# The evolution of traveling waves in a simple isothermal chemical system modeling quadratic autocatalysis with strong decay 

Sheng-Chen Fu ${ }^{\text {a,* }}$, Je-Chiang Tsai ${ }^{\text {b }}$<br>${ }^{\text {a }}$ Department of Mathematical Sciences, National Chengchi University, 64, S-2 Zhi-nan Road, Taipei 116, Taiwan<br>${ }^{\text {b }}$ Department of Mathematics, National Chung Cheng University, 168, University Road, Min-Hsiung, Chia-Yi 621, Taiwan

Received 11 March 2013; revised 2 October 2013
Available online 28 February 2014


#### Abstract

In this paper, we study a reaction-diffusion system for an isothermal chemical reaction scheme governed by a quadratic autocatalytic step $A+B \rightarrow 2 B$ and a decay step $B \rightarrow C$, where $A, B$, and $C$ are the reactant, the autocatalyst, and the inner product, respectively. Previous numerical studies and experimental evidences demonstrate that if the autocatalyst is introduced locally into this autocatalytic reaction system where the reactant $A$ initially distributes uniformly in the whole space, then a pair of waves will be generated and will propagate outwards from the initial reaction zone. One crucial feature of this phenomenon is that for the strong decay case, the formation of waves is independent of the amount of the autocatalyst $B$ introduced into the system. It is this phenomenon of KPP-type which we would like to address in this paper. To study the propagation of reactant and autocatalyst analytically, we first use the tail behavior of waves to construct a pair of generalized super-/sub-solutions for the approximate system of the autocatalytic reaction system. Note that the autocatalytic reaction system does not enjoy comparison principle. Together with a family of truncated problems, we can establish the existence of a family of traveling waves with the minimal speed. Second, we use this pair of generalized super-/sub-solutions to show that the propagation of waves is fully determined by the rate of decay of the initial data at infinity in the sense of Aronson-Weinberger formulation, which in turn confirms the aforementioned numerical and experimental results.


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[^0]Keywords: Reaction-diffusion system; Traveling wave; Global stability; Isothermal chemical reaction; Quadratic autocatalysis; Strong decay

## 1. Introduction

In this paper, we study a simple system to model the isothermal chemical reaction scheme governed by the quadratic autocatalytic step and the decay step (see [27,30,21,29], for example),

$$
\begin{array}{ll}
A+B \rightarrow 2 B & \left(\text { rate } k_{1} a b, \text { autocatalysis step }\right), \\
B \rightarrow C & \left(\text { rate } k_{2} b^{q}, \text { decay step }\right), \tag{1.1b}
\end{array}
$$

where $a$ and $b$ stand for the concentrations of the reactant $A$ and the autocatalyst $B$, respectively, $C$ is a stable product, $k_{1}$ and $k_{2}$ are the rate constants, and $q>0$ is the order of the decay. In general, the reaction order $q$ in realistic models is determined empirically, and so $q$ is not necessarily an integer. The quadratic autocatalytic reaction (1.1a) appears in several important chemical reactions such as the Belousov-Zhabotinskii reaction [16], and the radical chain branching in gas-phase oxidation reactions [14,26]. On the other hand, it is believed [12] that if the autocatalyst is not indefinitely stable and itself undergoes a further reaction, then the decay step (1.1b) must be added to the scheme. More details on the effect of the decay step (1.1b) can be found in $[13,21]$. Experimental results indicate that chemical reaction scheme (1.1) may support the generation of propagating waves [16,36]. Precisely, if the reactant $A$ initially distributes uniformly in the whole space, and a quantity of the autocatalyst, $B$, is added locally into this system, then waves of the reactant $A$ (and the autocatalyst $B$ ) will be generated and propagate outwards from this initial zone into the unreacted state. This is the phenomenon which we would like to explore in this paper.

### 1.1. The model

Assume that the mechanism of wave propagation is based on the interplay between chemical reaction and molecular diffusion. Then the governing equations for the reactant $A$ and the autocatalyst $B$ under the chemical reaction scheme (1.1) can be formulated in the reaction-diffusion setting, which reads

$$
\begin{align*}
u_{t} & =\delta u_{x x}-u v  \tag{1.2a}\\
v_{t} & =v_{x x}+u v-K v^{q} \tag{1.2b}
\end{align*}
$$

with the initial condition

$$
u(x, 0)=u^{*}, \quad \bar{v}(x, 0)= \begin{cases}\beta_{0} \cdot g_{0}(x), & |x| \leqslant l_{0}  \tag{1.2.ini}\\ 0, & |x|>l_{0}\end{cases}
$$

Here $u$ and $v$ are dimensionless concentrations of $A$ and $B$, respectively; $\delta$ is the ratio of the diffusion rate of $u$ to that of $v ; K$ measures the strength of the decay step (1.1b) relative to that
of the autocatalytic step (1.1a); $g_{0}(\cdot)$ is a positive continuous function on $\left[-l_{0}, l_{0}\right.$ ] with a maximum value of unity and $g_{0}( \pm l)=0 ; u^{*}$ is a positive constant; $\beta_{0}$ is a positive control parameter and measures the maximum concentration of the autocatalyst input; $x$ is the dimensionless distance and $t$ is the dimensionless time. We remark that system (1.2) with modification in reaction kinetics also arises in the context of epidemiology [22,3] and bio-reactor model [4].

In the context of the aforementioned wave phenomenon, we concern whether or not the solution of the initial problem (1.2)-(1.2.ini) can lead to the formation of waves. For the linear decay case $q=1$, the analysis of Merkin, Needham and Scott [27] indicates that for any $K \in\left(0, u^{*}\right)$ and $\beta_{0}>0$, the solution of the initial value problem (1.2)-(1.2.ini) with $\delta=1$ develops into a pair of diverging traveling waves with wave speed equal to $2 \sqrt{u^{*}-K}$ as $t \rightarrow \infty$, and thus there is a trigger mechanism for the initiation of waves for any positive $K \in\left(0, u^{*}\right)$ and $\beta_{0}>0$, while Needham [31] analytically shows that for $K>u^{*}$, the solution $(u, v)$ of the initial value problem (1.2)-(1.2.ini) with $\delta=1$ tends to $\left(u^{*}, 0\right)$ as $t \rightarrow \infty$. We remark that for the linear decay case $q=1$, the existence of traveling waves of system (1.2) has been established via the shooting arguments $[18,19,33,20]$. However, for the strong decay case $q>1$, the study of Needham and Merkin $[30,28]$ indicates that there is no restriction on $K$ for the initiation of waves and the wave speed is, in fact, equal to $2 \sqrt{u^{*}}$, which is independent of $K$. Moreover, the $u$-component of waves is nonzero at the rear of waves for the linear decay case $q=1$, whereas it is zero for the strong decay case $q>1$. Hence the strong decay case differs from the linear decay case in the initiation of waves and their associated wave profiles. For the weak decay case $q \in(0,1)$, the investigation of McCabe, Leach, and Needham [24,25] indicates that system (1.2) is an excitable system, and so waves are unique, which implies that the properties of waves of the weak decay case are different from those of the case with $q \geqslant 1$. Based on these discussion, we will mainly focus on the strong decay case $q>1$. Hereafter, we always assume that $q>1$. Note that the previous studies [30,28] for the strong decay case are based on the asymptotic analysis and numerical simulation, and so are not rigorous. Now we mention some results for the case without decay step (i.e., $K=0$ ). The existence of traveling waves was studied by Billingham and Needham [6] (see also [9]). The evolution of the solution of (1.2)-(1.2.ini) with $K=0$ was investigated by Billingham and Needham [7] via asymptotic analysis and numerical computation and by Chen and Qi [10] via a rigorous analysis. Finally, for the bounded spatial domain, the dynamics and the steady state solutions of system (1.2) was recently studied by Zhao, Wang, and Shi [37].

### 1.2. Main result

For the strong decay case $q>1$, the numerical evidence that a pair of diverging traveling waves would be generated no matter how small the initial input of autocatalyst suggests that the initial value problem (1.2)-(1.2.ini) has very similar dynamical behavior as the well-known Kolmogorov-Petrovsky-Piscounov equation (KPP) [23]. Hence one might expect that there is a family of waves of system (1.2) with minimal wave speed, and that under the assumption on the initial data which decays exponentially as $x \rightarrow \pm \infty$, the corresponding solution of system (1.2) develops into a pair of diverging traveling waves as $t \rightarrow \infty$ moving at the speed determined by the rate of decay of the initial data as $x \rightarrow \pm \infty$. In view of the above discussion, in this paper we would like to address two issues: (i) the existence of a family of waves of system (1.2) with minimal wave speed, and (ii) the evolution of system (1.2) with the initial data imposed by the following constraint:

$$
u(x, 0)=u^{*}, \quad v(x, 0): \begin{cases}\beta_{0} \cdot g_{0}(x), & |x| \leqslant l_{0}  \tag{1.3}\\ C_{1}^{ \pm} e^{-\lambda^{ \pm}|x|} \leqslant v(x, 0) \leqslant C_{2}^{ \pm} e^{-\lambda^{ \pm}|x|}, & \pm x>l_{0}\end{cases}
$$

where $\beta_{0}, u^{*}$ and $g_{0}$ are defined as in (1.2.ini), and $\lambda^{ \pm}, C_{1}^{ \pm}$and $C_{2}^{ \pm}$are positive constants. We remark that the $v$-component of the initial data (1.3) differs from that of the initial data (1.2.ini) only for $|x|>l_{0}$. In below, we will state the main results for these two issues.

First, we observe that for each $u_{*} \geqslant 0,\left(u_{*}, 0\right)$ is a homogeneous equilibrium point of system (1.2). Then for each given $u^{*}>0$, by a traveling wave solution of system (1.2) we mean a solution of system (1.2) of the form

$$
(u(x, t), v(x, t))=(U(z), V(z)), \quad z=x-c t,
$$

with the boundary condition $(U, V)(-\infty)=\left(u_{*}, 0\right)$ and $(U, V)(+\infty)=\left(u^{*}, 0\right)$, where the wave speed $c$ and $u_{*} \in\left[0, u^{*}\right)$ are constants to be determined, and the wave profile $(U, V) \in C^{2}(\mathbb{R}) \times$ $C^{2}(\mathbb{R})$ is a pair of nonnegative functions. Substituting the ansatz on $(U, V)$ into (1.2), we have

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}-U V=0,  \tag{1.4a}\\
& V^{\prime \prime}+c V^{\prime}+U V-K V^{q}=0 \tag{1.4b}
\end{align*}
$$

on $\mathbb{R}$, together with the boundary conditions

$$
\begin{equation*}
(U, V)(-\infty)=\left(u_{*}, 0\right), \quad(U, V)(+\infty)=\left(u^{*}, 0\right) \tag{1.5}
\end{equation*}
$$

Here the prime denotes the differentiation with respect to $z$. The main result on the existence of traveling waves for the case $q>1$ is stated as follows.

Theorem 1 (Existence of traveling waves). Let $q>1$ and $u^{*}>0$ be given. Then the following hold.
(i) Suppose that $(U, V)$ is a nonnegative solution $(U, V)$ of system (1.4)-(1.5). Then we have $(U, V)(-\infty)=(0,0)\left(i . e ., u_{*}=0\right)$.
(ii) For each $c<c^{*}=c^{*}\left(u^{*}\right):=2 \sqrt{u^{*}}$, there are no nonnegative solutions $(U, V)$ of system (1.4)-(1.5).
(iii) For each $c \geqslant c^{*}$, system (1.4)-(1.5) admits a nonnegative solution ( $U, V$ ). Moreover, $0<$ $U, V<u^{*}$ on $\mathbb{R}, U^{\prime}>0$ on $\mathbb{R}$, and $V^{\prime}>0$ on $\left(-\infty, \xi_{0}\right)$ and $V^{\prime}<0$ on $\left(\xi_{0},+\infty\right)$ for some $\xi_{0} \in \mathbb{R}$. Further, for $c>c^{*}$, we have $V(z)=\mathcal{O}\left(e^{-\lambda z}\right)$ as $z \rightarrow \infty$ where $\lambda$ is given by

$$
\begin{equation*}
\lambda=\lambda(c):=\frac{1}{2} \cdot\left(c-\sqrt{c^{2}-4 u^{*}}\right) . \tag{1.6}
\end{equation*}
$$

(iv) For $q \in(1,2]$, the nonnegative solution of system (1.4)-(1.5) is unique (up to a translation).

