

VALUATION OF RATCHET EQUITY-INDEXED ANNUITIES

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^b The author is grateful to the National Science Council of Taiwan for its financial support (project number NSC 96-2918-I-004-006) and to Santa Clara University (SCU) for the kind supports during the visit of the author.

ABSTRACT

Ratchet equity-indexed annuities (EIAs) are the most popular EIAs. Most ratchet EIAs credit returns to policyholders annually based on the higher of the calculated index return during the year and the minimum guaranteed rate. This annual reset feature allows the credited return to be locked in and thus the value of the policy will never decrease. In this paper we derive the pricing formulas under the standard Black-Scholes framework for more ratchet EIA products than the literature. Via these pricing formulas and numerical demonstration, we are able to analyze the impacts of various contract features on the policy value. These results can be helpful guides to ratchet EIA design and valuation.

JEL Classification: G22; G13

Keywords: equity-indexed annuities; option pricing

1. Introduction

Equity-indexed annuities (EIAs) are regarded as one of the most innovative products brought into the insurance market in years (Jaimungal, 2004). An EIA is a hybrid between a variable and a fixed annuity that allows the policyholder to participate in the potential appreciation of the stock market while eliminating the downside risk by a minimum return guarantee. The guarantee is usually high enough to meet the nonforfeiture laws so that the EIA does not need to be registered with the Securities Exchange Commission (SEC) and may enjoy tax deferring. EIAs thus have gained popularity. The sales increased from \$1.5 billion in 1996 to more than \$20 billion in 2007.¹

The three major product designs for EIAs, in the order of decreasing sales volumes, are ratchet, point-to-point, and lookback (including the high-water-mark and Asian-end designs).

The return of the point-to-point EIA is determined by the realized return of the linked index between two time points. Ratchet EIAs are more favorable because returns are credited periodically and the account value will never decrease once the return is credited. Another popular design is the high-water-mark that earns the highest return on the index attained during the life of the policy.

Pricing EIAs is a challenging problem due to the complex payoff structure and attracts

¹ Please see online reports on Advantage Compendium (<http://www.indexannuity.org>)

research attention. Tiong (2000) derive closed-form solutions for the values of the three major types of EIAs. Gerber and Shiu (2003) present closed-form formulas for pricing lookback options and dynamic guarantees. Lee (2003) proposes four product designs of EIAs to increase participation rates and derives associated pricing formulas. Extending the constant risk-free rate assumption in the previous papers to the short rate model of Vasicek (1977), Lin and Tan (2003) determine the participating rates numerically for the three major types of EIAs. Kijima and Wong (2007) adopt the extended Vasicek model and derive closed-form price formulas for ratchet EIAs. Jaimungal (2004) assumes that the underlying index follows a geometric Variance-Gamma process and develops closed-form expressions for prices of point-to-point and ratchet EIAs.

This paper focuses on the ratchet EIAs with annual reset, the most popular design of EIAs. We derive closed-form solutions for both compound and simple ratchet products with contract features of geometric return averaging and return cap. Under the ratchet contract design, the participation in the equity index is applied annually. The simple ratchet EIA adds the returns in each year together to give the final payout while the returns in the compound ratchet EIA are compounded. To reduce the costs of EIAs, the insurer may place a fixed upper limit, or cap, on the periodic return. It may also employ an averaging scheme in calculating the index return to reduce the volatility of credited returns and the costs of the guarantees. Two

commonly seen averaging schemes are geometric averaging and arithmetic averaging. The ratchet EIAs considered in this paper have a return cap. We analyze both compound and simple versions with and without geometric averaging, which enables us to investigate the impacts of these contract features on the value of the policy.

