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On the valuation of reverse mortgages with regular tenure payments

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ABSTRACT

For the valuation of reverse mortgages with tenure payments, this article proposes a specific analytic valuation framework with mortality risk, interest rate risk, and housing price risk that helps determine fair premiums when the present value of premiums equals the present value of contingent losses. The analytic valuation of reverse mortgages with tenure payments is more complex than the valuation with a lump sum payment. This study therefore proposes a dimension reduction technique to achieve a closed-form solution for reverse annuity mortgage insurance, conditional on the evolution of interest rates. The technique provides strong accuracy, offering important implications for lenders and insurers.

1. Introduction

Demographic aging constitutes a significant global problem. According to the Organization of Economic Cooperation and Development, the average aged dependency ratio (i.e., the ratio of the number of senior dependents over age 65 to the total workforce) reached 20.9% in 2000 and is expected to increase to 47% by 2050. Governments worldwide thus face increasing fiscal burdens, including the pressing question of how to increase the income of seniors effectively and inexpensively. Reverse mortgages (RMs) offer one such answer.

Unlike with conventional mortgages, RM lenders provide a lump sum or periodic payments to elderly homeowners, which enable them to convert their home equity into cash to support their retirement. In their assessment of the Connecticut Housing Finance Authority's reverse annuity mortgage (RAM) program, Klein and Sirmans (1994) conclude that annuity payments exert a demonstrable financial enhancement effect on borrowers, with an 88% average annual income increase. Mitchell and Piggott (2004) also explore the feasibility of RM markets in Japan and suggest that RMs could relieve the fiscal burden of traditional, state-funded retirement provision. Moreover, RMs can serve as an alternative option for elderly homeowners, enhancing the liquidity of their properties and improving their consumption, which in turn decreases financial pressures on governments (Hancock, 1998; Rowlingson, 2006).

Six types of payment options are available through RMs: lump sum, term, line of credit, modified term (which combines line of credit and term payments), tenure, and modified tenure (which combines tenure and line of credit). Of these, line of credit is the most popular payment option because of its high flexibility. However, RMs with tenure payments, such as the RAMs, offer relief to social security systems. By providing regular cash flows to the borrower, these mortgages increase the income replacement ratio for retirees. However, from financial institutions' perspective, issuing RMs is risky because of their non-recourse clauses—the lender may not access any other assets to reclaim the loan value except for the collateral housing property. In addition, compared with a lump sum payment RM, RAM suffers greater longevity risk and thus demands an insurance mechanism. In particular, the lender faces a "crossover risk", in which the outstanding balance might accumulate more quickly than the appreciation rate of housing value. The guarantee insurance of the Home Equity Conversion Mortgage (HECM) program, which is issued by the Federal Housing Administration (FHA), covers losses due to the non-recourse provision.

In this market, RAM lenders face three main types of risk: housing price risk, interest rate risk, and longevity risk. Previous research pertaining to housing price models suggests two perspectives. First, discrete time models assume that housing price returns exhibit autocorrelation (Case and Shiller, 1989; Hosios and Pesando, 1991; Ito and Hirono, 1993) and generalized autoregressive conditional heteroskedasticity (e.g., Nothaft et al., 1995; Chinloy et al., 1997; Chen et al., 2010a; and Li et al., 2010). Second, continuous time models argue that housing prices follow traditional,

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geometric Brownian motion (e.g., Kau et al., 1992, 1993, 1995; Szymanoski, 1994; Chinloy and Megbolugbe, 1994; Kau and Keenan, 1995, 1999; Hilliard and Reis, 1998; Yang et al., 1998; Bardhan et al., 2006; Ma et al., 2007; Wang et al., 2007; and Huang et al., 2011). However, housing prices have shifted substantially in recent years. Using Chicago Mercantile Exchange futures price data, Mizrach (2008) demonstrates that, on average, it requires approximately 69 jump risks to reach significance in a 315-day sample. Using the US national average new home price returns for single-family mortgages from January 1986 to June 2008, Chen et al. (2010b) identify 14 times that the monthly housing price changed more than 10% per month. In turn, we employ a jump diffusion model to capture the dynamics of the housing price.

As mortality rates continue to improve, longevity risks are critical, and thus a wide range of mortality models has been proposed. Among them, Lee and Carter (1992, hereinafter LC) model is perhaps the most popular because it is easy to implement and offers acceptable prediction errors (e.g., Koissi et al., 2006, Melnikov and Romaniuk, 2006; and Wang et al., 2011). Modifications of the LC model also allow for broader interpretations (Milevsky and Promislow, 2001; Brouhns et al., 2002; Renshaw and Haberman, 2003; Dahl, 2004; Cairns et al., 2004, 2009; Yang et al., 2010).

Because of their complex frameworks, RMs, and in particular RAMs, rarely appear in analytic studies. This research aims to develop a framework for pricing RAMs by considering stochastic mortality models, stochastic interest rates, and housing price models simultaneously. As noted by Phillips and Gwin (1992), a longer lifespan of RM, higher interest rates, or a real estate depression all can impose greater crossover risk. Yet existing efforts to price RM contracts tend to use periodic life tables, thus neglecting the dynamics of mortality rates (see Weinrobe, 1988; Chinloy and Megbolugbe, 1994; Szymanoski, 1994; Tse, 1995; and Zhai, 2000), or else price HECM models with a constant interest rate, thus ignoring the inherent dynamics of interest rates (Chinloy and Megbolugbe, 1994; Szymanoski, 1994; Chen et al., 2010a; Li et al., 2010). To fill the gap in extant research, we assume instead that the underlying mortality process follows the LC model and the interest rate follows the model of Cox et al. (1985, hereinafter CIR). In addition, after the subprime crisis, the jump risk of housing prices has drawn substantial attention (Chen et al., 2010b). We model housing prices as a jump diffusion process, and the complexity of our pricing model precludes the development of a closed-form solution. Conditional on the evolution of interest rates, we provide a closed-form solution for pricing RAM contracts. Using the closed-form solution, we can obtain a fair value of the RAM numerically by simulating only the interest rates. This dimension reduction algorithm can reduce simulation error significantly. Finally, for model comparison, we employ an autoregressive moving average–generalized autoregressive conditional heteroskedastic (ARMA–GARCH) model for housing price dynamics.

The contributions of this research are fourfold. First, we propose a pricing approach for deriving a fair level of periodic payments in the prevalent HECM program. Second, we consider the jump effect on housing prices in our valuation of RAM and show that unexpected shocks significantly affect the level of annuity payments. Third, we integrate the interest rate dynamic into the valuation of RAM; most research pertaining to HECM models uses a constant interest rate environment. Fourth, the proposed dimension reduction technique to achieve a closed-form solution for RAM insurance, conditional on the evolution of interest rates, provides an accurate approach.

The remainder of this article proceeds as follows. The next section specifies the valuation framework, including the structure of RAM insurance and the processes underlying interest rates, housing prices, and mortality rates. The valuation methodology appears in the third section. In Section 4, we use numerical results to examine how the properties of the amount of annuity payments change in response to various parameters; we also provide a robust analysis of our approach. Finally, in Section 5 we draw conclusions about our findings.

2. The model

In this section, we first describe the contract structure of RAMs and RM insurance. We then specify the model for the interest rate, housing prices, and mortality.

2.1. RAM contracts

We consider an RM with tenure payments and an insurance mechanism analogous to the HECM insurance structure in the United States. For example, RAMs constitute a home equity conversion type that provides regular annuity payments to the borrower, with no repayment of interest or principal due until the deadline, which arises when the homeowner sells the property, moves out permanently, or dies. When the loan is due and payable, the property is sold to repay the loan. The non-recourse debt means that the collateral property (i.e., house) is the only asset the lender may use to reclaim the loan. The lender faces a loss if the loan balance exceeds the value of the collateral property at the maturity date. Many banks and insurers thus are unwilling to enter the market, for fear of this non-recourse clause.

To encourage the financial industry to participate and offer RMs, some form of insurance is needed to cover its potential contingent losses. We investigate RMs with tenure payments, analogous to the US HECM program.[1] In this case, the insurer charges premiums to the lender with an up-front premium equal to $100\pi_0\%$ of the initial house value and an annual assessment equal to $100\pi_m\%$ of the outstanding balance. These charges are transformed by the borrower and accrued to the outstanding balance, along with the interest rate on the loan. In the prevailing HECM program, the up-front premium π_0 is 2%, and the annual premium π_m is 1.25%.[2] Using this predetermined insurance premium structure, we can evaluate the present value of expected claim losses and insurance premiums, thus determining the annuity payment in a condition in which the present value of expected claim losses equals the value of insurance premiums. For simplicity, we assume that the loan comes due and is payable only at the borrower's death. At this terminal date, the outstanding balance is payable, and the remaining value belongs to the heirs, if the property value is greater than the outstanding balance.

2.2. Interest rate model

Several models depict the local process for the short-term interest rate. To avoid the problem of a negative nominal interest

1 In real-life RM contracts in the United States, lenders generally limit the amount that can be borrowed to 40% of the value of the home, with a hard limit of $625,000 for FHA/HUD (Housing and Urban Development) RMs. The costs of a reverse mortgage vary, but in general people can expect to pay more than $6000, which include the following items: origination fee (maximum cap of $6000: 2% for the first $200,000, and 1% thereafter), mortgage insurance (2% of the real estate valuation), title insurance (varies by state), and legal fees. In some cases, a land/house survey, which costs around $400, is also required. These charges can be added to the RM itself, so the only up-front cash borrowers need is the costs of the real estate valuation, usually around $300. Regarding the interest rate, both fixed and adjustable rate RMs are available.

