 percent is between 19 and 20 years while the break-even maturity is between 18 and 19 years when the initial interest rate is 4 percent in Figure 4. The modified duration has a larger break-even maturity because the deterministic method using low interest rates results in larger reserves than the stochastic way does. The larger break-even maturity means that the zero-reserve line of the modified duration at 4 percent interest rate is on the right-hand side of the line of the effective duration with 4 percent initial interest rate. The modified duration therefore increases more slowly as the time to maturity approaches the larger break-even maturity from the left, which causes the modified duration to be smaller than the effective duration. On the other hand, the larger break-even maturity makes the modified duration decrease more rapidly in the negative domain as the maturity approaches the zero-reserve line from the right. The modified duration is thus smaller as well. The cases in which the modified duration is larger can also be explained using similar reasoning from the relative position of the zero-reserve lines. In short, the pattern of Figure 1 explains the difference in magnitude between modified duration and effective duration.

THE DURATIONS OF AGGREGATE RESERVES

The abnormal durations including negative durations and extreme durations found in the previous sections imply that a life insurer might have great difficulties in finding assets to match individual policies. For instance, how can managers find an asset with a negative duration of 67.29 or a very large duration of 321.64? One may further

infer the difficulties in hedging the insurer's DGAP. The inference is invalid, however, because it misses the underlying reason for the bizarre durations: irregular durations originate from the reserves being small and/or negative. The reserve liability of a life insurer, resulting from policies that are sold in different years, will have a normal duration figure because abnormal durations are weighted out by the normal duration figures that come with large reserves. We shall illustrate this statement using a pool consisting of the endowment policies that are analyzed in the above sections.

Let us assume that a life insurer's in-force policies contain equal numbers of 20-year endowment policies with the time to maturity ranging from 1 to 20 years. More specifically, the insurer has 1,000 20-year endowment policies in force from which the first premiums were just collected. The insurer also just collected the second premiums from the 1,000 20-year policies that are at the beginning of their second policy year. The 3rd, 4th, 5th, . . . , 20th premiums are collected at the same time from the 1,000 20-year policies that are at the beginning of the 3rd, 4th, 5th, . . . , 20th policy years, respectively. Therefore, the aggregate reserves are equal to $\sum_1^{20} 1,000 \times L_n$. The effective duration of the aggregate reserves will then be a weighted average of the durations calculated in the section "The Term Structure of Effective Duration."³²

The effective durations of the aggregate reserves calculated using five initial interest rates are listed in Table 4. The effective duration ranges from 9.32 to 5.29 with initial interest rates ranging from 0 percent to 8 percent, which is feasible for the asset-liability management. The negative durations and huge duration figures of some policy reserves do not result in an abnormal portfolio duration because these individual reserves are small. The roles they play are immaterial in determining the duration of the aggregate reserves. For instance, the reserve of the policy with 20 years to maturity calculated using the VAR model with the initial interest rate of 0 percent is \$16,120 with an effective duration of 260.76. This policy contributes little to the portfolio duration even though its reserve duration is huge because the policy's reserve accounts for only 0.19 percent of the aggregate reserves. Similarly, the policies with negative reserve durations have only a trivial impact on the portfolio duration because their reserves are all small relative to the aggregate reserves. For example, the aggregate magnitudes of the weights associated with negative durations are smaller than 1.5 percent when the initial interest rate is 8 percent. The policies sold 17 years ago or earlier account for about 40 percent of the aggregate reserves, but their reserve durations are smaller than 3. Therefore, the duration of the aggregate reserves as a weighted average of the individual reserve durations is within a normal range despite the presence of abnormal component durations.

The portfolio durations reported in Table 4 are consistent with those in Santomero and Babbel (1997). More specifically, the effective duration of the liabilities of the endowment policies is 8.5 in Santomero and Babbel (1997). Their figure is close to the case with the initial interest rate of 2 percent in Table 4. Albeit of the similarity in the final result, this article provides readers with detailed descriptions on how

³²Similarly, the modified duration of the aggregate reserves is a weighted average of the durations calculated in the section "The Term Structure of Modified Duration." We do not report the results about the modified duration because the effective duration is a better measure.