Our paper provides closed-form solutions for more ratchet EIA products than the literature in the standard Black-Scholes framework.² Tiong (2000) derive the pricing formulas only for the compound version without return averaging under the same framework. Hardy (2004) derives the valuation formula only for the compound version of the ratchet EIAs that have the return cap and no return averaging in the B-S framework. Lin and Tan (2003) determine the participating rates for the compound version with arithmetic return averaging under stochastic interest rates. The ratchet EIA considered by Jaimungal (2004) under a geometric Variance-Gamma index process is the compound version without return averaging. Under a stochastic interest rate environment, Kijima and Wong (2007) derive closed-form solutions for both compound and simple versions of ratchet EIAs with geometric averaging. They however have to resort to simulation for valuation when return cap is added onto the contract feature. Since we derive closed-form formulas for more ratchet products and analyze the impact of contract features on product valuation more comprehensively than the literature, this paper can

² Under the Black-Scholes framework, the linked index follows the geometric Brownian motion while the risk-free rate is assumed to be constant.

be a helpful guide for actuaries involved in ratchet EIAs.

The paper is organized as follows. We describe the ratchet EIA contracts under considerations in section 2. We briefly review the risk neutral valuation concepts in section 3 and proceed to provide the closed-form solutions for the EIAs contracts in section 4. Section 5 contains the numerical analyses about the impact of several contract features on product values. Conclusions and remarks are in section 6.

2. Product Specifications

The general formula used in calculating the return to be credited to the policy each year is:

$$\tilde{R}_t = 1 + \min(\max(\alpha(R_t - 1), f), c), \quad (1)$$

where R_t represents the calculated annual return on the index, f is the minimum guaranteed return rate, c is the cap rate, and α is the participation rate in the linked-index. The participation rate is usually less than 100%, which is reasonable in the sense that the investor sacrifices some of the upside potential for the downside protection of the minimum guarantee. The cap or ceiling rate c is the maximum rate that can be credited each year. Placing a cap on the credited return is a direct way to reduce the cost of product.

For the ratchet EIAs without return averaging, R_t is set as:

$$R_t = \frac{S(t)}{S(t-1)}, \quad t = 1, 2, \dots, T, \quad (2)$$

in which T denotes the maturity of a ratchet EIA contract and $S(t)$ denotes the linked-index at time $t \leq T$. We then can define the compound and simple versions of plain ratchet EIAs as follows.

Definition 1 The payoff at maturity T of a plain compound ratchet EIA based on an initial premium of \$1 at time 0 is:

$$R_{cr} = \prod_{t=1}^T \tilde{R}_t. \quad (3)$$

Definition 2 The payoff at maturity T of a plain simple ratchet EIA based on an initial premium of \$1 at time 0 is:

$$R_{sr} = 1 + \sum_{t=1}^T (\tilde{R}_t - 1) = 1 - T + \sum_{t=1}^T \tilde{R}_t. \quad (4)$$

The product design of participation and guarantee introduces the call-option feature to (ratchet) EIAs. To reduce the cost of this embedded option, the insurer may employ an averaging scheme to reduce the volatility of credited returns. We analyze two types of geometric averaging schemes in this paper. In the first type of geometric averaging (GA1), the annual return on the index is calculated as follows:

$$R_t^{(m)} = \left[\prod_{i=0}^{m-1} \frac{S(t-1 + \frac{i+1}{m})}{S(t-1 + \frac{i}{m})} \right]^{\frac{1}{m}}, \quad (5)$$

which represents the geometric average of m index returns over the t^{th} year. Define

$$\tilde{R}_t^{(m)} = 1 + \min(\max(\alpha(R_t^{(m)} - 1), f), c). \quad (6)$$

Then we have two ratchet EIAs with geometric return averaging.

Definition 3 The payoff at maturity T of a compound ratchet EIA with the first type of geometric averaging based on an initial premium of \$1 at time 0 is:

$$R_{cr}^{(m)} = \prod_{t=1}^T \tilde{R}_t^{(m)}. \quad (7)$$

Definition 4 The payoff at maturity T of a simple ratchet EIA with the first type of geometric averaging based on an initial premium of \$1 at time 0 is:

$$R_{sr}^{(m)} = 1 - T + \sum_{t=1}^T \tilde{R}_t^{(m)}. \quad (8)$$

We call the above two ratchet EIAs as GA1 compound ratchet and GA1 simple ratchet respectively in the following.