2 A modification to the HECM program on October 4, 2010, raised annual premiums for the HECM standard in the reverse mortgage program insured by the FHA from 0.5% to 1.25%. The initial premium rate remained the same (2%). The FHA asserts that the previous risk premium was underestimated before the modification.

rate, we assume that the time-t spot rate $r(t)$ for a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T})$ follows the CIR model:

$$dr(t) = \kappa_r(\theta_r - r(t))\,dt + \sigma_r\sqrt{r(t)}dW_r(t), \tag{1}$$

where F_t, $t \in [0, T]$, is the smallest sigma field, such that $r(t)$ and the housing price $H(t)$ is known and measurable; P is the physical (real-world) probability measure; $(F_t)_{t=0}^{T}$ is the right-continuous natural filtration, such that $F_t \subset F_u, t \le u$; θ_r is the long-term short interest rate; κ_r is the speed of reversion; σ_r is the instantaneous volatility; and $W_r(t)$ is a standard Brownian motion.

For the risk-neutral probability measure Q, the spot rate process defined in Eq. (1) ensures that the discounted zero coupon bond price follows a martingale, namely,

$$P(0,T) = E_Q\left[\frac{P(T,T)}{B(T)}\right] = E_Q\left[\exp\left(-\int_0^T r(u)\,du\right)\right], \tag{2}$$

where $B(t)$ is the money market account at time t, which satisfies

$$B(t) = \exp\left(\int_0^t r(u)\,du\right). \tag{3}$$

According to the CIR model (see also Svoboda, 2004), a standard Brownian motion under the risk-neutral probability measure Q, $W_r^Q(t)$, can be specified as follows:

$$dW_r^Q(t) = dW_r(t) + \frac{\vartheta_r\sqrt{r(t)}}{\sigma_r}dt, \tag{4}$$

where ϑ_r is the risk-premium parameter. Consequently, the time-t spot rate $r(t)$ becomes

$$dr(t) = \left(\kappa_Q - \theta_Q r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_r^Q(t), \tag{5}$$

where $\kappa_Q = \kappa_r\theta_r$ and $\theta_Q = \kappa_r + \vartheta_r$.

In a discrete-time setup, we assume that the spot rate between time t and $t + \Delta t$ is fixed at $r(t)$ but may vary from one band to the next. Consequently, the spot rate dynamic under the risk-neutral measure Q is governed by

$$r(t+\Delta t) - r(t) = \left(\kappa_Q - \theta_Q r(t)\right)\Delta t + \sigma_r\sqrt{r(t)}\Delta W_r^Q(t). \tag{6}$$

2.3. Housing price model

We assume that the housing price process follows a log-normal diffusion process with jumps. Specifically, in a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T})$, the housing price process is given by

$$\ln\left(\frac{H(t)}{H(0)}\right) = \int_0^t \mu_H(s)\,ds - \frac{1}{2}\sigma_H^2 t - \lambda\eta t + \sigma_H W_H(t) + \sum_{i=1}^{N(t)} J_i \tag{7}$$

where $\mu_H(t)$ is the annual rate of return for the house; σ_H is the volatility; $W_H(t)$ is a standard Brownian motion; N is a Poisson process with intensity λ; $\{J_i\}$ is a sequence of independent normal random variables, with mean θ_J and variance σ_J^2; and η is an adjustment term equal to the mean of $\exp(J_i)$ minus 1 (i.e., $\eta = \exp\left(\theta_J + \sigma_J^2/2\right) - 1$). In addition, the correlation coefficient between $W_r(t)$ and $W_H(t)$ is $\rho_{Hr}t$. The standard Brownian motions ($W_r(t)$ or $W_H(t)$), the Poisson process $N(t)$, and the normal random variables $\{J_i\}$ are assumed to be independent.

Regarding the housing price dynamics, Eq. (7) represents real-world housing price dynamics. For the valuation of the RM, we must obtain housing price dynamics for a risk-neutral measure. To achieve this goal, we employ an equivalent martingale measure using the conditional Esscher transform developed by Bühlmann et al. (1996). The Esscher transform has been widely applied to price financial and insurance securities in incomplete markets (Siu et al., 2004; Li et al., 2010; Chen et al., 2010a; Yang, 2011). We use the same conditional Esscher transform technique to price RMs with regular tenure payments. That is, we let $\{\xi_t | t = j\Delta t, j = 0, 1, \ldots, T/\Delta t\}$ be a F_t-adapted stochastic process:

$$\xi_T = \prod_{t=\Delta t}^{T} \frac{\exp(\varphi Y(t))}{E_P\left(\exp(\varphi Y(t)) | F_{t-\Delta t}\right)}, \tag{8}$$

where $Y(t) = \ln(H(t)/H(t-\Delta t))$. It is straightforward to verify that $E_P(\xi_T) = 1$ and $E_P(\xi_T|F_t) = \xi_t$. Equivalently, $\{\xi_t\}$ is a martingale under P. We define a new martingale measure Q by

$$\left.\frac{dQ}{dP}\right|_{F_T} = \xi_T. \tag{9}$$

Then, under the risk-neutral measure Q, the housing price dynamic can be rewritten as

$$\ln\left(\frac{H(t)}{H(0)}\right) = \int_0^t r(s)\,ds - \left(\frac{1}{2}\sigma_H^2 + \lambda_Q\eta_Q\right)t + \sigma_H W_H^Q(t) + \sum_{i=1}^{N(t)} J_i^Q, \tag{10}$$

where $W_H^Q(t)$ is a standard Brownian motion under Q and the correlation coefficient between $W_H^Q(t)$ and $W_r^Q(t)$ equals $\rho_{Hr}t$; N is a Poisson process with intensity $\lambda_Q = \lambda\exp\left(\theta_J\varphi + \varphi^2\sigma_J^2/2\right)$; $\{J_i^Q\}$ is a sequence of independent normal random variables, with mean $\theta_J^Q = \theta_J + \varphi\sigma_J^2$ and variance σ_J^2; and $\eta_Q = \exp\left(\theta_J^Q + \sigma_J^2/2\right) - 1$. The derivation of Eq. (10) based on the conditional Esscher transform appears in Appendix A.

To adjust for rental income, similar to Chen et al. (2010a) and Li et al. (2010), we can adjust the dynamics of the housing price process under the risk-neutral measure Q as follows:

$$\ln\left(\frac{H(t)}{H(0)}\right) = \int_0^t (r(s) - \delta(s))\,ds - \left(\frac{1}{2}\sigma_H^2 + \lambda_Q\eta_Q\right)t + \sigma_H W_H^Q(t) + \sum_{i=1}^{N(t)} J_i^Q, \tag{11}$$

where $\delta(t)$ is the rental rate (or maintenance yield) for the house.

2.4. Mortality model

We use the LC model to project the mortality process. The Census Bureau population forecast similarly has used it as a benchmark for long-term forecasts of US life expectancy. The two most recent Social Security Technical Advisory Panels suggest that trustees should adopt this method or tactics consistent with it (Lee and Miller, 2001). For our study, the central death rate for age x at time t, $m_{x,t}$, follows the process

$$\ln\left(m_{x,t}\right) = \alpha_x + \beta_x k_t + e_{x,t}, \tag{12}$$

where α_x refers to the average specific pattern of mortality for age group x; β_x describes the pattern of deviations for the age group x when the parameter k varies; k_t is a time-varying index that explains the change in mortality over time t; and $e_{x,t}$ describes the error term, which should be white noise with a zero mean and a relatively small variance (Lee, 2000). We fit the model using the singular value decomposition approximation proposed by Lee and Carter (1992).

We forecast the future values of k_t with an ARIMA(0, 1, 0) model, as is used almost exclusively in practice. According to this model, the dynamic of k_t takes the form:

$$k_t = k_{t-1} + z + \varepsilon_t, \tag{13}$$

where z is the drift parameter, equal to the average first difference in k_t, and ε_t is a sequence of independent and identically normal distributions with mean 0 and variance σ^2. Let the valuation date equal t_0; the values $k_1, \ldots, k_{t_0}$ are known and fitted by historical data to forecast the future k_{t_0+j}. Thus, according to Eq. (13),

$$k_{t_0+j} = k_{t_0} + jz + \sum_{i=1}^{j} \varepsilon_{t_0+i}. \tag{14}$$

Therefore, k_{t_0+j} follows a normal distribution with mean $k_{t_0} + jz$ and variance $j\sigma^2$, conditional on the information up to time t_0.