We make two remarks. First, the minimal speed $c^{*}$ of waves of system (1.2) is independent of the ratio $\delta$ of the diffusion rate of $u$ to that of $v$, and the decay parameter $K$. Second, the constraint on the parameter $q$ for the uniqueness of waves is technical, and the remaining case $q>2$ will be left as our future study. Also the proof for the uniqueness part is based on the scaling argument [28], and hence the proof is deferred to Appendix A (see Appendix A.3).


Fig. 1. The solution as a function of the spatial variable $x$ is plotted at $t=0, t=5, t=15$, and $t=25$. The $u$ component of the initial data $\left(u_{0}, v_{0}\right)$ is 1 . The $v$ component of the initial data $\left(u_{0}, v_{0}\right)$ is chosen so that $v_{0}$ is of the hump shape, $v_{0} \sim e^{0.3(x-100)}$ for $x$ close to the left end, and $v_{0} \sim e^{-0.3(x-100)}$ for $x$ close to the right end. The parameter values are $\delta=2, K=1.5$, and $q=1.5$.

Next, we turn to the long time behavior of the solution of system (1.2)-(1.3). As mentioned before, we expect that the solution of system (1.2)-(1.3) evolves into a pair of diverging traveling waves as $t \rightarrow \infty$ with wave speed determined by the rate of decay of the initial data as $x \rightarrow \pm \infty$ (see Fig. 1). Due to the lack of the comparison principle for system (1.2), it would be difficult to analytically establish such a convergence to a pair of diverging traveling waves in the usual sense if not completely impossible. However, one can still obtain the convergence to waves by adopting a less restrictive description. Indeed, one can characterize propagation of the solution by looking at the evolution of the leading edge. This approach is first introduced by Aronson and Weinberger $[1,2]$ and proven to be successful in a number of studies of wave propagation [35,15].

Before proceeding to state the convergence result, we make two remarks about the relation (1.6). To begin with, from the relation (1.6), one can verify that $\lambda(c)$ is decreasing in $c \in\left[2 \sqrt{u^{*}}, \infty\right)$. Moreover, $\lim _{c \rightarrow\left(2 \sqrt{u^{*}}\right)^{+}} \lambda(c)=\sqrt{u^{*}}$ and $\lim _{c \rightarrow+\infty} \lambda(c)=0$. Therefore, we have the decay rate $\lambda(c) \in\left(0, \sqrt{u^{*}}\right]$ for each admissible wave speed $c \geqslant 2 \sqrt{u^{*}}$. Next, it is easy to see that the relation (1.6) between the wave speed $c$ and the decay rate $\lambda$ is one-to-one correspondence. Motivated by this observation, we have the following definition.

Definition 1.1. For each $\lambda \in\left(0, \sqrt{u^{*}}\right)$, let the wave profile $(U, V)$ with $V(z)=\mathcal{O}\left(e^{-\lambda z}\right)$ as $z \rightarrow \infty$ and the associated wave speed $c$ established in Theorem 1 be denoted by $\left(U_{\lambda}, V_{\lambda}\right)$ and $c_{\lambda}$, respectively.

Now in the following theorem, we state the result that the solution of system (1.2)-(1.3) with $\lambda^{ \pm} \in\left(0, \sqrt{u^{*}}\right)$ develops into a pair of waves propagating at the speed $c_{\lambda^{ \pm}}$in the sense of Aronson and Weinberger [1,2]:

Theorem 2 (Evolution of traveling waves). Let $q>1$. Suppose that $(u, v)$ is the solution of system (1.2)-(1.3) with $\lambda^{ \pm} \in\left(0, \sqrt{u^{*}}\right)$. Then we have the following.
(i) For each $c>c_{\lambda^{+}}$and each $x \in \mathbb{R},(u, v)(x+c t, t) \rightarrow\left(u^{*}, 0\right)$ as $t \rightarrow \infty$;
(ii) There exists a pair of non-negative continuous functions $\left(\psi_{\lambda^{+}}^{+}, \phi_{\lambda^{+}}^{+}\right)$with $\left(\psi_{\lambda^{+}}^{+}, \phi_{\lambda^{+}}^{+}\right)(x) \rightarrow$ $\left(u^{*}, 0\right)$ as $x \rightarrow \infty$ such that the following hold:
(a) $\psi_{\lambda^{+}}^{+}$vanishes in $\left(-\infty, \zeta_{1}^{+}\right]$and is strictly increasing in $\left[\xi_{1}^{+}, \infty\right)$ for some constant $\zeta_{1}^{+}>0$, and

$$
u\left(x+c_{\lambda+} t, t\right) \geqslant \psi_{\lambda^{+}}^{+}(x), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty)
$$

(b) $\phi_{\lambda^{+}}^{+}$vanishes in $\left(-\infty, \xi_{1}^{+}\right]$, and is strictly increasing in $\left[\xi_{1}^{+}, \xi_{2}^{+}\right]$and strictly decreasing in $\left[\xi_{2}^{+}, \infty\right)$ for some constants $\xi_{2}^{+}>\xi_{1}^{+}>0$, and

$$
v\left(x+c_{\lambda+} t, t\right) \geqslant \phi_{\lambda^{+}}^{+}(x), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty) ;
$$

(iii) For each $t>0,(u, v)\left(x+c_{\lambda^{+}} t, t\right) \rightarrow\left(u^{*}, 0\right)$ as $x \rightarrow \infty$;
(iv) For each $c>c_{\lambda^{-}}$and each $x \in \mathbb{R},(u, v)(x-c t, t) \rightarrow\left(u^{*}, 0\right)$ as $t \rightarrow \infty$;
(v) There exists a pair of non-negative continuous functions $\left(\psi_{\lambda^{-}}^{-}, \phi_{\lambda^{-}}^{-}\right)$with $\left(\psi_{\lambda^{-}}^{-}, \phi_{\lambda^{-}}^{-}\right)(x) \rightarrow$ $\left(u^{*}, 0\right)$ as $x \rightarrow-\infty$ such that the following hold:
(c) $\psi_{\lambda^{-}}^{-}$vanishes in $\left[\zeta_{1}^{-}, \infty\right]$ and is strictly decreasing in $\left(-\infty, \zeta_{1}^{-}\right]$for some constant $\zeta_{1}^{-}<0$, and

$$
u\left(x-c_{\lambda^{-}} t, t\right) \geqslant \psi_{\lambda^{-}}^{-}(x), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty)
$$

(d) $\phi_{\lambda^{-}}^{-}$vanishes in $\left[\xi_{2}^{-}, \infty\right)$, and is strictly decreasing in $\left[\xi_{1}^{-}, \xi_{2}^{-}\right]$and strictly increasing in $\left(-\infty, \xi_{1}^{-}\right.$] for some constants $\xi_{1}^{-}<\xi_{2}^{-}<0$, and

$$
v\left(x-c_{\lambda-} t, t\right) \geqslant \phi_{\lambda^{-}}^{-}(x), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty) ;
$$

(vi) For each $t>0$, $(u, v)\left(x-c_{\lambda}-t, t\right) \rightarrow\left(u^{*}, 0\right)$ as $x \rightarrow-\infty$.

We make two remarks. First, roughly speaking, Theorem 2 states that (I) if one is in the moving coordinate $z=x-c t$ (reps. $z=x+c t$ ) with $c$ larger than the wave speed $c_{\lambda^{+}}$(resp. $c_{\lambda^{-}}$), then one will see the unstable state ( $u^{*}, 0$ ); (II) if one is in the moving coordinate $z=x-c t$ (reps. $z=x+c t$ ) with $c$ equal to the wave speed $c_{\lambda^{+}}$(resp. $c_{\lambda^{-}}$), then one will see a wave-like profile. Note that, in a rough sense, (I) and (II) suggest that the long time behavior of the solution $(u, v)$ of the initial value problem (1.2)-(1.3) is a pair of diverging traveling waves whose speed is determined by the rate of decay of the initial data as $x \rightarrow \pm \infty$ (see Fig. 1). From this, we can infer that $(u(x, t), v(x, t)) \rightarrow(0,0)$ as $t \rightarrow \infty$ for any given $x \in \mathbb{R}$. Second, we turn to the solution $(u, v)$ of the initial value problem (1.2)-(1.2.ini). Following the proof of Theorem 2(i), (iii), (iv) and (vi), the conclusion of Theorem 2(i), (iii), (iv) and (vi) with $c_{\lambda^{ \pm}}$replaced by $c^{*}$ holds for $(u, v)$. However, due to the restriction of the generalized sub-solution, we cannot deduce that the assertions (ii) and (iv) of Theorem 2 with $c_{\lambda^{ \pm}}$replaced by $c^{*}$ hold for $(u, v)$. Nevertheless, the conclusion of Theorem 2 suggests that system (1.2) possesses the wave propagation feature of KPP-type equation.

Now we briefly sketch our method for the proof of the results. Previous studies [9,18,19,33,20] on the existence of waves of system (1.2) for the linear decay case ( $q=1$ ) and the case without decay step $(K=0)$ are based on the dynamical system approach. However, such approaches cannot give any information about the evolution of the solution of system (1.2). Therefore, in this paper we will employ PDE approach, instead of dynamical system approach, to establish the existence of traveling waves, which in turn can describe the long time behavior of the solution of system (1.2)-(1.3). In fact, both of the proofs for Theorem 1 and Theorem 2 are based on a pair of the generalized super-/sub-solutions of system (1.2). Note that system (1.2) does not enjoy comparison principle. Hence traveling waves of system (1.2) do not qualify as super-/sub-solutions (unless they coincide), and so we cannot bound solutions of system (1.2) componentwise by translates of traveling waves. On the other hand, this pair of the generalized super-/sub-solutions are related to the solutions of a family of approximate linear inhomogeneous systems of system (1.2), and the construction of the generalized sub-solution is based on the generalized super-solution. With the aid of a family of truncated problems on the finite interval whose solutions can be proven to be sandwiched between the pair of the generalized super-/sub-solutions, one can establish the existence of a family of traveling waves with the minimal speed of system (1.2) through the limit process. We remark that the idea of the framework of the proof for the existence of waves is based on [5]. Next, via the comparison principle for a single equation, the solution of system (1.2)-(1.3) can be shown to be squeezed between the pair of the generalized super-/sub-solutions, from which the assertions of Theorem 2 can be proven.

The remaining parts of this paper are organized as follows. In Section 2, we derive basic properties of waves. Section 3 is devoted to the solutions of truncated problems of system (1.2). These approximate solutions are building blocks for the existence of traveling waves of system (1.2) which is established in Section 4. Finally, the asymptotic behavior of system (1.2)-(1.3) is investigated in Section 5.

## 2. Lower bound for the minimal speed of waves and decay rate of waves

To begin with, we establish the assertion of Theorem 1(ii) and the decay rate of $v$-component of waves near infinity.