In the second type of geometric averaging (GA2), the annual return on the index is calculated as follows:

$$R_t^\Delta = \left[\prod_{i=1}^n \frac{S\left(t-1+\frac{i}{n}\right)}{S(t-1)} \right]^{\frac{1}{n}}, \quad (9)$$

where n denotes the number of returns to be averaged over the t^{th} year. Let

$$\tilde{R}_t^\Delta = 1 + \min(\max(\alpha(R_t^\Delta - 1), f), c). \quad (10)$$

Then we have two more ratchet EIAs with geometric return averaging.

Definition 5 The payoff at maturity T of a compound ratchet EIA with the second type of geometric averaging based on an initial premium of \$1 at time 0 is:

$$R_{cr}^{\Delta} = \prod_{t=1}^T \tilde{R}_t^{\Delta}. \quad (11)$$

Definition 6 The payoff at maturity T of a simple ratchet EIA with the second type of geometric averaging based on an initial premium of \$1 at time 0 is:

$$R_{sr}^{\Delta} = 1 + \sum_{t=1}^T (\tilde{R}_t^{\Delta} - 1) = 1 - T + \sum_{t=1}^T \tilde{R}_t^{\Delta}. \quad (12)$$

The above two ratchet EIAs will be called as GA2 compound ratchet and GA2 simple ratchet respectively in the following.

3. Valuation of Ratchet EIAs via Risk Neutral Valuation Principle

We follow Hardy (2004), Lee (2003), Gerber and Shiu (2003), and Tiong (2000) in adopting the Black- Scholes assumptions for the linked index and interest rate. More specifically, we assume that the linked index $S(t)$ follows the geometric Brownian motion and the interest rate r is constant. Therefore, under the risk-neutral measure or martingale measure,

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)dz(t), \\ dB(t) &= rB(t)dt, \end{aligned}$$

where $z(t)$ is a standard Brownian motion, σ is the volatility of the linked index (assumed to be constant), and $B(t)$ denotes the money market account.

Assume that the market is complete. Then, the prices of the EIA contracts considered in section 2 can be represented as expectations according to the risk neutral valuation principle (see, for example, Harrison and Kreps 1979 and Harrison and Pliska 1981). More specifically, the price of a plain compound ratchet EIA contract can be represented as

$$V_{cr} = E\left[e^{-rT} R_{cr}\right], \quad (13)$$

and the price of a plain simple ratchet EIA contract as

$$V_{sr} = E\left[e^{-rT} R_{sr}\right]. \quad (14)$$

Similarly, the prices of GA1 compound and simple EIAs can be represented as

$$V_{cr}^{(m)} = E\left[e^{-rT} R_{cr}^{(m)}\right] \quad (15)$$

and

$$V_{sr}^{(m)} = E\left[e^{-rT} R_{sr}^{(m)}\right] \quad (16)$$

respectively. For GA2 compound and simple EIAs, the respective prices are

$$V_{cr}^{\Delta} = E\left[e^{-rT} R_{cr}^{\Delta}\right] \quad (17)$$

and

$$V_{sr}^{\Delta} = E\left[e^{-rT} R_{sr}^{\Delta}\right]. \quad (18)$$

4. Pricing formulas of Ratchet EIAs

4.1 Pricing Formulas for Plain Compound and Simple Ratchet EIAs

Suppose that the linked index pays a continuous dividend yield at a constant rate d per year. It is well known that, under the risk neutral measure (pricing measure), $\log(R_t)$ are independent normal random variables with parameters $r-d-\sigma^2/2$ and σ^2 (e.g., Hull, 2006). We can rewrite equation (1) as

$$\tilde{R}_t = (1-\alpha) + \alpha \min(\max(f_\alpha, R_t), c_\alpha),$$

in which $f_\alpha = 1 + f/\alpha$ and $c_\alpha = 1 + c/\alpha$. Let

$$X_t = \min(\max(f_\alpha, R_t), c_\alpha).$$

Then it is easy to see that X_t are independent censored lognormal random variables with censored values f_α and c_α .