Let the age-specific mortality rates be constant within bands of age and time, but they may vary from one band to the next. Given any integer age x_0 and calendar year t_0, we assume that

$$m_{x_0+\upsilon, t_0+\varsigma} = m_{x_0,t_0}, \quad \upsilon, \varsigma \in [0, 1). \tag{15}$$

Then, let ${}_np_{x_0,t_0}$ denote the n-year survival probability that an x_0-aged person in calendar year t_0 reaches age $x_0 + n$, which is

$$\begin{aligned} {}_np_{x_0,t_0} &= \exp\left(-\sum_{j=0}^{n-1} m_{x_0+j,t_0+j}\right) \\ &= \exp\left(-\sum_{j=0}^{n-1} \exp\left(\alpha_{x_0+j} + \beta_{x_0+j} k_{t_0+j}\right)\right). \end{aligned} \tag{16}$$

The distribution function of ${}_np_{x_0,t_0}$ under the real-world (physical) probability measure P is given by

$$\mathrm{F}_n(x) = \Pr\left({}_np_{x_0,t_0} \le x\right). \tag{17}$$

To change the probability measure from the real-world to a risk-neutral measure, Wang (2000) proposes a distortion operator:

$$F_n^{\tau}(x) = \Phi\left(\Phi^{-1}(\mathrm{F}_n(x)) + \tau\right), \tag{18}$$

where τ is a parameter called the "market price of risk", Φ is the cumulative standard normal distribution function, and F_n^{τ} is the risk-neutral distribution function of ${}_np_{x_0,t_0}$.

Denuit et al. (2007) employ the LC model to forecast mortality and price a risky coupon survivor bond, applying the Wang (2000) transform to consider the market price of risk for bearing mortality risk. According to Denuit et al. (2007), the expectation value of ${}_np_{x_0,t_0}$ under the Wang risk measure is

$$\begin{aligned} S(t_n) &= E_Q\left[{}_np_{x_0,t_0}\right] = \int_0^1 \left(1 - F_n^{\tau}(x)\right) dx \\ &= \int_0^1 \left(1 - \Phi\left(\Phi^{-1}(F_n(x)) + \tau\right)\right) dx. \end{aligned} \tag{19}$$

Using Eq. (19), we can transform the physical survival distribution of ${}_np_{x_0,t_0}$ into a risk-neutral version.

3. Pricing RAM contracts

In accordance with the principle that the present value of an insurance premium equals the present value of the expected loss, we can obtain the annuity payment for a RAM. The initial property value $H(0)$ serves to determine the annuity payment. An initial charge equals $100\pi_0$% of the initial house value at the inception of the contract, and the annual premium is $100\pi_m$% of the outstanding balance. The mortgage rate on the loan is assumed to float, equal to the short interest rate plus the spread π_r.

Let x_0 be the age of the borrower at the valuation date t_0, which we denote as 0 for simplicity, and let ω be the final age at which all lives end. The borrower's death occurs only at the end of each year. The mortality process and financial asset prices (i.e., short interest rate process and housing price process) also are assumed to be independent. Let a be the annual annuity payment of the RAM. We define $BAL(t_j)$ as the outstanding balance at the end of year $j, j = 1, \ldots, \omega - x_0 + 1$. The next year's outstanding balance is this year's outstanding balance and the annual premium charge, plus the annuity payment this year, with interest accrued. That is,

$$BAL\left(t_{j+1}\right) = \begin{cases} (\pi_0 H(0) + a)\exp\left(\int_0^{t_1} [r(s) + \pi_r]\, ds\right), & j = 0 \\ \left(BAL\left(t_j\right)(1 + \pi_m) + a\right)\exp\left(\int_{t_j}^{t_{j+1}} [r(s) + \pi_r]\, ds\right), & \\ \quad j = 1, \ldots, \omega - x_0. & \end{cases} \tag{20}$$

Eq. (20) then can be rearranged as follows:

$$BAL(t_j) = \sum_{i=0}^{j-1} a \exp\left(\int_{t_i}^{t_j} [r(s) + \pi_r]\, ds\right)\pi_{i,j}, \quad j = 1, \ldots, \omega - x_0 + 1, \tag{21}$$

where

$$\pi_{i,j} = \begin{cases} \left(1 + \pi_0 \dfrac{H(t_0)}{a}\right)(1 + \pi_m)^{j-1}, & i = 0 \\ (1 + \pi_m)^{j-1-i}, & i = 1, 2, \ldots, j - 1. \end{cases} \tag{22}$$

Similar to credit default swaps, the lender, who buys RAM insurance, makes a series of payments to the RAM insurer, who sells RAM insurance. In exchange, it receives a payoff if the loan balance of a RAM grows to exceed the property value in the event of a borrower's death. Without arbitrage opportunities, the present value of the premium charges should equal the present value of the total expected claim losses under the risk-neutral measure Q. The present value of RAM insurance premiums can be expressed as

$$\pi_0 H(0) + \sum_{j=1}^{\omega-x_0} E_Q\left[\frac{{}_{t_j}p_{x_0,t_0} BAL\left(t_j\right)\pi_m}{B\left(t_j\right)}\right]. \tag{23}$$

Through Eqs. (3) and (21), Eq. (23) becomes

$$\pi_0 H(0) + \sum_{j=1}^{\omega-x_0} S\left(t_j\right)\sum_{i=0}^{j-1}\left(a\, e^{\pi_r(t_j - t_i)}\pi_{i,j}\pi_m P(0, t_i)\right), \tag{24}$$

where $P(0, t_j)$ is the price of a zero coupon bond that matures at time t_i.

In contrast, the present value of the expected losses from future claims takes the form:

$$\sum_{j=1}^{\omega-x_0+1} E_Q\left[\frac{\left[{}_{t_{j-1}}p_{x_0,t_0} - {}_{t_j}p_{x_0,t_0}\right]\left[BAL\left(t_j\right) - H\left(t_j\right)\right]^+}{B\left(t_j\right)}\right]. \tag{25}$$

Or equivalently,

$$\sum_{j=1}^{\omega-x_0+1} \left(S\left(t_{j-1}\right) - S\left(t_j\right)\right) C\left(t_j\right), \tag{26}$$

where $C(t_j) = E_Q\left[[BAL(t_j) - H(t_j)]^+ / B(t_j)\right]$. We determine the annuity payment a when the present value of the insurance premiums, defined in Eq. (24), covers the present value of expected losses from future claims, defined in Eq. (26).

However, when the process for the short-term interest rate follows the CIR process defined in Eq. (1), the closed-form solutions of $C(t_j)$ are not available; rather, its value depends on the interest rate and housing price distributions. A standard approach to derive the value of $C(t_j)$ is to use a Monte Carlo simulation method, in which each simulation generates two possible paths for future interest rates and housing prices.

Instead, in Proposition 1 we offer a pricing method that can calculate $C(t_j)$ more effectively. First, we derive the closed-form solutions of $C(t_j)$, conditional on $BAL(t_j)/B(t_j)$. Second, we generate random numbers from $BAL(t_j)/B(t_j)$ to calculate the fair values of $C(t_j)$.

Proposition 1. *Let M be the number of simulation paths. The closed-form solutions of $C(t_j)$ can be obtained from*

$$C\left(t_j\right) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\left[x_{mj}\sum_{k=0}^{\infty}P_k\left(\lambda_Q t_j\right)\Phi\left(-d_{jk}\left(x_{mj}\right)\right) - H_0 e^{-\int_0^{t_j}\delta(s)ds}\sum_{k=0}^{\infty}P_k\left(\lambda_Q\left(1+\eta_Q\right)t_j\right)\times\Phi\left(-h_{jk}\left(y_{mj}\right)\right)\right], \tag{27}$$

where

$$P_k\left(\lambda\right) = \frac{e^{-\lambda}\left(\lambda\right)^k}{k!}, \tag{28}$$

$$d_{jk}\left(x_{mj}\right) = \frac{\ln\frac{H(t_0)}{x_{mj}} + \bar{r}_{jk} - \frac{1}{2}\sigma_{kj}^2}{\sigma_{kj}}, \quad h_{jk}\left(y_{mj}\right) = \frac{\ln\frac{H(t_0)}{y_{mj}} + \bar{r}_{jk} + \frac{1}{2}\sigma_{kj}^2}{\sigma_{kj}}, \tag{29}$$

$$\bar{r}_{jk} = -\int_0^{t_j}\delta\left(s\right)ds - \lambda_Q\eta_Q t_j + k\theta_J^Q + \frac{1}{2}k\sigma_J^2, \quad \textit{and}$$

$$\sigma_{kj}^2 = \sigma_H^2 t_j + k{\sigma_J}^2. \tag{30}$$

The x_{mj} and y_{mj} values are the mth random numbers drawn from the distribution of $BAL(t_j)/B(t_j)$, with the probability measures Q and $\tilde{R}$, respectively. The Radon–Nikodym derivative R satisfies

$$\frac{dR}{dQ} = \exp\left\{\sigma_H W_H^Q\left(t\right) - \frac{1}{2}\sigma_H^2 t\right\}, \tag{31}$$

and the Radon–Nikodym derivative $\tilde{R}$ (given $N(t) = k$) satisfies

$$\frac{d\tilde{R}}{dR} = \exp\left\{\sigma_J\sqrt{k}Z - \frac{1}{2}\sigma_J^2 k\right\}, \tag{32}$$

with Z as the standard normal distribution.

Appendix B provides the proof of Proposition 1 and the simulation procedure of x_{mj} and y_{mj}. According to this proposition, only one random variable, $BAL(t_j)/B(t_j)$, emerges when we calculate the value of $C(t_j)$. Because the distribution of $BAL(t_j)/B(t_j)$ is determined uniquely by the interest rate process, we can successfully reduce the simulation dimensions from two to one, which enhances the accuracy of the solution. We also can employ our approach for different types of RMs. For example, if the annuity payment, a, is zero except for the first payment, we can price RMs with a lump sum payment.

Table 1
Parameter estimation of spot rate and housing price.