Lemma 2.1. Suppose that $(U, V)$ is a nonnegative solution of system (1.4)-(1.5). Then we have
(i) $c \geqslant 2 \sqrt{u^{*}}$, and
(ii) $V(z)=\mathcal{O}\left(e^{-\lambda z}\right)$ as $z \rightarrow \infty$ where $\lambda$ is given by

$$
\lambda=\frac{1}{2} \cdot\left(c \pm \sqrt{c^{2}-4 u^{*}}\right) .
$$

Proof. Linearizing ( 1.4 b ) around $(1,0)$ leads to the equations

$$
\begin{align*}
& \delta u^{\prime \prime}+c u^{\prime}-u^{*} v=0  \tag{2.1a}\\
& v^{\prime \prime}+c v^{\prime}+u^{*} v=0 \tag{2.1b}
\end{align*}
$$

Note that (2.1b) has two eigenvalues

$$
\lambda_{1}=\frac{1}{2} \cdot\left(-c+\sqrt{c^{2}-4 u^{*}}\right), \quad \lambda_{2}=\frac{1}{2} \cdot\left(-c-\sqrt{c^{2}-4 u^{*}}\right) .
$$

Suppose that $c \leqslant-2 \sqrt{u^{*}}$. Then we have $\lambda_{i}>0, i=1,2$, and so $V(z)$ is unbounded as $z \rightarrow \infty$, which is a contradiction. Therefore, we have $c>-2 \sqrt{u^{*}}$. On the other hand, if $|c|<2 \sqrt{u^{*}}$ holds, then $\lambda_{1}$ and $\lambda_{2}$ form a complex conjugate pair. This would imply that $V(z)$ cannot be of the same sign for $z$ near infinity, a contradiction again. Hence we have $c \geqslant 2 \sqrt{u^{*}}$, which completes the proof of assertion (i). The assertion (ii) follows from the above linearized equation and the definitions of $\lambda_{1}$ and $\lambda_{2}$. The proof of this lemma is thus completed.

## 3. Truncated problem on the finite interval $[-l, l]$

To show the existence of a nonnegative solution ( $U, V$ ) of system (1.4)-(1.5), we introduce the function pairs $\left(U^{ \pm}(\cdot), V^{ \pm}(\cdot)\right)$ satisfying the following properties:
(i) $U^{ \pm}(\cdot)$ and $V^{ \pm}(\cdot)$ are continuous functions on $\mathbb{R}$.
(ii) $0 \leqslant U^{-}(\cdot) \leqslant U^{+}(\cdot)$ and $0 \leqslant V^{-}(\cdot) \leqslant V^{+}(\cdot)$ on $\mathbb{R}$.
(iii) $\left(U^{-}, V^{-}\right)(\infty)=\left(U^{+}, V^{+}\right)(\infty)=\left(u^{*}, 0\right)$.
(iv) $U^{-}(\cdot)$ and $V^{-}(\cdot)$ vanish in $(-\infty, 0)$ and are positive in $\left(z_{1}, \infty\right)$ for some $z_{1}>0$.

The strategy is to construct a family of solutions of (1.4) on a finite interval $I_{l}:=[-l, l]$ such that they are sandwiched between $\left(U^{-}, V^{-}\right)$and $\left(U^{+}, V^{+}\right)$. Then, by passing to the limit $l \rightarrow \infty$, we can obtain a nonnegative solution $(U, V)$ of (1.4) on the whole line $\mathbb{R}$ with $(U, V)(\infty)=\left(u^{*}, 0\right)$, which serves as a candidate of a nonnegative solution $(U, V)$ of system (1.4)-(1.5). Throughout this subsection, we always assume $c>c^{*}=2 \sqrt{u^{*}}$.

### 3.1. The setting of the truncated problem

To this end, for each $l>z_{1}$ we consider the truncated problem of system (1.4)-(1.5)

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}-U V=0 \quad \text { in }(-l, l), \\
& V^{\prime \prime}+c V^{\prime}+U V-K V^{q}=0 \quad \text { in }(-l, l) \tag{3.1}
\end{align*}
$$

together with the boundary conditions

$$
\begin{equation*}
(U, V)(-l)=\left(U^{-}, V^{-}\right)(-l), \quad(U, V)(l)=\left(U^{-}, V^{-}\right)(l) \tag{3.2}
\end{equation*}
$$

We will apply the Schauder fixed point theorem to show the existence of solutions of (3.1)-(3.2). For this, we first introduce the working space

$$
E:=\left\{(U, V) \in X:=C\left(I_{l}\right) \times C\left(I_{l}\right) \mid U^{-} \leqslant U \leqslant U^{+} \text {and } V^{-} \leqslant V \leqslant V^{+} \text {in } I_{l}\right\}
$$

which is closed and convex in the Banach space $X$ equipped with the norm $\left\|\left(\psi_{1}, \psi_{2}\right)\right\|_{X}=$ $\left\|\psi_{1}\right\|_{C\left(I_{l}\right)}+\left\|\psi_{2}\right\|_{C\left(I_{l}\right)}$. Since $U^{-}$and $V^{-}$are nonnegative, it follows that $U \geqslant 0$ and $V \geqslant 0$ for any $(U, V) \in E$. Next, we consider the mapping $\mathcal{F}=\left(\mathcal{F}_{1}, \mathcal{F}_{2}\right)$ on $E$ : given $\left(U_{0}, V_{0}\right) \in E$,

$$
\mathcal{F}_{1}\left(U_{0}, V_{0}\right):=U ; \quad \mathcal{F}_{2}\left(U_{0}, V_{0}\right):=V
$$

where $(U, V)$ is the classical solution of the boundary value problem

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}-U V_{0}=0 \quad \text { in }(-l, l)  \tag{3.3a}\\
& V^{\prime \prime}+c V^{\prime}+U_{0} V_{0}-K\left(V_{0}\right)^{q}=0 \quad \text { in }(-l, l)  \tag{3.3b}\\
& (U, V)(-l)=\left(U^{-}, V^{-}\right)(-l), \quad(U, V)(l)=\left(U^{-}, V^{-}\right)(l) \tag{3.3c}
\end{align*}
$$

One can easily see that any fixed point of $\mathcal{F}$ is a solution of the problem (3.1)-(3.2). Hence the existence of solutions of (3.1)-(3.2) is reduced to verifying that the mapping $\mathcal{F}$ satisfies the conditions of the Schauder fixed point theorem.

First, we have the basic properties of the mapping $\mathcal{F}$ as stated in the following lemma.
Lemma 3.1. The mapping $\mathcal{F}$ is well-defined and any function in $\mathcal{F}_{1}(E)$ is positive and strictly increasing in $(-l, l)$ with $l>z_{1}$; that is, for a given $\left(U_{0}, V_{0}\right) \in E$, (3.3) admits a unique solution $(U, V)$. In addition, $U>0$ and $U^{\prime}>0$ in $(-l, l)$ with $l>z_{1}$.

Proof. Note that the system (3.3) is not a coupled system and the equations for $U$ and $V$ are linear. The existence and uniqueness of $U$ and $V$ can be easily obtained by [17, Theorem 3.1 of Chapter 12]. Since $l>z_{1}$, the assumption on $U^{-}(\cdot)$ implies that $U^{-}(-l)=0$ and $U^{-}(l)>0$. Then it follows from the maximum principle that $U>0$ in $(-l, l)$.

Now it remains to show that $U^{\prime}>0$ in $(-l, l)$. Since $U(-l)=U^{-}(-l)=0$ and $U>0$ in $(-l, l)$, it follows that $U^{\prime}(-l) \geqslant 0$. Indeed, $U^{\prime}(-l)>0$ since otherwise $U(-l)=U^{\prime}(-l)=0$. Then the uniqueness gives that $U \equiv 0$, which contradicts the boundary condition $U(l)>0$. Using (3.3a), one can easily deduce that

$$
\left(e^{c z / \delta} U^{\prime}(z)\right)^{\prime}=\frac{1}{\delta} U(z) V_{0}(z) e^{c z / \delta}
$$

Then an integration of the above equation gives

$$
\begin{equation*}
e^{c z / \delta} U^{\prime}(z)=U^{\prime}(-l) e^{-c l / \delta}+\frac{1}{\delta} \int_{-l}^{z} U(\tau) V_{0}(\tau) e^{c \tau / \delta} d \tau \tag{3.4}
\end{equation*}
$$

which, together with the fact that $U, V_{0} \geqslant 0$ in $(-l, l)$ and $U^{\prime}(-l)>0$, implies that $U^{\prime}>0$ in $(-l, l)$. Hence we complete the proof of this lemma.

### 3.2. Super-/sub-solutions and maximum principle

To show that $\mathcal{F}$ maps into itself, we need to choose the function pairs ( $U^{ \pm}, V^{ \pm}$) carefully. For this, we will use the iteration process to construct the super-/sub-solutions of system (3.3) in Section 3.2.1. With the aid of this pair of super-/sub-solutions, we can verify that if ( $U_{0}, V_{0}$ ) is sandwiched between this pair of super-/sub-solutions, then so is $\mathcal{F}\left(U_{0}, V_{0}\right)$, which yields that $\mathcal{F}$ maps into itself. The idea of such a construction is motivated by [5]. Specifically, we first construct a super-solution $V^{+}$for the $V$-component, which is then used to build a sub-solution $U^{-}$for the $U$-component. The sub-solution $U^{-}$is in turn employed to produce a sub-solution $V^{-}$for the $V$-component. The super-solution $U^{+}$for the $U$-component is always chosen as the constant $u^{*}$. The maximum principle for the comparison between the solution of system (3.3) and the super-/sub-solutions $\left(U^{ \pm}, V^{ \pm}\right)$is given in Section 3.2.2.

### 3.2.1. Construction of super-/sub-solutions

Before proceeding to the construction of super-/sub-solutions, we remark that in order to facilitate the discussion of the long time behavior of the solution of system (1.2), the constructed super-/sub-solutions may depend on the auxiliary parameters $x_{0}$ and $x_{1}$ and the rate parameter $\lambda$.

Now we construct super-/sub-solutions. For simplicity, we set

$$
p(s):=s^{2}-c s+u^{*} .
$$

Since $c>2 \sqrt{u^{*}}$, the equation $p(s)=0$ has two positive roots $\lambda$ and $\lambda+d$, where

$$
\lambda=\frac{1}{2} \cdot\left(c-\sqrt{c^{2}-4 u^{*}}\right) \quad \text { and } \quad d=\sqrt{c^{2}-4 u^{*}}
$$

In addition, $p(s)<0$ when $s \in(\lambda, \lambda+d)$.
Lemma 3.2. For a fixed $x_{0} \in \mathbb{R}$, the function $V_{\lambda}^{+}\left(z ; x_{0}\right):=e^{-\lambda\left(z-x_{0}\right)}$ satisfies the equation

$$
\begin{equation*}
\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime}+u^{*} V_{\lambda}^{+}\left(z ; x_{0}\right)=0, \tag{3.5}
\end{equation*}
$$

for all $z \in \mathbb{R}$, where the prime denotes the differentiation with respect to $z$.
Proof. Since $p(\lambda)=0$, it follows that

$$
\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime}+u^{*} V_{\lambda}^{+}\left(z ; x_{0}\right)=p(\lambda) V_{\lambda}^{+}\left(z ; x_{0}\right)=0, \quad \forall z \in \mathbb{R}
$$

Remark. We note that the characteristic polynomial of Eq. (2.1b) is

$$
\tilde{p}(s):=s^{2}+c s+u^{*},
$$

which differs from $p(s)$ in the sign of the coefficient of the term $s$. Here we use the function $p(s)$, instead of $\tilde{p}(s)$, in order to make the rate constant $\lambda$ positive.

Select $0<\gamma<\min \{c / \delta, \lambda\}$. Then $c-\delta \gamma>0$ and $\gamma-\lambda<0$. Since $e^{(\gamma-\lambda) z} \rightarrow 0$ as $z \rightarrow \infty$, there exists $z_{0}=z_{0}\left(x_{0}\right)>0$ such that

$$
e^{(\gamma-\lambda) z+\lambda x_{0}} \leqslant \gamma(c-\delta \gamma), \quad \forall z \geqslant z_{0},
$$

which yields

$$
\begin{equation*}
\gamma(c-\delta \gamma) e^{-\gamma z} \geqslant V_{\lambda}^{+}\left(z ; x_{0}\right), \quad \forall z \geqslant z_{0} . \tag{3.6}
\end{equation*}
$$

Set $M=M\left(x_{0}\right):=u^{*} e^{\gamma z_{0}\left(x_{0}\right)}$. Then $M>u^{*}$ since $\gamma, z_{0}>0$. In the sequel, we retain the notation $z_{0}$.