Combining equations (3) and (13), we obtain

$$\begin{aligned} V_{cr} &= E \left[e^{-rT} \prod_{t=1}^T (1 - \alpha + \alpha X_t) \right] \\ &= e^{-rT} [1 - \alpha + \alpha E(X_t)]^T. \end{aligned} \quad (19)$$

Similarly,

$$\begin{aligned} V_{sr} &= E \left[e^{-rT} \left(1 - T + \sum_{t=1}^T \tilde{R}_t \right) \right] \\ &= e^{-rT} [(1 - \alpha T) + \alpha T E(X_t)]. \end{aligned} \quad (20)$$

Now we need only the explicit formula of $E(X_t)$ to derive the pricing formulas for V_{cr} and V_{sr} .

To compute $E(X_t)$, we first write

$$E(X_t) = f_\alpha P(R_t \leq f_\alpha) + E[R_t; f_\alpha \leq R_t \leq c_\alpha] + c_\alpha P(R_t \geq c_\alpha). \quad (21)$$

Representing R_t as $e^{r-d-\sigma^2/2+\sigma N(0,1)}$ and letting

$$d_1 = \frac{\log f_\alpha - r + d}{\sigma} + \frac{\sigma}{2}$$

and

$$d_2 = \frac{\log c_\alpha - r + d}{\sigma} + \frac{\sigma}{2},$$

we obtain

$$\begin{aligned} P(R_t \leq f_\alpha) &= P(N(0,1) \leq d_1) = \Phi(d_1), \\ P(R_t \geq c_\alpha) &= P(N(0,1) \geq d_2) = \Phi(-d_2), \end{aligned}$$

and

$$\begin{aligned} E[R_t; f_\alpha \leq R_t \leq c_\alpha] &= \int_{d_1}^{d_2} e^{r-d-\sigma^2/2+\sigma z} \phi(z) dz \\ &= e^{r-d} [\Phi(d_2 - \sigma) - \Phi(d_1 - \sigma)], \end{aligned} \quad (22)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density function and the cumulative distribution function of the standard normal random variable respectively. We thus get the explicit formula for $E(X_t)$ as:

$$EX_t = f_\alpha \Phi(d_1) + c_\alpha \Phi(-d_2) + e^{r-d} [\Phi(d_2 - \sigma) - \Phi(d_1 - \sigma)]. \quad (23)$$

The following two propositions become straightforward with equations (19), (20), and (23) in hands. Note that the pricing formula for the plain compound ratchet EIA have been obtained in Hardy (2004) as well.

Proposition 1 The pricing formula for the plain compound ratchet EIA is:

$$V_{cr} = e^{-rT} \left\{ 1 - \alpha + \alpha \left(f_\alpha \Phi(d_1) + c_\alpha \Phi(-d_2) + e^{r-d} [\Phi(d_2 - \sigma) - \Phi(d_1 - \sigma)] \right) \right\}^T, \quad (24)$$

in which

$$d_1 = \frac{\log f_\alpha - r + d}{\sigma} + \frac{\sigma}{2},$$

$$d_2 = \frac{\log c_\alpha - r + d}{\sigma} + \frac{\sigma}{2},$$

and $\Phi(\cdot)$ is the cumulative normal probability distribution.

Proposition 2 The pricing formula for the plain simple ratchet EIA is:

$$V_{sr} = e^{-rT} \left\{ 1 - \alpha T + \alpha T \left(f_\alpha \Phi(d_1) + c_\alpha \Phi(-d_2) + e^{r-d} [\Phi(d_2 - \sigma) - \Phi(d_1 - \sigma)] \right) \right\}. \quad (25)$$

with d_1 , d_2 , and $\Phi(\cdot)$ as defined in the above.