Panel A: spot rate				
κ_Q	θ_Q	σ_r	ϑ_r	
0.0114 (5.5651)	0.2137 (5.0839)	0.0648 (13.4054)	−0.0119 (−3.5501)	
Panel B: housing price				
σ_H	θ_J	σ_J	λ	φ
0.0739 (8.3052)	−0.0045 (−5.2842)	0.0344 (6.6075)	8.2223 (2.8926)	2.0280 (1.9227)

Values in parentheses denote t values.

4. Numerical results

For the numerical analyses of the impacts of longevity risk, interest rate risk, and housing price risk on the valuation of RAMs, we first describe the parameters for the dynamics of the interest rate, housing price, and mortality rate. Then, we present the numerical results for the annuity payments, as well as their sensitivity analyses.

First, we employ the three-month Treasury yield rate from January 1973 to December 2010, consistent with the period of housing price data, as a proxy for the spot rate.[3] Second, to test housing price dynamics empirically, we employ monthly observations of the national average prices of previously occupied homes for conventional single-family mortgages in the United States as a proxy for housing prices, using data from the Federal Housing Financial Board.[4] Third, with maximum likelihood estimation, we derive the corresponding parameters of the CIR model and the jump diffusion model in Table 1. We also calculate the correlation coefficient between spot rates and housing returns as equal to 0.0252. Fourth, we turn to the fit of the LC model with US men's mortality data, for an observation period from 1970 to 2005.[5] We let the market price of mortality risk (τ) equal −0.5.[6]

Using the estimated parameters in Table 1, we conduct a numerical analysis to assess the impacts of spot rates and housing prices on the RAM annuity payments. We illustrate the relevant parameters for the base case subsequently. Consider a RAM for a borrower aged 70 years. The interest rate spread (π_r) is 2%, and the initial risk-free interest rate is 0.14%. The initial housing price is assumed to be 100, so the annuity payment is a percentage of the initial property value. The rental rate is 2%. According to the modified HECM premium structure, the upfront premium (π_0) is 2%, and the annual premium (π_m) is 1.25%. Applying Proposition 1 with 10,000 simulation paths, the annual annuity payment based on the parameters is equal to 2.25. In addition, we consider the present value of total annuity payments by multiplying the corresponding annuity factor, calculated by the interest rate term structure and the mortality assumption, accompanied by market price of premium. The present value of total annuity payments therefore is 26.26.

3 See https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldAll.

4 Categories of homes include previously occupied, new, and all homes. Following Huang et al. (2011), we choose previously occupied home prices as the proxy for housing prices because a reverse mortgage loan gets repaid through the proceeds of the sale of the property.

5 Data source: Human Mortality Database (https://www.mortality.org/).

6 Using the market price of an annuity sold to a 65-year-old man in Belgium, Denuit et al. (2007) find that the market price of risk for men ranges from −0.4901 to −0.4449. We calculate the annuity payments in Table 4 by varying the market price of risk for comparison.

Table 2
Annual annuity payments and present values (PVs) of annuity.

Age	62	65	70	75	80
Annual annuity payment	1.43	1.70	2.25	3.00	4.06
PV of annuity	20.94	22.98	26.26	29.18	31.62
Annual premium rate (π_m)	0.005	0.00875	0.0125	0.01625	0.02
Annual annuity payment	1.98	2.13	2.25	2.36	2.45
PV of annuity	23.04	24.79	26.26	27.50	28.53
Interest rate spread (π_r)	0.01	0.015	0.02	0.025	0.03
Annual annuity payment	2.75	2.49	2.25	2.03	1.82
PV of annuity	32.03	29.05	26.26	23.65	21.22

4.1. Sensitivity analysis for base parameters

Table 2 contains the annual annuity payments and present values of the total annuity for different ages,[7] annual premium rates, and interest rate spreads. All else being equal, the lower the age, the lower are the annual annuity payment and the present value of annuity. In economic terms, the present value of the house is the sum of the present value of future rental incomes. According to the RAM mechanism, the borrower uses the anticipated rental income after his or her death in exchange for the annual annuity payment at inception. An older borrower can borrow more money, because his or her expected death is sooner, and the present value of the rental income after death is greater. As Table 2 shows, a higher annual premium rate contributes to a higher annual annuity payment and present value of annuity. After the subprime crisis, the FHA claimed that the risk associated with the HECM program had been underestimated and raised the annual premium while also reducing the amount that could be borrowed. Finally, higher interest rate spreads lead to a high mortgage rate on loans and, thus, to lower annual annuity payments and present value of annuity.

4.2. Sensitivity analysis for the parameters of interest rate and housing prices

We examine the sensitivity of the annual annuity payments and the present value of annuity by varying the parameters of the interest rate and housing price model in Table 3. First, in Panel A of Table 3, both higher correlation coefficients and higher volatilities in the spot rate and housing returns contribute to a higher risk profile in the RAM, which leads to lower annual annuity payments and present value of annuity. In addition, housing price volatility has the greatest impact, compared with correlation and interest rate volatility, on annual annuity payments, which implies that when pricing a RAM, it is crucial to estimate the volatility of housing prices precisely.

Second, in Panel B of Table 3, higher frequency (greater λ) and severity (lower θ_J) of housing price shocks may contribute to lower housing prices and thus lower annual annuity payments and present value of annuity. In addition, a higher level of σ_J should lead to greater uncertainty in housing prices, which contributes to lower annual annuity payments and present value of annuity.

Finally, Eq. (5) can be rewritten as follows:

$$d\,r\,(t) = \theta_Q \left(\frac{\kappa_Q}{\theta_Q} - r\,(t)\right) dt + \sigma_r \sqrt{r\,(t)} dW_r^Q\,(t)\,. \tag{5$'$}$$

In this case, θ_Q is the speed of reversion, and κ_Q/θ_Q is the long-term interest rate under the risk-neutral measure Q. As Panel C of Table 3 shows, when κ_Q/θ_Q is fixed, the higher the speed of reversion, the higher are the annual annuity payments and the present value of annuity. Similarly, a higher long-term interest rate increases the level of the interest rate on average, with a corresponding higher annual annuity payment and present value of annuity.

4.3. Sensitivity analysis for mortality improvement and decrement rate

In this subsection, we consider the effect of mortality improvements and decrements. We first consider the effect of the mortality projection by replacing the LC model with the period life table for 2005. As Table 4 shows, changing the underlying mortality projection increases the annual annuity payment, which indicates that it would be overestimated if we were to ignore the mortality improvement.

With Table 4, we also consider the sensitivity of the annual annuity payments by varying the level of the market price of risk. The higher the market price of risk (in absolute value), the lower is the annual annuity payment. However, compared with the volatility of housing returns or the parameters of the housing price shock, the impact of the market price of risk is trivial.

Finally, although we only depict death as a decrement in the base case, the decrement rate should be higher than that in the base case because of the possibility that the borrower will sell the property or move out. As a result, we examine the impact of the decrement rate on the annual annuity payment by increasing mortality rates. Table 4 reveals that the annuity payment increases 13.33% (25.78%) if the mortality rate increases 25% (50%), which means that a higher decrement rate leads to a higher annuity payment. Consequently, the annuity payment would be underestimated if the RAM insurers were to ignore the possibility that the borrower sells or moves away from the property.

4.4. Robust analyses

4.4.1. Our method versus the pure Monte Carlo method

In this subsection, we discuss the efficiency of the proposed approach by comparing it with the pure Monte Carlo method. To obtain the present value of the expected losses (Eq. (26)), a pure Monte Carlo approach must generate two possible paths for future interest rates and housing prices for each simulation. Unlike a pure Monte Carlo approach, we propose a simulation method in Proposition 1 to obtain the present value of the expected losses. First, using Eq. (B.7), we simulate the jth random numbers x_{mj} and y_{mj} from the distribution of $BAL(t_j)/B(t_j)$ according to the interest rate processes in Eqs. (B.8) and (B.15), respectively. Second, given x_{mj} and y_{mj} for $m = 1, \ldots, M$, we can obtain the numerical value of the expected loss $C(t_j)$ using the closed-form formula in Eq. (27). Consequently, our method is more accurate than the pure Monte Carlo approach because the simulated values x_{mj} and y_{mj} are determined only by the interest rates.

[7] Note that 62 years is the minimum age requirement in the HECM program.

Table 3
Sensitivity analyses for parameters of interest rate and housing price.

Panel A: impacts of volatilities					
σ_H	0.04	0.06	0.0739	0.08	0.10
Annual annuity payment	2.42	2.33	2.25	2.21	2.08
PV of annuity	28.17	27.15	26.26	25.83	24.27
σ_r	0.03	0.05	0.0648	0.07	0.09
Annual payment each year	2.27	2.26	2.25	2.25	2.24
PV of annuity	26.36	26.31	26.26	26.24	26.16
ρ_{Hr}	−0.5	−0.25	0.0252	0.25	0.5
Annual annuity payment	2.28	2.27	2.25	2.24	2.23
PV of annuity	26.59	26.44	26.26	26.12	25.96
Panel B: impacts of housing price shock					
λ	0	4	8.2223	12	16
Annual annuity payment	2.70	2.47	2.25	2.09	1.93
PV of annuity	31.47	28.76	26.26	24.31	22.49
θ_j	−0.04	−0.03	−0.02	−0.01	−0.0045
Annual annuity payment	1.91	2.06	2.18	2.24	2.25
PV of annuity	22.33	24.04	25.37	26.14	26.26
σ_j	0	0.02	0.0344	0.04	0.06
Annual annuity payment	2.69	2.53	2.25	2.12	1.63
PV of annuity	31.37	29.50	26.26	24.76	19.06
Panel C: impacts of spot rate					
θ_Q	0.1	0.2	0.2137	0.3	0.4
Annual payment each year	2.03	2.23	2.25	2.34	2.41
PV of annuity	25.71	26.22	26.26	26.44	26.56
κ_Q/θ_Q	0.02	0.04	0.0535	0.06	0.08
Annual payment each year	1.83	2.08	2.25	2.33	2.59
PV of annuity	25.02	25.79	26.26	26.47	27.07

Table 4
Sensitivity analyses for the decrements.