Lemma 3.3. The function $U_{\lambda}^{-}\left(z ; x_{0}\right):=\max \left\{0, u^{*}-M e^{-\gamma z}\right\}$ satisfies the inequality

$$
\begin{equation*}
\delta\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime}-U_{\lambda}^{-}\left(z ; x_{0}\right) V_{\lambda}^{+}\left(z ; x_{0}\right) \geqslant 0, \tag{3.7}
\end{equation*}
$$

for all $z \neq z_{0}$, where the prime denotes the differentiation with respect to $z$.

Proof. For $z<z_{0}$, the inequality (3.7) holds immediately since $U_{\lambda}^{-}\left(z ; x_{0}\right) \equiv 0$ in $\left(-\infty, z_{0}\right)$. For $z>z_{0}, U_{\lambda}^{-}\left(z ; x_{0}\right)=u^{*}-M e^{-\gamma z}$. Using (3.6) and the fact that $M>u^{*}$, we deduce that

$$
\begin{aligned}
& \delta\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime} \\
& \quad=M \gamma(c-\delta \gamma) e^{-\gamma z} \geqslant u^{*} V_{\lambda}^{+}\left(z ; x_{0}\right) \geqslant U_{\lambda}^{-}\left(z ; x_{0}\right) V_{\lambda}^{+}\left(z ; x_{0}\right) .
\end{aligned}
$$

Hence (3.7) holds.
Choose $0<\eta<\min \{\gamma, \lambda(q-1), d\}$. Then $\eta-\gamma<0, \lambda+\eta-\lambda q<0$, and $p(\lambda+\eta)<0$. For a fixed $x_{1} \in \mathbb{R}$, select

$$
\begin{equation*}
L=L\left(x_{1}, x_{0}\right)>\max \left\{\frac{M}{u^{*}} \cdot e^{\lambda x_{1}},-\frac{\left(M e^{\lambda x_{1}}+K e^{\lambda q x_{0}}\right)}{p(\lambda+\eta)}\right\} . \tag{3.8}
\end{equation*}
$$

Set

$$
\begin{equation*}
z_{1}=\left(\ln L-\lambda x_{1}\right) / \eta . \tag{3.9}
\end{equation*}
$$

Then $z_{1}>z_{0}>0$ since $z_{0}=\ln \left(M / u^{*}\right) / \gamma, L>\frac{M}{u^{*}} \cdot e^{\lambda x_{1}}$, and $\eta<\gamma$. In the sequel, we retain the notation $z_{1}$ and $L$.

Lemma 3.4. The function $V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right):=\max \left\{0, V_{\lambda}^{+}\left(z ; x_{1}\right)-L e^{-(\lambda+\eta) z}\right\}$ satisfies the inequality

$$
\begin{equation*}
\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}+U_{\lambda}^{-}\left(z ; x_{0}\right) V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)-K\left(\tilde{V}_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{q} \geqslant 0 \tag{3.10}
\end{equation*}
$$

for all $z \neq z_{1}$, where $\tilde{V}_{\lambda}^{+}\left(z ; x_{0}\right)=V_{\lambda}^{+}\left(z, x_{0}\right) \cdot H\left(z-z_{1}\right)$ with $H(\cdot)$ being the Heaviside function, and the prime denotes the differentiation with respect to $z$.

Proof. For $z<z_{1}$, the inequality (3.10) holds immediately since $V_{\lambda}^{-} \equiv 0$ in $\left(-\infty, z_{1}\right)$. For $z>z_{1}, V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)=V_{\lambda}^{+}\left(z ; x_{1}\right)-L e^{-(\lambda+\eta) z}$ and $U_{\lambda}^{-}\left(z ; x_{0}\right)=u^{*}-M e^{-\gamma z}$. A simple computation gives that

$$
\begin{aligned}
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}=\left(V_{\lambda}^{+}\left(z ; x_{1}\right)\right)^{\prime}+(\lambda+\eta) L e^{-(\lambda+\eta) z} \\
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}=\left(V_{\lambda}^{+}\left(z ; x_{1}\right)\right)^{\prime \prime}-(\lambda+\eta)^{2} L e^{-(\lambda+\eta) z}
\end{aligned}
$$

and

$$
\begin{aligned}
U_{\lambda}^{-}\left(z ; x_{0}\right) V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right) & =\left(u^{*}-M e^{-\gamma z}\right)\left(V_{\lambda}^{+}\left(z ; x_{1}\right)-L e^{-(\lambda+\eta) z}\right) \\
& \geqslant u^{*} V_{\lambda}^{+}\left(z ; x_{1}\right)-u^{*} L e^{-(\lambda+\eta) z}-M e^{\lambda x_{1}-(\lambda+\gamma) z}
\end{aligned}
$$

Together with (3.5) and definition of $p$, we get

$$
\begin{aligned}
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}+U_{\lambda}^{-}\left(z ; x_{0}\right) V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)-K\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{q} \\
& \quad \geqslant e^{-(\lambda+\eta) z}\left[-p(\lambda+\eta) L-M e^{\lambda x_{1}+(\eta-\gamma) z}-K e^{\lambda q x_{0}+(\lambda+\eta-\lambda q) z}\right] \\
& \geqslant e^{-(\lambda+\eta) z}\left[e^{\lambda x_{1}} M\left(1-e^{(\eta-\gamma) z}\right)+e^{\lambda q x_{0}} K\left(1-e^{(\lambda+\eta-\lambda q) z}\right)\right] \geqslant 0 \\
& \quad\left(\text { since } L>-\left(M e^{\lambda x_{1}}+K e^{\lambda q x_{0}}\right) / p(\lambda+\eta), \text { and } \eta-\gamma<0 \text { and } \lambda+\eta-\lambda q<0\right) .
\end{aligned}
$$

The proof of this lemma is thus completed.

### 3.2.2. The maximum principle

In this subsection, we will develop the maximum principle which allows us the compare the solution of system (3.3) with the super-/sub-solutions $\left(U^{ \pm}, V^{ \pm}\right)$constructed in the previous subsection.

To begin with, the following lemma is the maximum principle for a function $w \in C([a, b]) \cap$ $C^{2}((a, b))$ whose proof can be found in [32].

Lemma 3.5. Suppose that $w \in C([a, b]) \cap C^{2}((a, b))$ satisfies the differential inequality

$$
\begin{equation*}
w^{\prime \prime}(z)+g(z) w^{\prime}(z)+h(z) w(z) \leqslant 0, \quad \forall z \in(a, b), \tag{3.11}
\end{equation*}
$$

where $g$ and $h$ are functions in $(a, b)$ with $h \leqslant 0$. If $w(a) \geqslant 0$ and $w(b) \geqslant 0$, then $w \geqslant 0$ on $[a, b]$.

For a function $w \notin C([a, b]) \cap C^{2}((a, b))$ possessing some nice properties, we still have the following maximum principle. Since we cannot locate the proof of such results in the literature, we give the proof here.

Lemma 3.6. Let $z^{*} \in(a, b)$. Suppose that $w \in C([a, b])$ satisfies the following properties:
(i) both $w^{\prime}$ and $w^{\prime \prime}$ are continuous in $(a, b)$ except for $z^{*}$ and satisfy the differential inequality

$$
\begin{equation*}
w^{\prime \prime}(z)+A^{\star} w^{\prime}(z) \leqslant 0, \quad \forall z \in(a, b) \backslash\left\{z^{*}\right\} \tag{3.12}
\end{equation*}
$$

where $A^{\star}$ is a positive constant;
(ii) both $w^{\prime}\left(z^{*}+\right):=\lim _{z \rightarrow z^{*}+} w^{\prime}(z)$ and $w^{\prime}\left(z^{*}-\right):=\lim _{z \rightarrow z^{*}-} w^{\prime}(z)$ exist and

$$
\begin{equation*}
w^{\prime}\left(z^{*}+\right)-w^{\prime}\left(z^{*}-\right) \leqslant 0 \tag{3.13}
\end{equation*}
$$

If $w(a) \geqslant 0$ and $w(b) \geqslant 0$, then $w \geqslant 0$ on $[a, b]$.
Proof. Let

$$
\rho_{1}(z):=\frac{1}{A^{\star}}\left(e^{A^{\star}(z-a)}-1\right) \quad \text { and } \quad \rho_{2}(z):=\frac{1}{A^{\star}}\left(1-e^{A^{\star}(z-b)}\right)
$$

be the unique solutions of the second-order linear equation $\mathcal{L}[y]:=y^{\prime \prime}-A^{\star} y^{\prime}=0$ on $[a, b]$ such that

$$
\begin{equation*}
\rho_{1}(a)=0, \quad \rho_{1}^{\prime}(a)=1 \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho_{2}(b)=0, \quad \rho_{2}^{\prime}(b)=-1 . \tag{3.15}
\end{equation*}
$$

From definitions of $\rho_{1}$ and $\rho_{2}$, we see that

$$
\begin{gather*}
\rho_{1}>0, \quad \rho_{2}>0,  \tag{3.16}\\
\rho_{1}^{\prime}-A^{\star} \rho_{1}=1, \tag{3.17}
\end{gather*}
$$

and

$$
\begin{equation*}
\rho_{2}^{\prime}-A^{\star} \rho_{2}=-1 \tag{3.18}
\end{equation*}
$$

in $(a, b)$.
Now we claim that for any $z \in(a, b)$,

$$
\begin{equation*}
\rho_{1}\left(z^{*}\right) \rho_{2}(z)\left(w^{\prime}\left(z^{*}-\right)-w^{\prime}\left(z^{*}+\right)\right)-\left(\rho_{1}(z)+\rho_{2}(z)\right) w(z)+w(a) \rho_{2}(z)+w(b) \rho_{1}(z) \leqslant 0 . \tag{3.19}
\end{equation*}
$$

For this, we first consider $z^{*}<z<b$. Multiplying (3.12) by $\rho_{1}$, integrating the resulting inequality from $a$ to $z$, and then using integration by parts, we get that

$$
\begin{equation*}
\rho_{1}\left(z^{*}\right)\left(w^{\prime}\left(z^{*}-\right)-w^{\prime}\left(z^{*}+\right)\right)+\rho_{1}(z) w^{\prime}(z)-w(z)+w(a) \leqslant 0, \tag{3.20}
\end{equation*}
$$

where we have used (3.14), (3.17), and the fact that $\mathcal{L}\left[\rho_{1}\right]=0$. Similarly, multiplying (3.12) by $\rho_{2}$, integrating the resulting inequality from $z$ to $b$, and then using the integration by parts, we deduce that

$$
\begin{equation*}
-\rho_{2}(z) w^{\prime}(z)-w(z)+w(b) \leqslant 0 \tag{3.21}
\end{equation*}
$$

where we have used (3.15), (3.18), and the fact that $\mathcal{L}\left[\rho_{2}\right]=0$. Multiplying (3.20) and (3.21) by $\rho_{2}$ and $\rho_{1}$ respectively and then summing up, we finally get (3.19). For $a<z \leqslant z^{*}$, the inequality (3.19) can be obtained by a similar argument.

Finally, rearranging the inequality (3.19) and using (3.13), (3.16) and the assumption that $w(a) \geqslant 0$ and $w(b) \geqslant 0$, we discover that

$$
w(z) \geqslant \frac{w(a) \rho_{2}(z)+w(b) \rho_{1}(z)+\rho_{1}\left(z^{*}\right) \rho_{2}(z)\left(w^{\prime}\left(z^{*}-\right)-w^{\prime}\left(z^{*}+\right)\right)}{\rho_{1}(z)+\rho_{2}(z)} \geqslant 0
$$

for all $z \in(a, b)$. The proof of this lemma is therefore completed.

### 3.3. The verification of the Schauder fixed point theorem

In this section, we will use the super-/sub-solutions $V_{\lambda}^{+}\left(\cdot ; x_{0}\right), U_{\lambda}^{-}\left(\cdot ; x_{0}\right)$, and $V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ with $x_{0}=x_{1}=0$ established in Section 3.2.1 to verify the conditions of the Schauder fixed point theorem. For simplicity, in the remaining of this section, we assume $c>c^{*}, x_{0}=x_{1}=0$, and set $U_{\lambda}^{-}\left(z ; x_{0}\right), V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)$, and $V_{\lambda}^{+}\left(z ; x_{0}\right)$ as $U^{-}(z), V^{-}(z)$, and $V^{+}(z)$, respectively.