4.2 Pricing Formulas for GA1 Compound and Simple Ratchet EIAs

Under risk neutral measure, $\log(R_t^{(m)})$ are independent normal random variables with

mean $\mu_m \equiv \frac{1}{m} (r - d - \frac{\sigma^2}{2})$ and variance $\sigma_m^2 \equiv \frac{\sigma^2}{m^2}$.³ Let

$$X_t^{(m)} = \min(\max(f_\alpha, R_t^{(m)}), c_\alpha). \quad (26)$$

Combining equations (3) and (15), we obtain

³ Note that the case $m=1$ reduces to the case of no geometric index averaging and has already been discussed in the previous subsection.

$$\begin{aligned}
V_{cr}^{(m)} &= E \left[e^{-rT} \prod_{t=1}^T (1 - \alpha + \alpha X_t^{(m)}) \right] \\
&= e^{-rT} [1 - \alpha + \alpha E(X_t^{(m)})]^T,
\end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned}
V_{sr}^{(m)} &= E \left[e^{-rT} \left(1 - T + \sum_{t=1}^T \tilde{R}_t^{(m)} \right) \right] \\
&= e^{-rT} [(1 - \alpha T) + \alpha T E(X_t^{(m)})],
\end{aligned} \tag{28}$$

We can then obtain the pricing formulas as soon as we get the explicit formula for $E(X_t^{(m)})$.

Proposition 3 The pricing formula for the GA1 compound ratchet EIA is

$$V_{cr}^{(m)} = e^{-rT} \left\{ 1 - \alpha + \alpha \left[f_\alpha \Phi(d_1^{(m)}) + e^{\mu_m + \frac{1}{2}\sigma_m^2} [\Phi(d_2^{(m)} - \sigma_m) - \Phi(d_1^{(m)} - \sigma_m)] + c_\alpha \Phi(-d_2^{(m)}) \right] \right\}^T \tag{29}$$

where

$$\begin{aligned}
d_1^{(m)} &= \frac{\log f_\alpha - \mu_m}{\sigma_m} \\
d_2^{(m)} &= \frac{\log c_\alpha - \mu_m}{\sigma_m} \\
\mu_m &\equiv \frac{1}{m} \left(r - d - \frac{\sigma^2}{2} \right) \\
\sigma_m^2 &\equiv \frac{\sigma^2}{m^2}
\end{aligned}$$

Proof:

$$\begin{aligned}
EX_t^{(m)} &= f_\alpha P(R_t^{(m)} \leq f_\alpha) + E[R_t^{(m)} : f_\alpha \leq R_t^{(m)} \leq c_\alpha] + c_\alpha P(R_t^{(m)} \geq c_\alpha) \\
&= f_\alpha \Phi(d_1^{(m)}) + e^{\mu_m + \frac{1}{2}\sigma_m^2} [\Phi(d_2^{(m)} - \sigma_m) - \Phi(d_1^{(m)} - \sigma_m)] + c_\alpha \Phi(-d_2^{(m)})
\end{aligned} \tag{30}$$

Substitute $EX_t^{(m)}$ into (27) and the result follows.

Proposition 4 The pricing formula for the GA1 simple ratchet EIA is

$$V_{sr}^{(m)} = e^{-rT} \left\{ 1 - \alpha T + \alpha T \left[f_\alpha \Phi(d_1^{(m)}) + e^{\mu_m + \frac{1}{2}\sigma_m^2} [\Phi(d_2^{(m)} - \sigma_m) - \Phi(d_1^{(m)} - \sigma_m)] + c_\alpha \Phi(-d_2^{(m)}) \right] \right\} \quad (31)$$

where $d_1^{(m)}$, $d_2^{(m)}$, μ_m , and σ_m^2 are defined as in Proposition 3.

Proof: The proofs follow the same lines as the proofs of Proposition 3.