TABLE 4
The Effective Durations of the Aggregate Reserves Calculated Using Five Initial Interest Rates

Both Interest Rates and Surrender Rates Are Simulated Using the VAR Model (the Initial Surrender Rate Is 7%)													
Initial Surrender Rate		0%			2%			4%			6%		
Year(s) to Maturity	Number of Policies	Effective Duration	Reserve	Weight	Effective Duration	Reserve	Weight	Effective Duration	Reserve	Weight	Effective Duration	Reserve	Weight
1	1,000	0.97	965,353	11.16%	0.96	961,601	11.60%	0.95	952,827	12.47%	0.94	939,849	13.49%
2	1,000	1.93	893,934	10.34%	1.92	886,936	10.70%	1.89	870,855	11.40%	1.84	847,760	12.17%
3	1,000	2.90	825,896	9.55%	2.87	816,104	9.85%	2.81	794,002	10.39%	2.72	763,153	10.95%
4	1,000	3.88	760,850	8.80%	3.84	748,668	9.03%	3.73	721,649	9.45%	3.57	685,047	9.83%
5	1,000	4.90	698,754	8.08%	4.83	684,510	8.26%	4.66	653,519	8.55%	4.42	612,830	8.80%
6	1,000	5.96	639,522	7.40%	5.86	623,516	7.52%	5.61	589,382	7.71%	5.26	545,988	7.84%
7	1,000	7.09	582,658	6.74%	6.94	565,161	6.82%	6.60	528,608	6.92%	6.12	483,690	6.94%
8	1,000	8.30	527,889	6.10%	8.10	509,134	6.14%	7.65	470,811	6.16%	7.01	425,363	6.11%
9	1,000	9.64	475,201	5.50%	9.38	455,401	5.49%	8.79	415,879	5.44%	7.96	370,728	5.32%
10	1,000	11.14	424,558	4.91%	10.81	403,903	4.87%	10.06	363,681	4.76%	9.01	319,513	4.59%
11	1,000	12.87	375,711	4.34%	12.46	354,373	4.28%	11.53	313,932	4.11%	10.21	271,337	3.89%
12	1,000	14.89	329,567	3.81%	14.38	307,666	3.71%	13.25	267,344	3.50%	11.63	226,698	3.25%
13	1,000	17.36	284,768	3.29%	16.76	262,416	3.17%	15.42	222,563	2.91%	13.46	184,208	2.64%
14	1,000	20.52	241,236	2.79%	19.84	218,562	2.64%	18.33	179,526	2.35%	16.03	143,783	2.06%
15	1,000	24.77	198,903	2.30%	24.09	176,056	2.12%	22.61	138,210	1.81%	20.08	105,365	1.51%
16	1,000	30.73	158,517	1.83%	30.33	135,683	1.64%	29.54	99,403	1.30%	27.54	69,680	1.00%
17	1,000	40.03	119,477	1.38%	40.82	96,885	1.17%	43.57	62,576	0.82%	47.43	36,150	0.52%
18	1,000	56.93	81,644	0.94%	62.99	59,598	0.72%	90.12	27,700	0.36%	321.64	4,682	0.07%
19	1,000	94.23	47,316	0.55%	134.10	26,193	0.32%	-770.17	-2,922	-0.04%	-57.47	-22,584	-0.32%
20	1,000	260.76	16,120	0.19%	-897.89	-3,596	-0.04%	-67.29	-29,608	-0.39%	-23.74	-46,072	-0.66%
Portfolio duration		9.32		100.0%	8.67		100.0%	7.55		100.0%	6.34		100.0%

the figure is obtained. Table 4 further shows how the effective duration of aggregate reserves may change with the initial interest rate, which is useful for insurers to manage their interest rate risk dynamically.