Lemma 3.7. $\mathcal{F}$ maps $E$ into $E$.
Proof. For a given $\left(U_{0}, V_{0}\right) \in E$, let $(U, V):=\mathcal{F}\left(U_{0}, V_{0}\right)$. We first claim that $V^{-} \leqslant V \leqslant V^{+}$ on $I_{l}$. Since $0 \leqslant U^{-} \leqslant U_{0} \leqslant U^{+} \equiv u^{*}$ and $0 \leqslant V^{-} \leqslant V_{0} \leqslant V^{+}$, it follows that $U^{-} V^{-}-$ $K\left(V^{+}\right)^{q} \leqslant U_{0} V_{0}-K\left(V_{0}\right)^{q} \leqslant u^{*} V^{+}$. As a consequence, $V$ satisfies the following differential inequalities

$$
\begin{equation*}
V^{\prime \prime}+c V^{\prime}+U^{-} V^{-}-K\left(V^{+}\right)^{q} \leqslant 0, \tag{3.22}
\end{equation*}
$$

and

$$
\begin{equation*}
V^{\prime \prime}+c V^{\prime}+u^{*} V^{+} \geqslant 0 \tag{3.23}
\end{equation*}
$$

for all $z$ in $(-l, l)$. Now we consider the function $w_{1}=V-V^{-}$. From (3.3c), we know that $w_{1}(-l)=w_{1}(l)=0$. In addition, (3.10) and (3.22) give that $w_{1}^{\prime \prime}(z)+c w_{1}^{\prime}(z) \leqslant 0$ for all $z \in$ $(-l, l) \backslash\left\{z_{1}\right\}$. Note that $V^{\prime}\left(z_{1}+\right)=V^{\prime}\left(z_{1}-\right)$ due to $V \in C^{2}((-l, l))$. Together with the fact that $\left(V^{-}\right)^{\prime}\left(z_{1}-\right)=0$ and $\left(V^{-}\right)^{\prime}\left(z_{1}+\right)>0$, we obtain that $w_{1}^{\prime}\left(z_{1}+\right)-w_{1}^{\prime}\left(z_{1}-\right)<0$. Then it follows from Lemma 3.6 that $w_{1} \geqslant 0$ in $[-l, l]$. This implies that $V^{-} \leqslant V$ in $I_{l}$. With a similar argument and using Lemma 3.5, we also get that $V \leqslant V^{+}$in $I_{l}$.

Next, we show that $U^{-} \leqslant U$ in $I_{l}$. Since $U^{-} \equiv 0$ in $\left[-l, z_{0}\right]$ and $U \geqslant 0$ in $\left[-l, z_{0}\right]$, it follows that

$$
\begin{equation*}
U \geqslant U^{-} \quad \text { in }\left[-l, z_{0}\right] . \tag{3.24}
\end{equation*}
$$

Hence it remains to show that $U \geqslant U^{-}$in $\left(z_{0}, l\right]$. Since $V_{0} \leqslant V^{+}$, it follows that

$$
\begin{equation*}
\delta U^{\prime \prime}+c U^{\prime}-U V^{+} \leqslant 0 \quad \text { in }\left(z_{0}, l\right) \tag{3.25}
\end{equation*}
$$

Then (3.7) and (3.25) imply that the function $w_{2}:=U-U^{-}$satisfies $\delta w_{2}^{\prime \prime}+c w_{2}^{\prime}-V^{+} w_{2} \leqslant 0$ in $\left(z_{0}, l\right)$. In addition, from (3.24) and (3.3c), we know that $w_{2}\left(z_{0}\right) \geqslant 0$ and $w_{2}(l)=0$. Hence Lemma 3.5 asserts that $w_{2} \geqslant 0$ in $\left[z_{0}, l\right]$. Hence $U^{-} \leqslant U$ in $\left[z_{0}, l\right]$.

Finally, we show that $U \leqslant U^{+}$in $I_{l}$. Since $U^{+} \equiv u^{*}$ and $V_{0} \geqslant 0$, we see that $U^{+}$satisfies

$$
\delta\left(U^{+}\right)^{\prime \prime}+c\left(U^{+}\right)^{\prime}-U^{+} V_{0} \leqslant 0 \quad \text { in }(-l, l),
$$

and $U^{+}( \pm l)=u^{*} \geqslant U^{-}( \pm l)=U( \pm l)$. By a similar argument as the proof for $U^{-} \leqslant U$ in $\left[z_{0}, l\right]$, we get that $U \leqslant U^{+}$in $I_{l}$.

Lemma 3.8. $\mathcal{F}$ is a continuous mapping.

Proof. For given $\left(U_{0}, V_{0}\right)$ and $\left(\tilde{U}_{0}, \tilde{V}_{0}\right)$ in $E$, let

$$
\begin{equation*}
(U, V)=\mathcal{F}\left(U_{0}, V_{0}\right) \quad \text { and } \quad(\tilde{U}, \tilde{V})=\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right) \tag{3.26}
\end{equation*}
$$

We first consider the function $w_{1}:=U-\tilde{U}$. It is easy to see that $w_{1}(-l)=w_{1}(l)=0$ and

$$
w_{1}^{\prime \prime}+\frac{c}{\delta} w_{1}^{\prime}+g(z) w_{1}=h_{1}(z)
$$

where

$$
g(z)=-\frac{1}{\delta} V_{0}(z) \quad \text { and } \quad h_{1}(z)=\frac{1}{\delta} \tilde{U}(z)\left(V_{0}(z)-\tilde{V}_{0}(z)\right) .
$$

Note that

$$
-\frac{1}{\delta} e^{\lambda l} \leqslant g \leqslant 0 \quad \text { and } \quad\left|h_{1}\right| \leqslant \frac{u^{*}}{\delta} \cdot\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}
$$

since $0 \leqslant V_{0} \leqslant V^{+} \leqslant\left\|V^{+}\right\|_{C\left(I_{l}\right)}=e^{\lambda l}$ and $0 \leqslant \tilde{U} \leqslant U^{+} \equiv u^{*}$. In addition, from definition of $\lambda$, we know that the value of $\lambda$ depends only on $u^{*}$ and $c$. Then Lemma A. 1 in Appendix A asserts that there exists a positive constant $C_{1}$, depending only on $\delta, u^{*}, c$, and $l$, such that

$$
\left\|w_{1}\right\|_{C\left(I_{l}\right)} \leqslant C_{1} \cdot\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}
$$

which, together with definition of $w_{1}$, implies that

$$
\begin{equation*}
\|U-\tilde{U}\|_{C\left(I_{l}\right)} \leqslant C_{1} \cdot\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)} . \tag{3.27}
\end{equation*}
$$

Next, we consider the function $w_{2}=V-\tilde{V}$. One can easily see that $w_{2}$ satisfies $w_{2}(-l)=$ $w_{2}(l)=0$ and

$$
w_{2}^{\prime \prime}+c w_{2}^{\prime}=h_{2}(z),
$$

where

$$
h_{2}=\tilde{U}_{0} \tilde{V}_{0}-U_{0} V_{0}+K\left(V_{0}^{q}-\tilde{V}_{0}^{q}\right)
$$

Note that

$$
\begin{equation*}
h_{2}=\tilde{V}_{0}\left(\tilde{U}_{0}-U_{0}\right)+U_{0}\left(\tilde{V}_{0}-V_{0}\right)+K\left(V_{0}^{q}-\tilde{V}_{0}^{q}\right) . \tag{3.28}
\end{equation*}
$$

Since $0 \leqslant V_{0}, \tilde{V}_{0} \leqslant\left\|V^{+}\right\|_{C\left(I_{l}\right)}=e^{\lambda l}$, we can apply the mean-value theorem to get that

$$
\left|V_{0}^{q}-\tilde{V}_{0}^{q}\right| \leqslant q e^{\lambda l(q-1)}\left|\tilde{V}_{0}-V_{0}\right| .
$$

Together with the fact that

$$
\left|\tilde{V}_{0}\right| \leqslant e^{\lambda l} \quad \text { and } \quad\left|U_{0}\right| \leqslant u^{*},
$$

we deduce from (3.28) that

$$
\left|h_{2}\right| \leqslant e^{\lambda l}\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left(u^{*}+K q e^{\lambda l(q-1)}\right)\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)} .
$$

Then Lemma A. 1 in Appendix A asserts that there exists a positive constant $C_{2}$, depending only on $u^{*}, c, K, q$, and $l$, such that

$$
\left\|w_{2}\right\|_{C\left(I_{l}\right)} \leqslant C_{2}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right)
$$

which, together with definition of $w_{2}$, implies that

$$
\begin{equation*}
\|V-\tilde{V}\|_{C\left(I_{l}\right)} \leqslant C_{2}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right) \tag{3.29}
\end{equation*}
$$

Finally, we use (3.26), (3.27), (3.29), and definition of the norm $\|\cdot\|_{X}$ to deduce that

$$
\begin{align*}
\| \mathcal{F} & \left(U_{0}, V_{0}\right)-\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right) \|_{X} \\
\quad & =\|(U, V)-(\tilde{U}, \tilde{V})\|_{X} \\
& =\|U-\tilde{U}\|_{C\left(I_{l}\right)}+\|V-\tilde{V}\|_{C\left(I_{l}\right)} \\
& \leqslant C_{3}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right) \\
& =C_{3}\left\|\left(U_{0}, V_{0}\right)-\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}, \tag{3.30}
\end{align*}
$$

where $C_{3}=C_{1}+C_{2}$. Thus, for a given $\epsilon>0$, we choose $0<\delta_{1}<\epsilon / C_{3}$. Then, by (3.30), we have

$$
\left\|\mathcal{F}\left(U_{0}, V_{0}\right)-\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}<\epsilon,
$$

for any $\left(U_{0}, V_{0}\right),\left(\tilde{U}_{0}, \tilde{V}_{0}\right) \in E$ such that $\left\|\left(U_{0}, V_{0}\right)-\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}<\delta_{1}$. This shows that $\mathcal{F}$ is a continuous mapping. Hence the proof of this lemma is completed.

## Lemma 3.9. $\mathcal{F} E$ is precompact.

Proof. For a given sequence $\left\{\left(U_{0, n}, V_{0, n}\right)\right\}_{n \in \mathbb{N}}$ in $E$, let $\left(U_{n}, V_{n}\right)=\mathcal{F}\left(U_{0, n}, V_{0, n}\right)$. Since $U^{-}$ and $U^{+}$are bounded in $I_{l}$, we can easily see from definition of the set $E$ and Lemma 3.7 that the sequences

$$
\left\{U_{0, n}\right\}, \quad\left\{V_{0, n}\right\}, \quad\left\{U_{n}\right\}, \quad\left\{V_{n}\right\}, \quad\left\{U_{n} V_{0, n}\right\}, \quad\left\{U_{0, n} V_{0, n}\right\} \quad \text { and } \quad\left\{V_{0, n}^{q}\right\}
$$

are uniformly bounded in $I_{l}$. Then, by Lemma A. 2 in Appendix A, it follows that the sequences

$$
\left\{U_{n}^{\prime}\right\} \quad \text { and }\left\{V_{n}^{\prime}\right\}
$$

are also uniformly bounded in $I_{l}$. Therefore, we can use Arzela-Ascoli theorem to get a subsequence $\left\{\left(U_{n_{j}}, V_{n_{j}}\right)\right\}$ of $\left\{\left(U_{n}, V_{n}\right)\right\}$ such that

$$
\left(U_{n_{j}}, V_{n_{j}}\right) \rightarrow(U, V),
$$

uniformly in $I_{l}$ as $j \rightarrow \infty$, for some $(U, V) \in E$. Hence the set $\mathcal{F} E$ is precompact.
Since $\mathcal{F}$ is a continuous mapping of $E$ into itself such that the image $\mathcal{F} E$ is precompact, it follows from the Schauder fixed point theorem that $\mathcal{F}$ has a fixed point, which is a nonnegative solution of system (3.1)-(3.2). So we have the following lemma.