4.3 Pricing Formulas for GA2 Compound and Simple Ratchet EIAs

Equation (9) can be rewritten as

$$\begin{aligned} R_t^\Delta &= \left[\frac{S\left(t - \frac{n-1}{n}\right)}{S(t-1)} \cdot \frac{S\left(t - \frac{n-2}{n}\right)}{S(t-1)} \cdots \frac{S\left(t - \frac{1}{n}\right)}{S(t-1)} \frac{S(t)}{S(t-1)} \right]^{\frac{1}{n}} \\ &= [Y_1 \cdot Y_2 \cdots Y_{n-1} \cdot Y_n]^{\frac{1}{n}} \\ &= \left[\prod_{k=1}^n Y_k \right]^{\frac{1}{n}} \end{aligned} \quad (32)$$

Each Y_k follows lognormal distribution with parameters $\mu_y = \frac{k}{n}(r - d - \frac{\sigma^2}{2})$ and $\sigma_y^2 = \frac{k}{n}\sigma^2$.

However, Y_k are not independent and difficult to analyze. Let

$$Z_1 \equiv \log(Y_1), \quad Z_2 \equiv \log(Y_2) - \log(Y_1), \dots, \quad Z_n \equiv \log(Y_n) - \log(Y_{n-1}).$$

It is easy to show that Z_i are Brownian motion increments and thus are independent and normally

distributed with mean $\mu_Z = \frac{1}{n}(r-d - \frac{\sigma^2}{2})$ and variance $\sigma_Z^2 = \frac{\sigma^2}{n}$. Taking log on both sides of equation (32), we get

$$\begin{aligned}\log R_t^\Delta &= \frac{1}{n} \sum_{k=1}^n \log Y_k \\ &= \frac{1}{n} [Z_1 + (Z_1 + Z_2) + \dots + (Z_1 + \dots + Z_n)]\end{aligned}\quad (33)$$

Then it follows that $\log(R_t^\Delta)$ are independent normal random variables with mean

$$\mu_\Delta = \frac{n+1}{2n}(r-d - \frac{\sigma^2}{2}) \text{ and variance } \sigma_\Delta^2 = \frac{(n+1)(2n+1)}{6n^2}\sigma^2.$$

Proposition 5 The pricing formula for the GA2 compound ratchet EIA is

$$V_{cr}^\Delta = e^{-rT} \left\{ 1 - \alpha + \alpha \left[f_\alpha \Phi(d_1^\Delta) + e^{\mu_\Delta + \frac{1}{2}\sigma_\Delta^2} [\Phi(d_2^\Delta - \sigma_\Delta) - \Phi(d_1^\Delta - \sigma_\Delta)] + c_\alpha \Phi(-d_2^\Delta) \right] \right\}^T \quad (34)$$

where

$$\begin{aligned}d_1^\Delta &= \frac{\log f_\alpha - \mu_\Delta}{\sigma_\Delta} \\ d_2^\Delta &= \frac{\log c_\alpha - \mu_\Delta}{\sigma_\Delta} \\ \mu_\Delta &= \frac{n+1}{2n}(r-d - \frac{\sigma^2}{2}) \\ \sigma_\Delta^2 &= \frac{(n+1)(2n+1)}{6n^2}\sigma^2\end{aligned}$$

Proof: Define $X_t^\Delta = \min(\max(f_\alpha, R_t^\Delta), c_\alpha)$.

By equation (17), the pricing formula for V_{cr}^Δ is

$$V_{cr}^\Delta = e^{-rT} [1 - \alpha + \alpha E(X_t^\Delta)]^T,$$

and

$$\begin{aligned} EX_t^\Delta &= f_\alpha P(R_t^\Delta \leq f_\alpha) + E[R_t^\Delta : f_\alpha \leq R_t^\Delta \leq c_\alpha] + c_\alpha P(R_t^\Delta \geq c_\alpha) \\ &= f_\alpha \Phi(d_1^\Delta) + e^{\mu_\Delta + \frac{1}{2}\sigma_\Delta^2} [\Phi(d_2^\Delta - \sigma_\Delta) - \Phi(d_1^\Delta - \sigma_\Delta)] + c_\alpha \Phi(-d_2^\Delta). \end{aligned}$$

The result then follows by straightforward calculations.