This article further analyzes the durations of the aggregate reserves coming from a thriving and a declining insurer. Suppose that the number of in-force policies with m years to maturity ($m \in N$ and $1 \leq m \leq 20$) is 5 percent smaller than the number of policies with $m - 1$ years to maturity. This in-force policy pool implies that the businesses of these insurers have been declining for two decades and that the effective duration of aggregate reserves of this pool ranges from 7.25 to 4.39 with initial interest rates ranging from 0 percent to 8 percent. The decreases in the portfolio durations when compared with Table 4 are due to the fact that old policies that have small durations but large reserves account for a higher portion of the aggregate reserves when the business has been declining. If the policies with the m -year maturity are 10 percent more than the policies with $m - 1$ years to maturity, then the duration of the aggregate reserves would range from 15.28 to 8.13. These results imply that the policy reserve liability of a life insurance company with declining/growing businesses is less/more sensitive to changes in the interest rate and demands smaller/larger duration assets in the asset-liability management.

CONCLUSIONS

Life insurance businesses are significantly exposed to changes in interest rates because the contracts usually last for long periods and have guaranteed minimal credit rates. High leverage ratios of life insurance companies aggravate the threat from interest rate variations. To measure/manage the interest rate risk, a life insurer usually has to calculate its DGAP that in turn demands the calculations of asset durations and liability durations. The durations of various assets have been studied extensively in the literature, but the durations of life insurance liabilities received limited attention and remained obscure. This article analyzes the liability duration in detail to fill the hole of the literature and the results have meaningful implications to the asset-liability management of life insurers.

In calculating the modified duration of the policy reserves for an endowment life insurance, this study identified features of policy reserve duration not previously recognized in the literature. The modified duration may be negative, be larger than the policy's maturity, and/or have extreme figures. This study further identified an interesting term structure of modified duration with respect to time to maturity. The term structure consists of a pair of curves separated by a zero-reserve line. One curve is in the positive domain and the duration increases with the maturity to infinity; the other is in the negative domain and the duration decreases with the maturity to negative infinity.

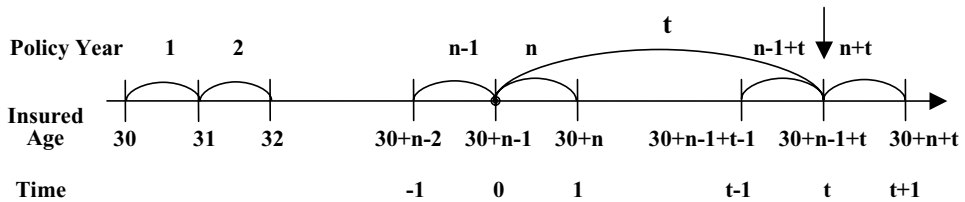
This pattern of duration is derived from the definition and pattern of the policy reserve. The reserve of a life insurance policy is the difference between the present value of expected cash outflows (e.g., death/survival benefits, surrender payments, interest rate dividends, commissions, and expenses) and that of expected cash inflows (premiums paid by the policyholder in the future). It changes with the time to maturity and can be negative, zero, or positive. When the reserve is negative, its duration is negative. If the reserve is close to zero, its duration will be a large figure.

The effective duration of the policy reserve has the same pattern of term structure. Using a model capturing the relation between the surrender rate and interest rate, this article calculated the effective duration of policy reserves for policies with different maturities. It was found that some effective durations are negative and some have extreme figures. In addition, it was found that the modified duration of policy reserve is not necessarily larger than the effective duration. This newly identified term structure of the duration helps to explain the irregular differences between modified duration and effective duration.

The above results may seem alarming to life insurance companies and regulators as they imply that the interest rate risk of a life insurance company may not be manageable. Life insurers would have tremendous difficulties in finding assets to match liabilities with negative and/or huge durations. The concern is true at the micro level but not at the macro level. Our further analyses on the durations of aggregate reserves indicate that the duration of an insurer's policy reserve liability may still be within a feasible range. It was found that the aggregate reserve comprising the policy reserves of the endowment policies in different policy years have durations smaller than 15 under reasonable growth/declining assumptions. The dramatic changes result from the fact that the policy reserves with large duration values are small and account for a small percentage of the aggregate reserve. Therefore, the interest rate risk of a life insurance company can still be managed even when some individual policy reserves exhibit abnormal sensitivities to fluctuations in interest rates. Finally, it was found that a company with a higher growth rate in underwriting life insurance should seek longer-duration assets to match their liabilities.