Lemma 3.10. If $c>c^{*}$, then system (3.1)-(3.2) admits a solution ( $U, V$ ) satisfying

$$
\begin{equation*}
0 \leqslant U^{-} \leqslant U \leqslant u^{*} \quad \text { and } \quad 0 \leqslant V^{-} \leqslant V \leqslant V^{+} \tag{3.31}
\end{equation*}
$$

on $I_{l}$, and $U^{\prime}>0$ in $(-l, l)$.

## 4. Existence of traveling waves of system (1.2)

Again, for simplicity, throughout this section, we assume $x_{0}=x_{1}=0$, and write $U_{\lambda}^{-}\left(z ; x_{0}\right)$, $V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)$, and $V_{\lambda}^{+}\left(z ; x_{0}\right)$ as $U^{-}(z), V^{-}(z)$, and $V^{+}(z)$, respectively.

Now we are in a position to prove Theorem 1. First, we defer the proof of the assertion of Theorem 1(i) and (iv) to Appendix A. Second, the proof of the assertion of Theorem 1(ii) is given in Lemma 2.1(i). Hence we only need to establish the existence and the associated properties of traveling wave solutions. We divide the proof into two parts. Precisely, the proof for the case of non-critical waves $\left(c>c^{*}\right)$ is given in Section 4.1, and the case of the critical wave $\left(c=c^{*}\right)$ is shown in Section 4.2.

### 4.1. Existence of non-critical waves of system (1.2)

We first establish the existence of non-critical traveling waves of system (1.2).
Lemma 4.1. Let $q>1$. For a given $u^{*}>0$, if $c>c^{*}$, then system (1.4)-(1.5) admits a solution ( $U, V$ ) satisfying (3.31) on $\mathbb{R}$. Moreover, $0<U, V<u^{*}$ on $\mathbb{R}, U^{\prime}>0$ on $\mathbb{R}, V^{\prime}>0$ on $\left(-\infty, \xi_{0}\right)$ and $V^{\prime}<0$ on $\left(\xi_{0},+\infty\right)$ for some $\xi_{0} \in \mathbb{R}$, and $(U, V)(-\infty)=(0,0)$. Further, we have $V(z)=$ $\mathcal{O}\left(e^{-\lambda z}\right)$ as $z \rightarrow \infty$ where $\lambda$ is given by (1.6).

Proof. Let $\left\{l_{n}\right\}_{n \in \mathbb{N}}$ be an increasing sequence in $\left(z_{1}, \infty\right)$ such that $l_{n} \rightarrow \infty$ as $n \rightarrow \infty$ and let $\left(U_{n}, V_{n}\right), n \in \mathbb{N}$, be a solution of system (3.1)-(3.2) with $l=l_{n}$. For any fixed $N \in \mathbb{N}$, since the function $V^{+}$is bounded above in $\left[-l_{N}, l_{N}\right]$, it follows from (3.31) that the sequences

$$
\left\{U_{n}\right\}_{n \geqslant N}, \quad\left\{V_{n}\right\}_{n \geqslant N}, \quad\left\{U_{n} V_{n}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}^{q}\right\}_{n \geqslant N}
$$

are uniformly bounded in $\left[-l_{N}, l_{N}\right]$. Then we can use [11, Lemma 3.3] to infer that the sequences

$$
\left\{U_{n}^{\prime}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}^{\prime}\right\}_{n \geqslant N}
$$

are also uniformly bounded in $\left[-l_{N}, l_{N}\right]$. Using (3.1), we can express $U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$ in terms of $U_{n}$, $V_{n}, U_{n}^{\prime}$ and $V_{n}^{\prime}$. Differentiating (3.1), we can use the resulting equations to express $U_{n}^{\prime \prime \prime}$ and $V_{n}^{\prime \prime \prime}$ in terms of $U_{n}, V_{n}, U_{n}^{\prime}, V_{n}^{\prime}, U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$. Consequently, the sequences

$$
\left\{U_{n}^{\prime \prime}\right\}_{n \geqslant N}, \quad\left\{V_{n}^{\prime \prime}\right\}_{n \geqslant N}, \quad\left\{U_{n}^{\prime \prime \prime}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}^{\prime \prime \prime}\right\}_{n \geqslant N}
$$

are uniformly bounded in $\left[-l_{N}, l_{N}\right]$. With the aid of Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence $\left\{\left(U_{n_{j}}, V_{n_{j}}\right)\right\}$ of $\left\{\left(U_{n}, V_{n}\right)\right\}$ such that

$$
U_{n_{j}} \rightarrow U, \quad U_{n_{j}}^{\prime} \rightarrow U^{\prime}, \quad U_{n_{j}}^{\prime \prime} \rightarrow U^{\prime \prime}
$$

and

$$
V_{n_{j}} \rightarrow V, \quad V_{n_{j}}^{\prime} \rightarrow V^{\prime}, \quad V_{n_{j}}^{\prime \prime} \rightarrow V^{\prime \prime},
$$

uniformly in any compact interval of $\mathbb{R}$ as $n \rightarrow \infty$, for some functions $U$ and $V$ in $C^{2}(\mathbb{R})$. Then it is easy to see that $(U, V)$ is a nonnegative solution of system (1.4) and satisfies (3.31) and $U^{\prime} \geqslant 0$ over $\mathbb{R}$. From the definitions of $U^{-}$and $V^{+}$, we see that $U^{-}(z) \rightarrow u^{*}$ and $V^{+}(z) \rightarrow 0$ as $z \rightarrow \infty$. This, together with (3.31), implies that $V(z)=\mathcal{O}\left(e^{-\lambda z}\right)$ as $z \rightarrow \infty$ where $\lambda$ is given by (1.6). Hence we have

$$
\begin{equation*}
(U, V)(+\infty)=\left(u^{*}, 0\right) \tag{4.1}
\end{equation*}
$$

Furthermore, we claim that $U, V>0$ and $U^{\prime}>0$ over $\mathbb{R}$. For contradiction, we assume that $U\left(\tilde{z}_{1}\right)=0$ for some $\tilde{z}_{1} \in \mathbb{R}$. In this case, $U^{\prime}\left(\tilde{z}_{1}\right)=0$. Then the uniqueness gives that $U \equiv 0$, which contradicts the fact that $U(\infty)=u^{*}>0$. Hence $U>0$ over $\mathbb{R}$. Arguing as above and noting that $V \geqslant V^{-}>0$ on $\left(z_{1}, \infty\right)$, we also have $V>0$ over $\mathbb{R}$. To prove $U^{\prime}>0$ over $\mathbb{R}$, we also use a contradictory argument and assume that $U^{\prime}\left(\tilde{z}_{2}\right)=0$ for some $\tilde{z}_{2} \in \mathbb{R}$. In this case, we have $U^{\prime \prime}\left(\tilde{z}_{2}\right)=0$ since $U^{\prime} \geqslant 0$ over $\mathbb{R}$. Using (1.4), we get $U\left(\tilde{z}_{2}\right) V\left(\tilde{z_{2}}\right)=0$, which contradicts the positivity of $U$ and $V$.

Now it remains to show that $V<u^{*}$ over $\mathbb{R}$ and $(U, V)(-\infty)=(0,0)$. We divide the proof into four steps:

Step 1: We claim that

$$
\begin{equation*}
U^{\prime}(+\infty)=0 \quad \text { and } \quad V^{\prime}(+\infty)=0 \tag{4.2}
\end{equation*}
$$

Integrating Eq. (1.4a) from $s$ to $z$ gives that

$$
\begin{equation*}
U^{\prime}(z)=e^{-\frac{c}{\delta}(z-s)} U^{\prime}(s)+\frac{1}{\delta} e^{-\frac{c}{\delta} z} \int_{s}^{z} e^{\frac{c}{\delta} \tau} U(\tau) V(\tau) d \tau \tag{4.3}
\end{equation*}
$$

From the equality (4.3) one immediately deduces that, by fixing $s$ and letting $z \rightarrow \infty$,

$$
\limsup _{z \rightarrow \infty}\left|U^{\prime}(z)\right| \leqslant \frac{1}{\delta}\left(\max _{\tau \geqslant s}(U(\tau) V(\tau))\right) \cdot \limsup _{z \rightarrow \infty} e^{-\frac{c}{\delta} z} \int_{s}^{z} e^{\frac{c}{\delta} \tau} d \tau \leqslant \frac{1}{c} \max _{\tau \geqslant s}(U(\tau) V(\tau))
$$

for $s \in \mathbb{R}$. Together with the fact that $U(\infty) V(\infty)=0$, we can deduce that $U^{\prime}(\infty)=0$. Similarly, integrating Eq. (1.4b) from 0 to $z$ and arguing as above, we also get $V^{\prime}(\infty)=0$.

Step 2: We claim that

$$
\begin{equation*}
U(-\infty)=u_{*} \quad \text { and } \quad U^{\prime}(-\infty)=0 \tag{4.4}
\end{equation*}
$$

for some $u_{*} \in\left[0, u^{*}\right)$. Since $U$ is increasing and $0<U \leqslant u^{*}$, it follows that $u_{*}:=U(-\infty)$ exists and $0 \leqslant u_{*} \leqslant u^{*}$. Moreover, $U(-\infty) \neq u^{*}$ since $U^{\prime}>0$ over $\mathbb{R}$. Hence $u_{*} \in\left[0, u^{*}\right)$. Now we show that $U^{\prime}(-\infty)=0$. Integrating Eq. (1.4a) from $z$ to $\infty$ and recalling that $U(\infty)=u^{*}$ and $U^{\prime}(\infty)=0$, we get

$$
\begin{equation*}
-\delta U^{\prime}(z)+c\left[u^{*}-U(z)\right]=\int_{z}^{\infty} U(\tau) V(\tau) d \tau \tag{4.5}
\end{equation*}
$$

Since $U>0$ and $U^{\prime}>0$, Eq. (4.5) implies that

$$
\int_{z}^{\infty} U(\tau) V(\tau) d \tau \leqslant c u^{*}
$$

Hence the improper integral

$$
\int_{-\infty}^{\infty} U(\tau) V(\tau) d \tau
$$

converges. Letting $z \rightarrow-\infty$ in (4.5) and recalling the fact that $U(-\infty)$ exists, we infer that $U^{\prime}(-\infty)$ exists. Furthermore, since $U^{\prime}>0$, it follows that $U^{\prime}(-\infty) \geqslant 0$. Indeed, $U^{\prime}(-\infty)=0$. Otherwise, $U^{\prime}(-\infty)>0$, which implies $U(-\infty)=-\infty$, a contradiction to the fact that $U(-\infty)$ exists.