Proposition 6 The pricing formula for the GA2 simple ratchet EIA is

$$V_{sr}^\Delta = e^{-rT} \left\{ 1 - \alpha T + \alpha T \left[f_\alpha \Phi(d_1^\Delta) + e^{\mu_\Delta + \frac{1}{2}\sigma_\Delta^2} [\Phi(d_2^\Delta - \sigma_\Delta) - \Phi(d_1^\Delta - \sigma_\Delta)] + c_\alpha \Phi(-d_2^\Delta) \right] \right\} \quad (35)$$

where d_1^Δ , d_2^Δ , μ_Δ , and σ_Δ^2 are defined as in Proposition 5.⁴

Proof: By equation (18), the pricing formula for V_{sr}^Δ , is

$$V_{sr}^\Delta = e^{-rT} [1 - \alpha T + \alpha TE(X_t^\Delta)].$$

The proofs then follow the same lines as the proofs of Proposition 5.

5. Numerical Examples

Although ratchet options are appealing to the investor, they are also imposing potential high guarantee cost on the insurance companies. The first set of numerical examples is to demonstrate various contract designs and their cost to the insurance companies. We begin with a description of the common and typical parameter values of this set of examples: contract

⁴ Note that V_{cr}^Δ and V_{sr}^Δ reduce to V_{cr} and V_{sr} respectively when $n = 1$.

maturity $T = 7$, initial investment $P = 100$, floor rate $f = 0$ (i.e. premium guarantee), the volatility of the linked-index $\sigma = 25\%$, interest rate $r = 6\%$, dividend rate of the linked-index $d = 2\%$.

Then we show the prices of various contract designs under different combinations of participation rate α and ceiling rate c in Tables 1 ~ 6. From these results, we can see the impacts of participation rate α , ceiling rate c , and the methods of averaging returns on the cost of the contracts. In particular, if we select the guarantee budget for the contract, then we have various possible combinations of participation rate α , ceiling rate c , and the methods of averaging returns to choose. Moreover, among these choices of the same cost, we can also discover which contract design is easier to hedge its risk.

Table 1: Prices of plain compound ratchet EIA contracts.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	85.94	93.01	98.15	104.11	106.65
0.8	87.60	96.60	104.04	114.57	120.59
1.0	88.66	99.00	108.22	122.89	132.90
1.2	89.39	100.71	111.29	129.51	143.47
1.4	89.92	101.99	113.63	134.84	152.47

Table 2: Prices of plain simple ratchet EIA contracts.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	83.69	89.11	92.85	96.96	98.66
0.8	85.00	91.74	96.92	103.73	107.38
1.0	85.82	93.45	99.69	108.74	114.40
1.2	86.38	94.64	101.67	112.52	119.99
1.4	86.79	95.52	103.14	115.45	124.48

Table 3: Prices of GA1 compound ratchet EIA contracts with $m = 2$.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	80.10	82.09	82.74	82.97	82.98
0.8	82.68	86.45	88.28	89.37	89.52
1.0	84.47	89.85	93.08	95.77	96.40
1.2	85.77	92.49	97.12	101.86	103.43
1.4	86.74	94.58	100.48	107.48	110.42

Table 4: Prices of GA1 simple ratchet EIA contracts with $m = 2$.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	78.91	80.57	81.11	81.29	81.30
0.8	81.06	84.09	85.52	86.37	86.48
1.0	82.51	86.73	89.17	91.14	91.59
1.2	83.55	88.73	92.11	95.43	96.51
1.4	84.32	90.27	94.48	99.21	101.11

Table 5: Prices of GA2 compound ratchet EIA contracts with $n = 4$.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	82.96	87.07	89.23	90.73	91.01
0.8	85.15	91.31	95.40	99.42	100.71
1.0	86.59	94.35	100.21	107.29	110.40
1.2	87.60	96.59	103.96	114.13	119.63
1.4	88.35	98.30	106.93	120.01	128.17

Table 6: Prices of GA2 simple ratchet EIA contracts with $n = 4$.