APPENDIX

The expected net cash outflow of the endowment policy at time t ($1 \leq t < 20 - n + 1$) for the policy that is at the beginning of its policy year n (or equivalently at the insured's age of $30 + n - 1$; $1 \leq n < 20$) after the n th net premium has been received, $E(CF_t|n)$, is defined based on the following time line:



First bracket term of Equation (1) (at the end of policy year $n - 1 + t$)

Death benefit (based on ${}_{t-1}p_{30+n-1}^{(r)} \times q_{30+n-1+t-1}^{(d)}$)

F

Surrender value (based on ${}_{t-1}p_{30+n-1}^{(r)} \times q_t^{(s)}$)

S_{n-1+t}

Second term of Equation (1) (at the beginning of policy year $n + t$)

Premium (based on ${}_tp_{30+n-1}^{(r)}$)

P

Commission (based on ${}_tp_{30+n-1}^{(r)}$)

$CMR_{n+t} \times P$

Fixed expense (based on ${}_tp_{30+n-1}^{(r)}$)

$FExp_{n+t}$

Variable cost (based on ${}_tp_{30+n-1}^{(r)}$)

$VarCost \times P$

REFERENCES

- Babbel, D. F., 1995, Asset-Liability Matching in the Life Insurance Industry, in: E. I. Altman and I. T. Vanderhoof, eds., *The Financial Dynamics of the Insurance Industry* (New York: Irwin Professional Publishing), pp. 239-255.
- Babbel, D. F., C. Merrill, and W. Panning, 1997, Default Risk and the Effective Duration of Bonds, *Financial Analysts Journal*, 53(1): 35-44.
- Bierwag, G. O., 1987, *Duration Analysis: Managing Interest Rate Risk* (Cambridge, MA: Ballinger Publishing Company).
- Bierwag, G. O., C. J. Corrado, and G. G. Kaufman, 1990, Computing Durations for Bond Portfolios, *Journal of Portfolio Management*, 17(1): 51-55.
- Bierwag, G. O., C. J. Corrado, and G. G. Kaufman, 1992, Durations for Portfolios of Bonds Priced on Different Term Structures, *Journal of Banking and Finance*, 16(4): 705-714.
- Bierwag, G. O., and I. J. Fooladi, 2006, Duration Analysis: An Historical Perspective, *Journal of Applied Finance*, 16(2): 144-160.
- Bierwag, G. O., and G. G. Kaufman, 1985, Duration Gaps for Financial Institutions, *Financial Analysts Journal*, 41(2): 68-71.
- Bierwag, G. O., and G. G. Kaufman, 1992, Duration Gaps With Futures and Swaps for Managing Interest Rate Risk at Depository Institutions, *Journal of Financial Services Research*, 5(3): 217-234.
- Bierwag, G. O., and G. G. Kaufman, 1996, Managing Interest Rate Risk With Duration Gaps to Achieve Multiple Targets, *Journal of Financial Engineering*, 5(1): 53-73.
- Bierwag, G. O., G. G. Kaufman, and A. Toevs, 1983, Duration: Its Development and Use in Bond Portfolio Management, *Financial Analysts Journal*, 39(4): 15-35.
- Black, K., Jr., and H. Skipper, Jr., 2000, *Life and Health Insurance*, 13th edition (Upper Saddle River, NJ: Prentice Hall).
- Bowers, N. L., H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt, 1997, *Actuarial Mathematics*, 2nd edition (Schaumburg, IL: Society of Actuaries).
- Briys, E., and F. de Varenne, 1997, On the Risk of Insurance Liabilities: Debunking Some Common Pitfalls, *Journal of Risk and Insurance*, 64(4): 673-694.
- Briys, E., and F. de Varenne, 2001, *Insurance From Underwriting to Derivatives: Asset Liability Management in Insurance Companies* (New York: John Wiley & Sons).
- Cornell, B., 2000, Equity Duration, Growth Options, and Asset Pricing, *Journal of Portfolio Management*, 26(3): 105-111.
- Engle, R. F., and C. W. J. Granger, 1987, Co-integration and Error Correction Representation, Estimation and Testing, *Econometrica*, 55(2): 251-276.
- Fooladi, I. J., and G. S. Roberts, 2004, Macrohedging for Financial Institutions: Beyond Duration, *Journal of Applied Finance*, 14(1): 11-19.
- Fooladi, I. J., G. S. Roberts, and F. Skinner, 1997, Duration for Bonds With Default Risk, *Journal of Banking and Finance*, 21(1): 1-16.
- Hamelink, F., B. Macgregor, N. Nanthakumaran, and A. Orr, 2002, A Comparison of UK Equity and Property Duration, *Journal of Property Research*, 19(1) 61-80.