	Annual annuity payment
Period table for 2005	2.49
LC model with $\tau = 0$	2.29
LC model with $\tau = -0.5$ (base case)	2.25
LC model with $\tau = -1$	2.22
1.25 times of mortality ($\tau = -0.5$)	2.55
1.5 times of mortality ($\tau = -0.5$)	2.83

To examine pricing efficiency, we depict 100 repeated, simulated results of annuity payment in Fig. 1, each of which we obtain using 10,000 simulation paths. The results of our pricing method are more concentrated than those of a pure Monte Carlo approach. The standard deviations of the 100 simulated results are 0.00036 for the proposed method and 0.01 for the pure Monte Carlo approach. Therefore, reducing the simulation dimensions from two to one, our proposed approach seems to provide a more accurate and credible solution.

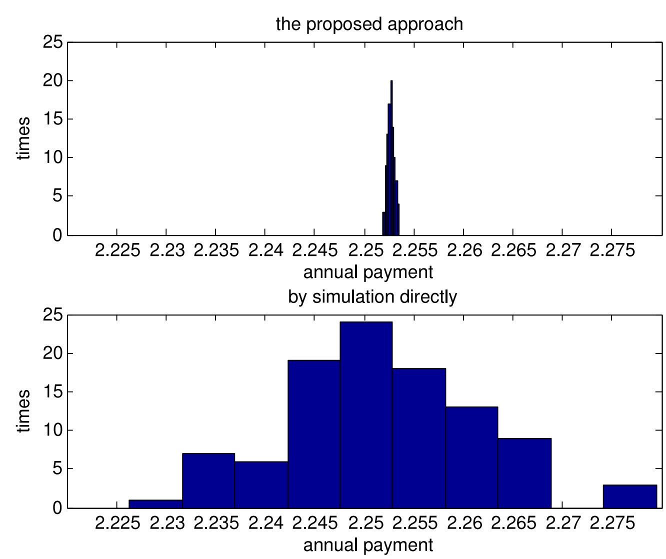

Fig. 1. Comparison of proposed method with the pure Monte Carlo method.

4.4.2. Model comparison: Jump diffusion model versus ARMA–GARCH model

We also fit an ARMA(s, m)–GARCH(p, q) model proposed by Bollerslev (1986) to the housing price return data. Two specifications support the ARMA(s, m)–GARCH(p, q) model: the conditional mean and the conditional variance. The conditional mean model for the difference in the logarithm of housing prices on a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T})$ can be expressed as

$$Y(t) = \ln\left(\frac{H(t)}{H(t-\Delta t)}\right) = \mu_H(t) + \varepsilon_H(t), \tag{33}$$

where $\mu_H(t) = c + \sum_{i=1}^{s} \tau_i Y(t - i\Delta t) + \sum_{j=1}^{m} \zeta_j \varepsilon_H(t - j\Delta t)$ is the conditional mean function, given the time $(t - \Delta t)$ information $F_{t-\Delta t}$; s is the order of the autocorrelation terms; m is the order of the moving average terms; τ_i is the ith-order autocorrelation coefficient; and ζ_j is the jth-order moving average coefficient. In addition, $\varepsilon_H(t)$ is a Gaussian innovation with conditional variance $h(t)$, given the information $F_{t-\Delta t}$, as follows:

$$h(t) = w + \sum_{i=1}^{q} \alpha_i \varepsilon_H^2 (t - i\Delta t) + \sum_{j=1}^{p} \beta_j h(t - j\Delta t), \tag{34}$$

where p is the order of the GARCH terms; q is the order of the ARCH term; α_i is the ith-order ARCH coefficient; and β_j is the jth-order GARCH coefficient.

Table 5
Parameter estimation of the ARMA(0,1)–GARCH(1,1) model.

c	ζ_1	w	α_1	β_1
0.0052 (6.2844)	−0.3971 (−8.4983)	1.93×10^{-5} (2.4925)	0.9012 (58.1093)	0.0844 (5.3428)

Values in parentheses denote t values.

Table 6
Model comparison: jump diffusion model versus ARMA–GARCH model.

Age	62	65	70	75	80
Jump diffusion					
Annual annuity payment	1.43	1.70	2.25	3.00	4.06
PV of annuity	20.94	22.98	26.26	29.18	31.62
ARMA–GARCH					
Annual annuity payment	1.86	2.17	2.81	3.65	4.80
PV of annuity	27.21	29.42	32.76	35.46	37.38

To obtain the housing price dynamics under a risk-neutral measure, we also employ an equivalent martingale measure using the conditional Esscher transform developed by Bühlmann et al. (1996). Let $\{\xi_t | t = j\Delta t, j = 0, 1, \ldots, T/\Delta t\}$ be an F_t-adapted stochastic process, defined as

$$\xi_T = \prod_{t=\Delta t}^{T} \frac{\exp(\varphi(t) Y(t))}{E_P(\exp(\varphi(t) Y(t)) | F_{t-\Delta t})}. \tag{35}$$

Then, we define a new martingale measure Q_φ by

$$\left. \frac{dQ_\varphi}{dP} \right|_{F_T} = \xi_T. \tag{36}$$

Under the risk-neutral measure Q_φ, the housing return becomes

$$\begin{aligned} Y(t) &= \ln\left(\frac{H(t)}{H(t-\Delta t)}\right) \\ &= r(t-\Delta t)\Delta t - \frac{1}{2}h(t) + \varepsilon_H^Q(t), \end{aligned} \tag{37}$$

where $\varepsilon_H^Q(t) = \varepsilon_H(t) - \varphi(t) h(t)$ follows a normal distribution with mean 0 and variance $h(t)$ under the martingale measure Q_φ. The correlation coefficient between $\varepsilon_H^Q(t)/\sqrt{h(t)}$ and $W_r^Q(t)/\sqrt{t}$ is equal to ρ_{Hr}. Appendix C provides the derivation of Eq. (37), based on the conditional Esscher transform. In addition, similar to the modification of Eq. (11), Eq. (37) is also adjusted by the rental rate.

We fit the ARMA(0,1)–GARCH(1,1) model, the best model according to the Bayesian information criterion, to the same housing price data. Table 5 exhibits the corresponding calibrated parameters. Table 6 presents the numerical results of the ARMA(0,1)–GARCH(1,1) model, as well as those from the jump diffusion model. Apparently, the annual annuity payments of the ARMA(0,1)–GARCH(1,1) model are larger than those of the jump diffusion model, which means that model risk is considerable when pricing the fair annuity payment for RAM contracts.[8]

5. Conclusion

Demographic aging challenges economies worldwide and thus cannot be neglected. As an effective method to increase the income of seniors, RMs can decrease the fiscal burden on social security systems. Among the various types of RMs, the RAM provides regular cash flows and can provide superior benefits in terms of enhancing social security.

However, the complexity of RAM pricing models precludes a closed-form formula from solving the underlying problem; annuity payments must be calculated with traditional Monte Carlo simulation methods. This article offers a closed-form solution, conditional on the evolution of interest rates, to calculate a fair RAM value. Using the closed-form solution, we can numerically determine the fair RAM value, which depends solely on the distribution of interest rates, by simulating interest rates. Our alternative approach reduces the number of dimensions in the simulation and thus increases solution accuracy.

This study simultaneously considers the dynamics of interest rate, housing prices, and mortality rate in the valuation of RAM; these considerations are rare in extant RM literature. From the numerical results, we determine that the impacts of annual premium rate, long-term interest rate, and the housing price shock on fair annuity payments are significant. We examine the effects of the mortality projection and decrement on RAM contracts. The annual annuity payment will be overestimated if RAM insurers ignore mortality improvements but underestimated if they ignore the possibility that the borrower sells the property or moves out. Finally, similar to Li et al. (2010), we find significant model risk for pricing the fair RAM annuity payments. Thus, it is crucial for lenders and insurers to model the housing price process suitably.

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Appendix A. Proof of Eq. (10)

When housing prices follow the jump diffusion process from Eq. (7) (e.g., Matsuda, 2004), the moment-generating function of $\ln(H(t)/H(0))$ takes the form:

$$\begin{aligned} & E_P(\exp(\omega \ln(H(t)/H(0)))) \\ &\quad = \exp\left(\omega \int_0^t \mu_H(s)\, ds\right) \exp(t\psi_H(\omega)), \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} \psi_H(\omega) = {} & \lambda\left(\exp\left(\omega\theta_J + \frac{\omega^2\sigma_J^2}{2}\right) - 1\right) \\ & - \omega\left(\frac{1}{2}\sigma_H^2 + \lambda\eta\right) + \frac{\omega^2\sigma_H^2}{2}. \end{aligned} \tag{A.2}$$

Because $\ln(H(t)/H(t-\Delta t))$ and $\ln(H(t-\Delta t)/H(0))$ are independent, we have

$$\begin{aligned} E_P(\exp(\omega Y(t))) &= E_P\left(\exp\left(\omega \ln\left(\frac{H(t)}{H(t-\Delta t)}\right)\right)\right) \\ &= \exp\left(\omega \int_{t-\Delta t}^t \mu_H(s)\, ds\right) \\ &\quad \times \exp(\Delta t\psi_H(\omega)). \end{aligned} \tag{A.3}$$

[8] Li et al. (2010) employ an ARMA–EGARCH model and geometric Brownian motion to capture the dynamics of housing prices and calculate the guaranteed cost for a single-payment reverse mortgage. They also find that model risk is significant when valuing the guaranteed cost.