$\alpha \backslash c$	0.1	0.15	0.2	0.3	0.4
0.6	81.28	84.58	86.26	87.41	87.61
0.8	83.06	87.85	90.87	93.74	94.64
1.0	84.20	90.10	94.29	99.08	101.10
1.2	85.00	91.73	96.86	103.46	106.81
1.4	85.58	92.95	98.84	107.04	111.77

The second set of numerical examples is to demonstrate the sensitivities of the pricing formulas to the parameters σ and r . In these examples, we set participation rate $\alpha = 1$ and ceiling rate $c = 20\%$. Tables 7~12 show the prices of ratchet EIA contracts under various interest rate and linked-index's volatility levels. From these results, we see the contract values are not very sensitive to interest rate and linked-index's volatility risks; and the pricing formulas can be effective tools for hedging interest rate and linked-index's volatility risks.

Table 7: Prices of plain compound ratchet EIA contracts.

$r \backslash \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	108.55	112.05	113.86	114.69	114.91
5.5%	106.38	109.46	111.00	111.64	111.73
6.0%	104.28	106.95	108.22	108.68	108.64
6.5%	102.23	104.50	105.51	105.79	105.64
7.0%	100.24	102.12	102.87	102.99	102.72

Table 8: Prices of plain simple ratchet EIA contracts.

$r \setminus \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	101.87	104.26	105.47	106.02	106.16
5.5%	99.44	101.52	102.54	102.96	103.02
6.0%	97.08	98.86	99.69	99.98	99.96
6.5%	94.77	96.26	96.91	97.10	97.00
7.0%	92.51	93.74	94.22	94.30	94.13

Table 9: Prices of GA1 compound ratchet EIA contracts with $m = 2$.

$r \setminus \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	90.44	94.88	98.29	100.75	102.45
5.5%	88.19	92.42	95.65	97.95	99.53
6.0%	86.01	90.04	93.08	95.24	96.69
6.5%	83.91	87.73	90.59	92.60	93.94
7.0%	81.87	85.49	88.18	90.05	91.28

Table 10: Prices of GA1 simple ratchet EIA contracts with $m = 2$.

$r \setminus \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	88.37	91.88	94.48	96.31	97.55
5.5%	86.02	89.34	91.79	93.49	94.63
6.0%	83.75	86.88	89.17	90.75	91.80
6.5%	81.54	84.49	86.63	88.10	89.06
7.0%	79.40	82.17	84.17	85.52	86.40

Table 11: Prices of GA2 compound ratchet EIA contracts with $n = 4$.

$r \setminus \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	97.75	102.56	105.76	107.81	109.06
5.5%	95.46	99.98	102.94	104.82	105.95
6.0%	93.24	97.47	100.21	101.92	102.92
6.5%	91.09	95.03	97.55	99.10	99.99
7.0%	89.00	92.66	94.97	96.37	97.14

Table 12: Prices of GA2 simple ratchet EIA contracts with $n = 4$.

$r \setminus \sigma$	0.15	0.2	0.25	0.3	0.35
5.0%	94.08	97.64	99.92	101.36	102.23
5.5%	91.65	94.96	97.07	98.37	99.15
6.0%	89.29	92.36	94.29	95.47	96.16
6.5%	86.99	89.83	91.60	92.66	93.26
7.0%	84.76	87.38	88.98	89.93	90.45

6. Conclusions and Suggestions

In this paper, we derive the pricing formulas for various ratchet EIA contracts under the Black-Scholes assumptions. These formulas can be a useful tool for designing ratchet EIA contract in terms of controlling guarantee cost and various market variable risks, such as interest rate level and linked-index's volatility.

Although the formulas are derived under simple assumptions, they could have good hedging capacity for more general assumptions for linked-index and interest rate. For example, delta hedge under simple Black-Scholes assumptions are proved to be a powerful method of hedging

equity-linked insurance product risk; see chapter 7 and 8 of Hardy (2003) for more detail.

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