Step 3: We show that $V<u^{*}$ on $\mathbb{R}$. Summing up (1.4a) and (1.4b), we deduce that

$$
\begin{equation*}
\delta U^{\prime \prime}+V^{\prime \prime}+c\left(U^{\prime}+V^{\prime}\right)=K V^{q} \geqslant 0 \quad \text { on } \mathbb{R}, \tag{4.6}
\end{equation*}
$$

which implies that the function $\delta U^{\prime}+V^{\prime}+c(U+V)$ is nondecreasing. Since (4.1) and (4.2) imply that $\delta U^{\prime}+V^{\prime}+c(U+V) \rightarrow c u^{*}$ as $z \rightarrow \infty$, we obtain that

$$
\delta U^{\prime}+V^{\prime}+c(U+V) \leqslant c u^{*} \quad \text { on } \mathbb{R},
$$

and therefore,

$$
\begin{equation*}
\delta U^{\prime}+V^{\prime}+c\left(U+V-u^{*}\right) \leqslant 0 \quad \text { on } \mathbb{R} . \tag{4.7}
\end{equation*}
$$

For $\delta \geqslant 1$, we consider the function $W_{1}:=U+V-u^{*}$. Since $U \leqslant u^{*}$, we can use (3.31) to get that

$$
\begin{equation*}
W_{1}(z) \leqslant V(z) \leqslant V^{+}(z) \leqslant e^{-\lambda z}, \quad \forall z \in \mathbb{R} . \tag{4.8}
\end{equation*}
$$

In addition, since $U^{\prime}>0$, we can use (4.7) to deduce that

$$
W_{1}^{\prime}+c W_{1}=(1-\delta) U^{\prime} \leqslant 0 \quad \text { on } \mathbb{R},
$$

which follows that $\left[e^{c z} W_{1}(z)\right]^{\prime} \leqslant 0$ over $\mathbb{R}$. This, together with (4.8), implies that

$$
e^{c z} W_{1}(z) \leqslant e^{c z^{*}} W_{1}\left(z^{*}\right) \leqslant e^{(c-\lambda) z^{*}}
$$

for any $-\infty<z^{*}<z<\infty$. Letting $z^{*} \rightarrow-\infty$ in the above inequality and noting that $e^{(c-\lambda) z^{*}} \rightarrow 0$ due to $\lambda<c$, we get $W_{1}(z) \leqslant 0$ and therefore $U+V \leqslant u^{*}$ on $\mathbb{R}$. This, together with the fact that $U>0$, implies that $V<u^{*}$ on $\mathbb{R}$. Now we consider the case $0<\delta<1$. Set $W_{2}:=\delta U+V-u^{*}$. Since $U \leqslant u^{*}$ and $\delta<1$, it follows from (3.31) that

$$
W_{2}(z) \leqslant V(z) \leqslant V^{+}(z) \leqslant e^{-\lambda z}, \quad \forall z \in \mathbb{R}
$$

In addition, since $c>0, \delta<1$, and $U>0$, we can use (4.7) to deduce that

$$
W_{2}^{\prime}+c W_{2}=c(\delta-1) U \leqslant 0 \quad \text { on } \mathbb{R}
$$

Arguing as the proof for $W_{1} \leqslant 0$, we can easily get $W_{2} \leqslant 0$ and therefore $\delta U+V \leqslant u^{*}$ on $\mathbb{R}$. This, together with the fact that $U>0$, implies that $V<u^{*}$ on $\mathbb{R}$.

Step 4: We show that $V(-\infty)=0$. To this end, we first claim that $B^{\star}:=\left(V^{\prime}+c V\right)(-\infty)$ exists. Integrating (4.6) over $\mathbb{R}$ and using (4.1), (4.2), and (4.4), we obtain that

$$
\begin{equation*}
c\left(u^{*}-u_{*}\right)-\left(V^{\prime}+c V\right)(-\infty)=K \int_{-\infty}^{\infty} V^{q}(z) d z \tag{4.9}
\end{equation*}
$$

Note that the improper integral

$$
\begin{equation*}
\int_{-\infty}^{\infty} V^{q}(z) d z \tag{4.10}
\end{equation*}
$$

converges since otherwise it diverges to $\infty$ and therefore $\left(V^{\prime}+c V\right)(-\infty)=-\infty$. This, together with the boundedness of $V$, implies that $V^{\prime}(-\infty)=-\infty$, which contradicts the boundedness of $V$. Hence the limit $\left(V^{\prime}+c V\right)(-\infty)$ exists.

Now we show that $V(-\infty)=0$. Since $V>0$ on $\mathbb{R}$, the convergence of the improper integral (4.10) implies that $\liminf _{z \rightarrow-\infty} V(z)=0$. Recall that $V$ is bounded. For contradiction we assume that $\xi:=\lim \sup _{z \rightarrow-\infty} V(z)$ is a positive number. Select two sequences $\left\{s_{n}\right\}_{n \in \mathbb{N}} \searrow-\infty$ and $\left\{t_{n}\right\}_{n \in \mathbb{N}} \searrow-\infty$ such that $s_{n}>t_{n}>s_{n+1}, V\left(s_{n}\right)<\xi / 2, V\left(t_{n}\right)>\xi / 2$ for all $n \in \mathbb{N}$, and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} V\left(s_{n}\right)=0 \quad \text { and } \quad \lim _{n \rightarrow \infty} V\left(t_{n}\right)=\xi \tag{4.11}
\end{equation*}
$$

For each $n \in \mathbb{N}$, the continuity of $V$ implies that there exist $s_{n}^{*} \in\left[s_{n+1}, s_{n}\right]$ and $t_{n}^{*} \in\left[t_{n+1}, t_{n}\right]$ such that

$$
V\left(s_{n}^{*}\right)=\max _{z \in\left[s_{n+1}, s_{n}\right]} V(z) \quad \text { and } \quad V\left(t_{n}^{*}\right)=\min _{z \in\left[t_{n}+1, t_{n}\right]} V(z) .
$$

It is easy to see that $s_{n}^{*}$ and $t_{n}^{*}$ are critical points, so that

$$
\begin{equation*}
V^{\prime}\left(s_{n}^{*}\right)=0 \quad \text { and } \quad V^{\prime}\left(t_{n}^{*}\right)=0 \tag{4.12}
\end{equation*}
$$

Since $s_{n+1} \in\left[t_{n+1}, t_{n}\right]$, the minimality of $V$ at $t_{n}^{*}$ gives that $0<V\left(t_{n}^{*}\right) \leqslant V\left(s_{n+1}\right)$. Together with (4.11), we get that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} V\left(t_{n}^{*}\right)=0 \tag{4.13}
\end{equation*}
$$

Using (4.12), (4.13), and the fact that $B^{\star}=\left(V^{\prime}+c V\right)(-\infty)=\lim _{n \rightarrow \infty}\left(V^{\prime}+c V\right)\left(t_{n}^{*}\right)$, we get $B^{\star}=0$. Hence $\lim _{n \rightarrow \infty}\left(V^{\prime}+c V\right)\left(s_{n}^{*}\right)=\left(V^{\prime}+c V\right)(-\infty)=0$. Then, by (4.12), we conclude that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} V\left(s_{n}^{*}\right)=0 \tag{4.14}
\end{equation*}
$$

Since $t_{n} \in\left[s_{n+1}, s_{n}\right]$, the maximality of $V$ at $s_{n}^{*}$ gives that $V\left(s_{n}^{*}\right) \geqslant V\left(t_{n}\right) \geqslant 0$. Together with (4.14), we get $\lim _{n \rightarrow \infty} V\left(t_{n}\right)=0$, which contradicts (4.11). Therefore, we have $V(-\infty)=0$.

Finally, the proof of the assertion that $U(-\infty)=0$ (i.e., $u_{*}=0$ ) follows a similar argument as that of [28, Proposition 4.2]. For the reader's convenience, we sketch it in Appendix A. 2 of Appendix A. For the assertion that $V^{\prime}>0$ on $\left(-\infty, \xi_{0}\right)$ and $V^{\prime}<0$ on $\left(\xi_{0},+\infty\right)$ for some $\xi_{0} \in \mathbb{R}$, the proof follows from that of [30], and so we omit it. Hence the proof of Theorem 1(iii) for the case of non-critical waves is completed.

### 4.2. Existence of critical waves of system (1.2)

Next we establish the existence of traveling waves of system (1.2) with critical speed $c=c^{*}$.
Lemma 4.2. Let $q>1$. For a given $u^{*}>0$, if $c=c^{*}$, then system (1.4)-(1.5) admits a solution $(U, V)$ on $\mathbb{R}$. Moreover, $0<U, V<u^{*}$ on $\mathbb{R}, U^{\prime}>0$ on $\mathbb{R}, V^{\prime}>0$ on $\left(-\infty, \xi_{0}\right)$ and $V^{\prime}<0$ on $\left(\xi_{0},+\infty\right)$ for some $\xi_{0} \in \mathbb{R}$, and $(U, V)(-\infty)=(0,0)$.

Proof. Firstly, we select a sequence $\left\{c_{n}\right\}_{n \in \mathbb{N}} \searrow c^{*}$. For each $n \in \mathbb{N}$, Lemma 3.2 asserts that the function $V_{n}^{+}(z):=e^{-\lambda_{n} z}$, where

$$
\lambda_{n}=\frac{1}{2} \cdot\left(c_{n}-\sqrt{c_{n}^{2}-4 u^{*}}\right)
$$

satisfies

$$
\left(V_{n}^{+}\right)^{\prime \prime}+c_{n}\left(V_{n}^{+}\right)^{\prime}+u^{*} V_{n}^{+}=0, \quad \forall z \in \mathbb{R}
$$

Choose $0<\gamma^{*}<\min \left\{c^{*} / \delta, \lambda_{1}\right\}$. Then $c^{*}-\delta \gamma^{*}>0$ and $\gamma^{*}-\lambda_{1}<0$. Since $e^{\left(\gamma^{*}-\lambda_{1}\right) z} \rightarrow 0$ as $z \rightarrow \infty$, there exists $z_{0}^{*}>0$ such that

$$
e^{\left(\gamma^{*}-\lambda_{1}\right) z} \leqslant \gamma^{*}\left(c^{*}-\delta \gamma^{*}\right), \quad \forall z \geqslant z_{0}^{*}
$$

which, together with the fact that $\lambda_{n}>\lambda_{1}$ and $c_{n}>c^{*}$, yields

$$
e^{\left(\gamma^{*}-\lambda_{n}\right) z} \leqslant \gamma^{*}\left(c_{n}-\delta \gamma^{*}\right), \quad \forall z \geqslant z_{0}^{*} .
$$

Hence,

$$
\left(c_{n}-\delta \gamma^{*}\right) \gamma^{*} e^{-\gamma^{*} z} \geqslant V_{n}^{+}(z), \quad \forall z \geqslant z_{0}^{*}
$$

Set $M^{*}=u^{*} e^{\gamma^{*} z_{0}^{*}}$. Then $M^{*}>u^{*}$ since $\gamma^{*}, z_{0}^{*}>0$. Hence Lemma 3.3 asserts that the function $\left(U^{*}\right)^{-}(z):=\max \left\{0, u^{*}-M^{*} e^{-\gamma^{*} z}\right\}$ satisfies the inequality

$$
\delta\left(\left(U^{*}\right)^{-}\right)^{\prime \prime}+c_{n}\left(\left(U^{*}\right)^{-}\right)^{\prime}-\left(U^{*}\right)^{-} V_{n}^{+} \geqslant 0, \quad \forall z \neq z_{0}^{*} \text { and } n \in \mathbb{N} .
$$

Then, for each $n \in \mathbb{N}$, Lemma 4.1 asserts that there exists a solution $\left(U_{n}, V_{n}\right)$ to system (1.4)-(1.5) with $c=c_{n}$ such that

$$
\begin{gather*}
0<U_{n}(z), V_{n}(z) \leqslant u^{*},  \tag{4.15}\\
U_{n}^{\prime}(z)>0, \tag{4.16}
\end{gather*}
$$

and

$$
\begin{equation*}
0 \leqslant\left(U^{*}\right)^{-}(z) \leqslant U_{n}(z) \leqslant u^{*}, \quad \text { and } \quad 0 \leqslant V_{n}(z) \leqslant V_{n}^{+}(z) \tag{4.17}
\end{equation*}
$$

for all $z \in \mathbb{R}$. Thus the sequences

$$
\left\{U_{n}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}\right\}_{n \geqslant N}
$$

are uniformly bounded in $\mathbb{R}$ and also we have

$$
\begin{align*}
& \delta U_{n}^{\prime \prime}+c_{n} U_{n}^{\prime}-U_{n} V_{n}=0 \\
& V_{n}^{\prime \prime}+c_{n} V_{n}^{\prime}+U_{n} V_{n}-K V_{n}^{q}=0 \tag{4.18}
\end{align*}
$$

in $\mathbb{R}$. For each $n \in \mathbb{N}$, since $U_{n}^{\prime}( \pm \infty)=0$ and $V_{n}^{\prime}( \pm \infty)=0$, there exists $\xi_{n}^{*}$ and $\eta_{n}^{*}$ such that

$$
\begin{equation*}
U_{n}^{\prime}\left(\xi_{n}^{*}\right)=\left\|U_{n}^{\prime}\right\|_{L(\mathbb{R})} \quad \text { and } \quad\left|V_{n}^{\prime}\left(\eta_{n}^{*}\right)\right|=\left\|V_{n}^{\prime}\right\|_{L(\mathbb{R})} \tag{4.19}
\end{equation*}
$$