In addition, because the discount housing price under the martingale measure Q is a martingale, we have

$$H(t-\Delta t) = E_Q\left(\frac{B(t-\Delta t)}{B(t)}H(t)\middle| F_{t-\Delta t}\right) = E_Q\left(\exp\left(-\int_{t-\Delta t}^{t} r(u)\,du\right)H(t)\middle| F_{t-\Delta t}\right). \quad (A.4)$$

We assume that the spot rate between $t-\Delta t$ and t is fixed at $r(t-\Delta t)$ but may vary from one band to the next. Consequently, $B(t) = B(t-\Delta t)e^{r(t-\Delta t)\Delta t}$. According to Lemma 5.2.2 of Shreve (2004), we obtain

$$\begin{aligned} H(t-\Delta t) &= e^{-r(t-\Delta t)\Delta t}E_Q\left(H(t)\mid F_{t-\Delta t}\right) \\ &= e^{-r(t-\Delta t)\Delta t}E_P\left(\frac{\xi_t}{\xi_{t-\Delta t}}H(t)\mid F_{t-\Delta t}\right) \\ &= e^{-r(t-\Delta t)\Delta t}E_P\left(\frac{\exp(\varphi Y(t))}{E_P\left(\exp(\varphi Y(t))\mid F_{t-\Delta t}\right)}\right. \\ &\quad \left.\times\ H(t-\Delta t)\exp(Y(t))\mid F_{t-\Delta t}\right) \\ &= H(t-\Delta t)\,e^{-r(t-\Delta t)\Delta t} \\ &\quad \times \frac{E_P\left(\exp((\varphi+1)Y(t))\mid F_{t-\Delta t}\right)}{E_P\left(\exp(\varphi Y(t))\mid F_{t-\Delta t}\right)} \\ &= H(t-\Delta t)\,e^{\int_{t-\Delta t}^{t}\mu_H(s)ds - r(t-\Delta t)\Delta t} \\ &\quad \times \exp\left(\Delta t\left(\psi_H(\varphi+1)-\psi_H(\varphi)\right)\right). \end{aligned} \quad (A.5)$$

Or equivalently,

$$\begin{aligned} r(t-\Delta t)\Delta t &= \int_{t-\Delta t}^{t}\mu_H(s)\,ds + \left(\psi_H(\varphi+1)-\psi_H(\varphi)\right)\Delta t \\ &= \int_{t-\Delta t}^{t}\mu_H(s)\,ds + \left(\lambda_Q\eta_Q - \lambda\eta + \varphi\sigma_H^2\right)\Delta t. \end{aligned} \quad (A.6)$$

In addition, the characteristic function of $\ln(H(t)/H(t-\Delta t))$ under the risk-neutral measure is of the form:

$$\begin{aligned} &E_Q\left(\exp(i\omega Y(t))\mid F_{t-\Delta t}\right) \\ &= E_P\left(\frac{\exp(\varphi Y(t))}{E_P\left(\exp(\varphi Y(t))\mid F_{t-\Delta t}\right)}\exp(i\omega Y(t))\middle| F_{t-\Delta t}\right) \\ &= \exp\left(i\omega\int_{t-\Delta t}^{t}\mu_H(s)\,ds\right. \\ &\quad \left.+\Delta t\left(\psi_H(\varphi+i\omega)-\psi_H(\varphi)\right)\right), \end{aligned} \quad (A.7)$$

where

$$\begin{aligned} &i\omega\int_{t-\Delta t}^{t}\mu_H(s)\,ds + \Delta t\left(\psi_H(\varphi+i\omega)-\psi_H(\varphi)\right) \\ &= \lambda\Delta t\,e^{\theta_J\varphi+\frac{\varphi^2\sigma_J^2}{2}}\left(e^{i\omega\left(\theta_J+\varphi\sigma_J^2\right)-\frac{1}{2}\omega^2\sigma_J^2}-1\right) \\ &\quad + i\omega\left(\int_{t-\Delta t}^{t}\mu_H(s)\,ds - \left(\frac{1}{2}\sigma_H^2+\lambda\eta\right)\Delta t\right) \\ &\quad + \frac{\sigma_H^2}{2}\Delta t\left(2i\omega\varphi-\omega^2\right) \\ &= \lambda_Q\Delta t\left(e^{i\omega\theta_J^Q-\frac{1}{2}\omega^2\sigma_J^2}-1\right) \\ &\quad + i\omega\left(\int_{t-\Delta t}^{t}\mu_H(s)\,ds + \left(\varphi\sigma_H^2-\frac{1}{2}\sigma_H^2-\lambda\eta\right)\Delta t\right) \\ &\quad - \frac{\omega^2\sigma_H^2}{2}\Delta t \\ &= \Delta t\left\{\lambda_Q\left(e^{i\omega\theta_J^Q-\frac{1}{2}\omega^2\sigma_J^2}-1\right)\right. \\ &\quad \left.+ i\omega\left(r(t-\Delta t)-\frac{1}{2}\sigma_H^2-\lambda_Q\eta_Q\right)-\frac{\omega^2\sigma_H^2}{2}\right\}. \end{aligned} \quad (A.8)$$

In view of Eq. (A.8), because $B(t) = B(t-\Delta t)e^{r(t-\Delta t)\Delta t}$, the characteristic function of $Y(t)$ under the martingale measure Q governs the following expression:

$$\begin{aligned} \ln\left(\frac{H(t)}{H(t-\Delta t)}\right) &= \left(r(t-\Delta t)-\frac{1}{2}\sigma_H^2-\lambda_Q\eta_Q\right)\Delta t \\ &\quad + \sigma_H\left(W_H^Q(t)-W_H^Q(t-\Delta t)\right) \\ &\quad + \left(\sum_{i=1}^{N(t)}J_i - \sum_{i=1}^{N(t-\Delta t)}J_i\right) \\ &= \ln\left(\frac{B(t)}{B(t-\Delta t)}\right) - \left(\frac{1}{2}\sigma_H^2+\lambda_Q\eta_Q\right)\Delta t \\ &\quad + \sigma_H\left(W_H^Q(t)-W_H^Q(t-\Delta t)\right) \\ &\quad + \left(\sum_{i=1}^{N(t)}J_i - \sum_{i=1}^{N(t-\Delta t)}J_i\right). \end{aligned} \quad (A.9)$$

Or equivalently,

$$\begin{aligned} \ln\left(\frac{H(t)}{B(t)}\right) &= \ln\left(\frac{H(t-\Delta t)}{B(t-\Delta t)}\right) - \left(\frac{1}{2}\sigma_H^2+\lambda_Q\eta_Q\right)\Delta t \\ &\quad + \sigma_H\left(W_H^Q(t)-W_H^Q(t-\Delta t)\right) \\ &\quad + \left(\sum_{i=1}^{N(t)}J_i - \sum_{i=1}^{N(t-\Delta t)}J_i\right). \end{aligned} \quad (A.10)$$

Consequently, we can obtain Eq. (10) by iterating Eq. (A.10). This completes the proof.

Appendix B. Proof of Proposition 1 and the simulation procedure

$$\begin{aligned} C(t_j) &= E_Q\left[\frac{\left[BAL(t_j)-H(t_j)\right]^+}{B(t_j)}\right] \\ &= E_Q\left[\frac{BAL(t_j)\,1_D}{B(t_j)}\right] - E_Q\left[\frac{H(t_j)\,1_D}{B(t_j)}\right], \end{aligned} \quad (B.1)$$

where $D = \left\{BAL(t_j) \geq H(t_j)\right\}$. The first term of Eq. (B.1) can be rewritten as:

$$\begin{aligned} &E_Q\left[E_Q\left[\frac{BAL(t_j)\,1_D}{B(t_j)}\middle|\frac{BAL(t_j)}{B(t_j)}\right]\right] \\ &= \int_0^\infty E_Q\left[x1_D\middle|\frac{BAL(t_j)}{B(t_j)}=x\right]f_{\frac{BAL(t_j)}{B(t_j)}}(x)\,dx, \end{aligned} \quad (B.2)$$

where $f_{BAL(t_j)/B(t_j)}(x)$ is the probability density function of $BAL(t_j)/B(t_j)$. Let M be the total number of simulation paths. Eq. (B.2) can be rewritten as:

$$E_Q\left[\frac{BAL\left(t_j\right)1_D}{B\left(t_j\right)}\right] = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}E_Q\left[x_{mj}1_D\left|\frac{BAL\left(t_j\right)}{B\left(t_j\right)}=x_{mj}\right.\right], \tag{B.3}$$

where x_{mj} is the mth random number drawn from the distribution of $BAL(t_j)/B(t_j)$ under the probability measure Q. Using Ito's lemma, the housing price dynamic under Q takes the form:

$$\frac{H\left(t_j\right)}{B\left(t_j\right)} = H(0)\exp\left\{-\int_0^{t_j}\delta(s)\,ds - \lambda_Q\eta_Q t_j - \frac{1}{2}\sigma_H^2 t_j + \sigma_H W_H^Q\left(t_j\right) + \sum_{n=1}^{N(t_j)}J_n\right\}. \tag{B.4}$$