- Haugen, R. A., and D. W. Wichern, 1974, The Elasticity of Financial Assets, *Journal of Finance*, 29(4): 1229-1240.
- Hayre, L., and H. Chang, 1997, Effective and Empirical Durations of Mortgage Securities, *Journal of Fixed Income*, 6(4): 17-33.
- Hevert, K. T., R. M. McLaughlin, and R. A. Taggart, Jr., 1998, Growth Options and Equity Duration, *Journal of Portfolio Management*, 25(1): 43-50.
- Hopewell, M. H., and G. G. Kaufman, 1973, Bond Price Volatility and Term to Maturity: A Generalized Specification, *American Economic Review*, 63(4): 749-753.
- Jacoby, G., 2003, A Duration Model for Defaultable Bonds, *Journal of Financial Research*, 26(1): 129-146.
- Kalotay, A. J., 1984, An Analysis of Original Issue Discount Bonds, *Financial Management*, 13(3): 29-38.
- Kalotay, A. J., G. O., Williams, and F. J. Fabozzi, 1993, A Model for Valuing Bonds and Embedded Options, *Financial Analysts Journal*, 49(3): 35-46.
- Kuo, W., C. Tsai, and W. Chen, 2003, An Empirical Study on the Lapse Rate: The Cointegration Approach, *Journal of Risk and Insurance*, 70(3): 489-508.
- Leibowitz, M. L., E. H. Sorensen, R. D. Arnott, and H. N. Hanson, 1989, A Total Differential Approach to Equity Duration, *Financial Analysts Journal*, 45(5): 30-37.
- Little, P. K., 1984, Negative Cash Flows, Duration, and Immunization: A Note, *Journal of Finance*, 39(1): 283-288.
- Macaulay, F., 1938, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856* (New York: National Bureau of Economic Research).
- Morgan, G. E., 1986, Floating Rate Securities and Immunization: Some Further Results, *Journal of Financial and Quantitative Analysis*, 21(1): 87-94.
- Redington, F. M., 1952, Review of the Principles of Life-Office Valuations, *Journal of the Institute of Actuaries*, 78(3): 286-340.
- Reilly, F. K., D. J. Wright, and R. R. Johnson, 2007, Analysis of the Interest Rate Sensitivity of Common Stocks, *Journal of Portfolio Management*, 33(3): 85-108.
- Samuelson, P.A., 1945, The Effect of Interest Rate Increases on the Banking System, *American Economic Review*, 35(1): 16-27.
- Santomero, A. M., and D. F. Babbel, 1997, Financial Risk Management by Insurers: An Analysis of the Process, *Journal of Risk and Insurance*, 64(2): 231-270.
- Tsai, C., W. Kuo, and W. Chen, 2002, Early Surrender and the Distribution of Policy Reserves, *Insurance: Mathematics, and Economics*, 31(3): 429-445.
- Tuckman, B., 1996, *Fixed Income Securities: Tools for Today's Markets* (New York: John Wiley & Sons).