Therefore,

$$\begin{aligned}
E_Q&\left[x_{mj}1_D\left|\frac{BAL\left(t_j\right)}{B\left(t_j\right)}=x_{mj}\right.\right] \\
&= x_{mj}\Pr_Q\left\{x_{mj}\geq\frac{H\left(t_j\right)}{B\left(t_j\right)}\right\} \\
&= x_{mj}\Pr_Q\left\{x_{mj}\geq H(0)\exp\left\{-\int_0^{t_j}\delta(s)\,ds-\lambda_Q\eta_Q t_j - \frac{1}{2}\sigma_H^2 t_j+\sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{N(t_j)}J_n\right\}\right\} \\
&= x_{mj}\Pr_Q\left\{\ln\frac{x_{mj}}{H(0)}\geq-\int_0^{t_j}\delta(s)\,ds-\lambda_Q\eta_Q t_j - \frac{1}{2}\sigma_H^2 t_j+\sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{N(t_j)}J_n\right\}.
\end{aligned} \tag{B.5}$$

Recall that $\{J_n\}$ is a sequence of independent normal random variables with mean $\theta_J^Q = \theta_J + \varphi\sigma_J^2$ and variance σ_J^2. Let $J_n = \theta_J^Q + \sigma_J Z_n \sim N\left(\theta_J^Q, \sigma_J^2\right)$, where $Z_n \sim N(0,1)$ and Z_i is independent of Z_j $(i\neq j)$. Eq. (B.5) can be rewritten as:

$$\begin{aligned}
x_{mj}&\Pr_Q\left\{\ln\frac{x_{mj}}{H(0)}\geq-\int_0^{t_j}\delta(s)\,ds-\lambda_Q\eta_Q t_j-\frac{1}{2}\sigma_H^2 t_j + \sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{N(t_j)}\left[\theta_J^Q+\sigma_J Z_n\right]\right\} \\
&= x_{mj}\Pr_Q\left\{-\ln\frac{H(0)}{x_{mj}}+\int_0^{t_j}\delta(s)\,ds + \left(\lambda_Q\eta_Q+\frac{1}{2}\sigma_H^2\right)t_j\geq\sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{N(t_j)}\left[\theta_J^Q+\sigma_J Z_n\right]\right\} \\
&= x_{mj}\sum_{k=0}^{\infty}\frac{e^{-\lambda_Q t_j}\left(\lambda_Q t_j\right)^k}{k!} \\
&\quad\times\Pr_Q\left\{-\left(\ln\frac{H(0)}{x_{mj}}+k\theta_J^Q-\int_0^{t_j}\delta(s)\,ds - \left(\lambda_Q\eta_Q+\frac{1}{2}\sigma_H^2\right)t_j\right)\geq\sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{k}\sigma_J Z_n\right\} \\
&= x_{mj}\sum_{k=0}^{\infty}\frac{e^{-\lambda_Q t_j}\left(\lambda_Q t_j\right)^k}{k!}\Phi\left(-d_{jk}\left(x_{mj}\right)\right).
\end{aligned} \tag{B.6}$$

To describe the simulation procedure for generating the random number x_{mj}, we first rewrite $BAL(t_j)/B(t_j)$ as follows:

$$\begin{aligned}
\frac{BAL\left(t_j\right)}{B\left(t_j\right)} &= \frac{\sum_{i=0}^{j-1}a\exp\left\{\int_{t_i}^{t_j}\left[r(s)+\pi_r\right]ds\right\}\pi_{i,j}}{\exp\left\{\int_0^{t_j}r(s)\,ds\right\}} \\
&= \sum_{i=0}^{j-1}ae^{\pi_r\left(t_j-t_i\right)}\pi_{i,j}\exp\left(-\int_0^{t_i}r(s)\,ds\right).
\end{aligned} \tag{B.7}$$

After generating the possible values of $\int_0^{t_i}r(s)\,ds$ for $i = 1,\ldots,j-1$, we can draw the random number x_{mj} from the distribution of $BAL(t_j)/B(t_j)$, according to Eq. (B.7). Because we assume that the spot rate between $t-\Delta t$ and t is fixed at $r(t-\Delta t)$ but may vary from one band to the next, we have $\int_0^{t_i}r(s)\,ds = \sum_{n=1}^{t_i/\Delta t}r((n-1)\Delta t)\Delta t$. Applying Eq. (6), we can generate the spot rate dynamic under the risk-neutral measure Q as follows:

$$\begin{aligned}
r(n\Delta t) &= r((n-1)\Delta t)+\left(\kappa_Q-\theta_Q r((n-1)\Delta t)\right)\Delta t \\
&\quad+\sigma_r\sqrt{r((n-1)\Delta t)\Delta t}\,\varepsilon_n,
\end{aligned} \tag{B.8}$$

where ε_n is a standard normal random variable. Let ε_n^m with $n = 1,2,\ldots,\left(\frac{\omega-x_0+1}{\Delta t}-1\right)$ and $m = 1,\ldots,M$ be the nth standard normal random number in the mth simulation path. Using $\left\{\varepsilon_n^m\,\middle|\,n=1,2,\ldots,\left(\frac{\omega-x_0+1}{\Delta t}-1\right)\right\}$, we can obtain the random values of $\int_0^{t_i}r(s)\,ds$ with Eq. (B.8) and obtain x_{mj} with Eq. (B.7) for $j = 1,\ldots,(\omega-x_0+1)$.

The second term of Eq. (B.1) can be rewritten, according to Eq. (B.4), as:

$$E_Q\left[H(0)\exp\left\{-\int_0^{t_j}\delta(s)\,ds-\lambda_Q\eta_Q t_j-\frac{1}{2}\sigma_H^2 t_j + \sigma_H W_H^Q\left(t_j\right)+\sum_{n=1}^{N(t_j)}J_n\right\}1_D\right]. \tag{B.9}$$

Then, we define

$$\frac{dR}{dQ} = \xi_{t_j}^R = \exp\left\{\sigma_H W_H^Q\left(t_j\right)-\frac{1}{2}\sigma_H^2 t_j\right\}. \tag{B.10}$$

By Girsanov's theorem, $dW_H^R(t) = dW_H^Q(t)-\sigma_H dt$, and Eq. (B.9) becomes

$$\begin{aligned}
&H(0)\,e^{-\int_0^{t_j}\delta(s)ds-\lambda_Q\eta_Q t_j}\sum_{k=0}^{\infty}\frac{e^{-\left(\lambda_Q t_j\right)}\left(\lambda_Q t_j\right)^k}{k!} \\
&\quad\times E_R\left[\exp\left\{\sum_{n=1}^{k}\left(\theta_J^Q+\sigma_J Z_n\right)\right\}1_D\left|N\left(t_j\right)=k\right.\right].
\end{aligned} \tag{B.11}$$

Let $d\tilde{R}/dR = \exp\left\{\sigma_J\sqrt{k}Z - \frac{1}{2}\sigma_J^2 k\right\}$. Again, by Girsanov's theorem, $Z^* = Z - \sigma_J\sqrt{k}$ and $W_{\tilde{R}}(t) = W_R(t)$. Eq. (B.11) thus can be rewritten as:

$$\begin{aligned}
&H(0)\, e^{-\int_0^{t_j}\delta(s)ds-\lambda_Q\eta_Q t_j}\sum_{k=0}^{\infty}\frac{e^{-(\lambda_Q t_j)}\left(\lambda_Q t_j\right)^k}{k!}\\
&\quad\times E_R\left[e^{\sum_{n=1}^{k}\left(\theta_J^Q+\sigma_J Z_n\right)}1_D\left|N\left(t_j\right)=k\right.\right]\\
&= H(0)\, e^{-\int_0^{t_j}\delta(s)ds-\lambda_Q\eta_Q t_j}\sum_{k=0}^{\infty}\frac{e^{-(\lambda_Q t_j)}\left(\lambda_Q t_j\right)^k}{k!}\\
&\quad\times E_R\left[e^{k\theta_J^Q+\sigma_J\sqrt{k}Z}1_D\left|N\left(t_j\right)=k\right.\right]\\
&= H(0)\, e^{-\int_0^{t_j}\delta(s)ds-\lambda_Q\eta_Q t_j}\sum_{k=0}^{\infty}\frac{e^{-(\lambda_Q t_j)}\left(\lambda_Q t_j\right)^k}{k!}\\
&\quad\times E_{\tilde{R}}\left[e^{k\theta_J^Q+\frac{1}{2}\sigma_J^2 k}1_D\left|N\left(t_j\right)=k\right.\right]\\
&= H(0)\, e^{-\int_0^{t_j}\delta(s)ds}\sum_{k=0}^{\infty}\frac{e^{-\lambda_Q(1+\eta_Q)t_j}\left(\lambda_Q t_j\right)^k}{k!}\\
&\quad\times E_{\tilde{R}}\left[\left(1+\eta_Q\right)^k 1_D\left|N\left(t_j\right)=k\right.\right]\\
&= H(0)\, e^{-\int_0^{t_j}\delta(s)ds}\sum_{k=0}^{\infty}\frac{e^{-\lambda_Q(1+\eta_Q)t_j}\left(\lambda_Q\left(1+\eta_Q\right)t_j\right)^k}{k!}\\
&\quad\times\int_0^{\infty}E_{\tilde{R}}\left(1_D\left|N\left(t_j\right)=k,\frac{BAL\left(t_j\right)}{B\left(t_j\right)}=y\right.\right)f_{\frac{BAL(t_j)}{B(t_j)}}(y)\,dy\\
&= \lim_{M\to\infty}\frac{1}{M}H(0)\, e^{-\int_0^{t_j}\delta(s)ds}\sum_{k=0}^{\infty}\left\{P_k\left(\lambda_Q\left(1+\eta_Q\right)t_j\right)\right.\\
&\quad\left.\times\sum_{m=1}^{M}E_{\tilde{R}}\left[1_D\left|N\left(t_j\right)=k\,,\frac{BAL\left(t_j\right)}{B\left(t_j\right)}=y_{mj}\right.\right]\right\}\\
&= \lim_{M\to\infty}\frac{1}{M}H(0)\, e^{-\int_0^{t_j}\delta(s)ds}\sum_{k=0}^{\infty}\left\{P_k\left(\lambda_Q\left(1+\eta_Q\right)t_j\right)\right.\\
&\quad\left.\times\sum_{m=1}^{M}\Pr_{\tilde{R}}\left[y_{mj}\geq\frac{H\left(t_j\right)}{B\left(t_j\right)}\left|N\left(t_j\right)=k\right.\right]\right\},
\end{aligned}\tag{B.12}$$

where y_{mj} is the mth random number drawn from the distribution of $BAL(t_j)/B(t_j)$ under the probability measure $\tilde{R}$. Using Ito's lemma, conditional on $N\left(t_j\right)=k$, the housing price dynamic under $\tilde{R}$ takes the form:

$$\begin{aligned}
\left.\frac{H\left(t_j\right)}{B\left(t_j\right)}\right|_{N(t_j)=k} &= H(0)\, e^{-\int_0^{t_j}\delta(s)ds-\lambda_Q\eta_Q t_j+\frac{1}{2}\sigma_H^2 t_j+\sigma_H W_H^{\tilde{R}}(t_j)}\\
&\quad\times e^{k\theta_J^Q+\sigma_J\sqrt{k}\left(Z^*+\sigma_J\sqrt{k}\right)}.
\end{aligned}\tag{B.13}$$

Using Eq. (B.13), we have

$$\begin{aligned}
&\Pr_{\tilde{R}}\left[y_{mj}\geq\frac{H\left(t_j\right)}{B\left(t_j\right)}\left|N\left(t_j\right)=k\right.\right]\\
&= \Pr_{\tilde{R}}\left[-\frac{\ln\frac{H(0)}{y_{mj}}+\bar{r}_{jk}+\frac{1}{2}\sigma_{kj}^2}{\sigma_{kj}}\geq\frac{\sigma_H W_H^{\tilde{R}}\left(t_j\right)+\sigma_J\sqrt{k}Z^*}{\sigma_{kj}}\right]\\
&= \Phi\left(-h_{jk}\left(y_{mj}\right)\right).
\end{aligned}\tag{B.14}$$

Similarly, for the simulation procedure of y_{mj}, the dynamic of $r(t)$ under $\tilde{R}$ takes the form:

$$\begin{aligned}
&r(n\Delta t)\\
&\quad= r((n-1)\Delta t)+\Big(\kappa_Q+\rho_{Hr}\sigma_r\sigma_H\sqrt{r((n-1)\Delta t)}\\
&\qquad-\theta_Q r((n-1)\Delta t)\Big)\Delta t\\
&\qquad+\sigma_r\sqrt{r((n-1)\Delta t)\Delta t}\,\varepsilon_n^{\tilde{R}}
\end{aligned}\tag{B.15}$$

where $\varepsilon_n^{\tilde{R}}$ is a standard normal random variable under $\tilde{R}$. Similarly, let $\varepsilon_n^{\tilde{m}}$ with $n = 1, 2, \ldots, (\frac{\omega-x_0+1}{\Delta t}-1)$ and $m = 1, \ldots, M$ be the nth standard normal random number in the mth simulation path. Using $\left\{\tilde{\varepsilon}_n^m\left|n=1,2,\ldots,(\frac{\omega-x_0+1}{\Delta t}-1)\right.\right\}$, we can obtain the random values of $\int_0^{t_i}r(s)\,ds$ using Eq. (B.15) and can determine y_{mj} on the basis of Eq. (B.7) for $j = 1, \ldots, \omega - x_0 + 1$.

Appendix C. Proof of Eq. (37)

We assume that the spot rate between $t-\Delta t$ and t is fixed at $r(t-\Delta t)$. Under the ARMA(s, m)–GARCH(p, q) process in Eqs. (33) and (34), and following the procedure from Appendix A, we have

$$\begin{aligned}
H(t-\Delta t) &= e^{-r(t-\Delta t)\Delta t}E_{Q_\varphi}\left(H(t)\right|F_{t-\Delta t})\\
&= e^{-r(t-\Delta t)\Delta t}E_P\left(\frac{\xi_t}{\xi_{t-\Delta t}}H(t)\right|\left.F_{t-\Delta t}\right)\\
&= H(t-\Delta t)\,e^{-r(t-\Delta t)\Delta t}\\
&\quad\times\frac{E_P\left(\exp\left((\varphi(t)+1)Y(t)\right)\right|F_{t-\Delta t})}{E_P\left(\exp\left(\varphi(t)Y(t)\right)\right|F_{t-\Delta t})}.
\end{aligned}\tag{C.1}$$

Or equivalently,

$$e^{r(t-\Delta t)\Delta t} = \frac{E_P\left(\exp\left((\varphi(t)+1)Y(t)\right)\right|F_{t-\Delta t})}{E_P\left(\exp\left(\varphi(t)Y(t)\right)\right|F_{t-\Delta t})}.\tag{C.2}$$

Because $Y(t)$ is normally distributed with mean $\mu_H(t)$ and variance $h(t)$, given the information $F_{t-\Delta t}$, we obtain

$$\begin{aligned}
e^{r(t-\Delta t)\Delta t} &= \frac{\exp\left((\varphi(t)+1)\mu_H(t)+\frac{1}{2}(\varphi(t)+1)^2h(t)\right)}{\exp\left(\varphi(t)\mu_H(t)+\frac{1}{2}\varphi(t)^2h(t)\right)}\\
&= \exp\left(\mu_H(t)+\left(\varphi(t)+\frac{1}{2}\right)h(t)\right).
\end{aligned}\tag{C.3}$$

Equivalently, $\mu_H(t) = r(t-\Delta t)\Delta t - \left(\varphi(t)+\frac{1}{2}\right)h(t)$. Similarly, the characteristic function of $\varepsilon_H(t)$ under martingale measure Q_φ is of the form:

$$\begin{aligned}
&E_{Q_\varphi}\left(\exp\left(i\omega\varepsilon_H(t)\right)\right|F_{t-\Delta t})\\
&\quad= E_P\left(\frac{\xi_t}{\xi_{t-\Delta t}}e^{i\omega\varepsilon_H(t)}\right|\left.F_{t-\Delta t}\right)\\
&\quad= \frac{E_P\left(e^{\varphi(t)Y(t)}e^{i\omega\varepsilon_H(t)}\right|F_{t-\Delta t})}{E_P\left(\exp\left(\varphi(t)Y(t)\right)\right|F_{t-\Delta t})}\\
&\quad= \frac{\exp\left(\varphi(t)\mu_H(t)\right)E_P\left(e^{(\varphi_t+i\omega)\varepsilon_H(t)}\right|F_{t-\Delta t})}{\exp\left(\varphi(t)\mu_H(t)+\frac{1}{2}\varphi(t)^2h(t)\right)}\\
&\quad= \frac{\exp\left(\frac{1}{2}(\varphi(t)+i\omega)^2h(t)\right)}{\exp\left(\frac{1}{2}\varphi(t)^2h(t)\right)}\\
&\quad= \exp\left(i\omega\varphi(t)h(t)-\frac{1}{2}\omega^2h(t)\right).
\end{aligned}\tag{C.4}$$

Consequently, $\varepsilon_H(t)$ under the martingale measure Q_φ becomes normally distributed, with mean $\varphi(t)h(t)$ and variance $h(t)$, given the information $F_{t-\Delta t}$. That is, given the information $F_{t-\Delta t}$, $\varepsilon_H^Q(t) = \varepsilon_H(t) - \varphi(t)h(t)$ follows a normal distribution with mean 0 and variance $h(t)$ under the martingale measure Q_φ. As a result, Eq. (33) can be rewritten as:

$$\begin{aligned} Y(t) &= \ln\left(\frac{H(t)}{H(t-\Delta t)}\right) = \mu_H(t) + \varepsilon_H(t) \\ &= r(t-\Delta t)\,\Delta t - \left(\varphi(t) + \frac{1}{2}\right)h(t) \\ &\quad + \left(\varepsilon_H^Q(t) + \varphi(t)h(t)\right) \\ &= r(t-\Delta t)\,\Delta t - \frac{1}{2}h(t) + \varepsilon_H^Q(t). \end{aligned} \tag{C.5}$$

This completes the proof of Appendix C.

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