Then $U_{n}^{\prime \prime}\left(\xi_{n}^{*}\right)=0$ and $V_{n}^{\prime \prime}\left(\eta_{n}^{*}\right)=0$. Together with (4.18) and (4.15), we obtain that

$$
\begin{equation*}
U_{n}^{\prime}\left(\xi_{n}^{*}\right)=\frac{1}{c_{n}} U_{n}\left(\xi_{n}^{*}\right) V_{n}\left(\xi_{n}^{*}\right) \leqslant \frac{1}{c^{*}}\left(u^{*}\right)^{2} \tag{4.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|V_{n}^{\prime}\left(\eta_{n}^{*}\right)\right|=\frac{1}{c_{n}}\left|-U_{n}\left(\eta_{n}^{*}\right) V_{n}\left(\eta_{n}^{*}\right)+K V_{n}\left(\eta_{n}^{*}\right)^{q}\right| \leqslant \frac{1}{c^{*}}\left[\left(u^{*}\right)^{2}+K\left(u^{*}\right)^{q}\right] . \tag{4.21}
\end{equation*}
$$

Hence (4.19), (4.20), and (4.21) give that

$$
\left\|U_{n}^{\prime}\right\|_{L(\mathbb{R})} \leqslant \frac{\left(u^{*}\right)^{2}}{c^{*}} \quad \text { and } \quad\left\|V_{n}^{\prime}\right\|_{L(\mathbb{R})} \leqslant \frac{1}{c^{*}}\left[\left(u^{*}\right)^{2}+K\left(u^{*}\right)^{q}\right], \quad \forall n \in \mathbb{N} .
$$

This means that the sequences

$$
\left\{U_{n}^{\prime}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}^{\prime}\right\}_{n \geqslant N}
$$

are also uniformly bounded in $\mathbb{R}$. Note that the sequence $\left\{c_{n}\right\}_{n} \geqslant{ }_{N}$ is bounded. Using (4.18), we can express $U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$ in terms of $c_{n}, U_{n}, V_{n}, U_{n}^{\prime}$ and $V_{n}^{\prime}$. Differentiating (4.18), we can use the resulting equations to express $U_{n}^{\prime \prime \prime}$ and $V_{n}^{\prime \prime \prime}$ in terms of $c_{n}, U_{n}, V_{n}, U_{n}^{\prime}, V_{n}^{\prime}, U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$. Consequently, the sequences

$$
\left\{U_{n}^{\prime \prime}\right\}_{n \geqslant N}, \quad\left\{V_{n}^{\prime \prime}\right\}_{n \geqslant N}, \quad\left\{U_{n}^{\prime \prime \prime}\right\}_{n \geqslant N} \quad \text { and } \quad\left\{V_{n}^{\prime \prime \prime}\right\}_{n \geqslant N}
$$

are uniformly bounded in $\mathbb{R}$. With the aid of Arzela-Ascoli theorem, we can use a diagonal process to get a subsequence $\left\{\left(U_{n_{j}}, V_{n_{j}}\right)\right\}$ of $\left\{\left(U_{n}, V_{n}\right)\right\}$ such that

$$
U_{n_{j}} \rightarrow U, \quad U_{n_{j}}^{\prime} \rightarrow U^{\prime}, \quad U_{n_{j}}^{\prime \prime} \rightarrow U^{\prime \prime}
$$

and

$$
V_{n_{j}} \rightarrow V, \quad V_{n_{j}}^{\prime} \rightarrow V^{\prime}, \quad V_{n_{j}}^{\prime \prime} \rightarrow V^{\prime \prime}
$$

uniformly in any compact interval of $\mathbb{R}$ as $n \rightarrow \infty$, for some functions $U$ and $V$ in $C^{2}(\mathbb{R})$. Then it is easy to see that $(U, V)$ is a solution of system (1.4). Moreover, (4.15), (4.16), and (4.17) implies that

$$
\begin{gathered}
0 \leqslant U(z), V(z) \leqslant u^{*} \\
U^{\prime}(z) \geqslant 0
\end{gathered}
$$

and

$$
\begin{equation*}
0 \leqslant\left(U^{*}\right)^{-}(z) \leqslant U(z) \leqslant u^{*}, \quad \text { and } \quad 0 \leqslant V(z) \leqslant\left(V^{*}\right)^{+}(z) \tag{4.22}
\end{equation*}
$$

for all $z \in \mathbb{R}$, where $\left(V^{*}\right)^{+}(z):=\lim _{n \rightarrow \infty} V_{n}^{+}(z)=e^{-c^{*} / 2 z}$. From the definitions of $\left(U^{*}\right)^{-}$and $\left(V^{*}\right)^{+}$, we see that $\left(U^{*}\right)^{-}(z) \rightarrow u^{*}$ and $\left(V^{*}\right)^{+}(z) \rightarrow 0$ as $z \rightarrow \infty$. This, together with (4.22), implies that $(U, V)(+\infty)=\left(u^{*}, 0\right)$.

Now we show that $(U(-\infty), V(-\infty))=(0,0)$. To do this, we first claim that $V \not \equiv 0$. Indeed, using (4.9) with $V=V_{n_{j}}$ and recalling that $u_{*}=0$, we have $c_{n_{j}} u^{*}=K \int_{-\infty}^{\infty} V_{n_{j}}^{q}(z) d z$ for $j \in \mathbb{N}$, which, together with Lebesgue dominated convergence theorem, yields

$$
\int_{-\infty}^{\infty} V^{q}(z) d z=c^{*} u^{*} / K>0
$$

Hence $V \not \equiv 0$. Next we claim that $V>0$ on $\mathbb{R}$. For contradiction, we assume that $V\left(z_{1}^{\star}\right)=0$ for some $z_{1}^{\star} \in \mathbb{R}$. Since $V(\cdot) \geqslant 0$ on $\mathbb{R}$, we have $V^{\prime}\left(z_{1}^{\star}\right)=0$. Then the uniqueness theorem for differential equations gives that $V \equiv 0$, which contradicts the fact that $V \not \equiv 0$. Hence $V>0$ over $\mathbb{R}$. Then with the aid of the fact that $V>0$ on $\mathbb{R}$, we can follow the argument in the paragraph right after Eq. (4.1) to deduce that $U>0$ and $U^{\prime}>0$ over $\mathbb{R}$. Finally, we can follow the proof of Lemma 4.1 to infer that $0<U, V<u^{*}$ on $\mathbb{R}, U^{\prime}>0$ on $\mathbb{R}, V^{\prime}>0$ on $\left(-\infty, \xi_{0}\right)$ and $V^{\prime}<0$ on $\left(\xi_{0},+\infty\right)$ for some $\xi_{0} \in \mathbb{R}$, and $(U, V)(-\infty)=(0,0)$. This completes the proof of Theorem 1 (iii) for the case of the critical wave.

## 5. Propagation of traveling waves

In this section, we give the proof of Theorem 2. The proof is based on the construction of positive functions propagating with the speed $c_{\lambda}$.

### 5.1. Comparison lemmas

To begin with, we can use the argument of [30] to establish the invariant region for the solution of system (1.2).

Lemma 5.1. Let $(u, v)$ be the solution of system (1.2) on $\mathbb{R} \times[0, T]$ with the initial data ( $u_{0}, v_{0}$ ) satisfying that $0 \leqslant u_{0}(\cdot) \leqslant u^{*}$ and $v_{0}(\cdot) \geqslant 0$ on $\mathbb{R}$. Then there exists a constant $\chi>0$, independent of $T$, such that

$$
0 \leqslant u(x, t) \leqslant u^{*} \quad \text { and } \quad 0 \leqslant v(x, t) \leqslant \chi \quad \text { for all } x \in \mathbb{R} \text { and } t \in[0, T]
$$

Proof. First, using the argument of [30, p. 270], we have that $0 \leqslant u(x, t) \leqslant u^{*}$ and $v(x, t) \geqslant 0$ for all $x \in \mathbb{R}$ and $t \in[0, T]$.

To establish the upper bound for $v$, we use the above assertion for $(u, v)$ and Eq. (1.2a) to deduce that $v(x, t)$ is a sub-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+u^{*} v-K v^{q} \tag{5.1}
\end{equation*}
$$

for $(x, t) \in \mathbb{R} \times(0, T]$. Since $q>1$, Eq. (5.1) is the generalized KPP equation. Taken together, we can find a constant $\chi>0$, independent of $T$, such that $v(x, t) \leqslant \chi$ for $(x, t) \in \mathbb{R} \times(0, T]$. This completes the proof of this lemma.

We remark that with the use of Lemma 5.1, we can follow the standard argument of [34] to establish the global existence and regularity of solutions of system (1.2) with the initial condition (1.3).

Next we show that if the $v$-component of the initial data (1.3) is squeezed between $V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ and $V_{\lambda}^{+}\left(\cdot ; x_{0}\right)$ for some $x_{0}$ and $x_{1} \in \mathbb{R}$, then the solution of system (1.2) with the initial data (1.3) is squeezed between $\left(U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right), V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)\right)$ and $\left(u^{*}, V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)\right)$ for all $t>0$.

Lemma 5.2. Let $(u, v)$ be the solution of system (1.2) on $\mathbb{R} \times[0, \infty)$ with initial data $\left(u_{0}, v_{0}\right)$ satisfying

$$
\begin{equation*}
u_{0}(x)=u^{*} \quad \text { and } \quad V_{\lambda}^{-}\left(x ; x_{1}, x_{0}\right) \leqslant v_{0}(x) \leqslant V_{\lambda}^{+}\left(x ; x_{0}\right) \tag{5.2}
\end{equation*}
$$

for all $x \in \mathbb{R}$, and for some $x_{0}, x_{1} \in \mathbb{R}$ and $\lambda \in\left(0, \sqrt{u^{*}}\right)$. Then we have

$$
\begin{align*}
& U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right) \leqslant u(x, t) \leqslant u^{*}  \tag{5.3a}\\
& V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right) \leqslant v(x, t) \leqslant V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right) \tag{5.3b}
\end{align*}
$$

for all $(x, t) \in \mathbb{R} \times[0, \infty)$.
Proof. We first establish (5.3b). To do this, we use Lemma 5.1 to deduce that $0 \leqslant u(x, t) \leqslant u^{*}$ and $v(x, t) \geqslant 0$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. Hence from Eq. (1.2b), $v(x, t)$ is a sub-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+u^{*} v \tag{5.4}
\end{equation*}
$$

From Lemma 3.2, $V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)$ is a solution of Eq. (5.4). Together with the fact that $v_{0}(\cdot) \leqslant$ $V_{\lambda}^{+}\left(\cdot ; x_{0}\right)$ on $\mathbb{R}$, we can apply the comparison principle to Eq. (5.4) to deduce that $v(x, t) \leqslant$ $V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$.

Now we prove $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. For this, we use Eq. (1.2a) and the right-hand side inequality of (5.3b) to conclude that $u(x, t)$ is a super-solution of the equation

$$
\begin{equation*}
u_{t}=\delta u_{x x}-u V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right) \tag{5.5}
\end{equation*}
$$

From Lemma 3.3, $U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ is a sub-solution of Eq. (5.5). Hence, together with the fact that $u_{0}(\cdot) \geqslant U_{\lambda}^{-}\left(\cdot ; x_{0}\right)$ on $\mathbb{R}$, we can apply the comparison principle to Eq. (5.5) to conclude that $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$.

Finally we show $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. To do this, we recall the definition of $z_{1}$ given in Section 3.2.1. For each $t \geqslant 0$, we set $x_{t}:=c_{\lambda} t+z_{1}$. Then for $(x, t)$ with $x \leqslant x_{t}$, since $V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)=0$, it is obvious that $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$. Next we consider the region $\left\{(x, t): x \geqslant x_{t}, t \geqslant 0\right\}$. Note that Eq. (5.3a), the right-hand side inequality of (5.3b), and Eq. (1.2b) yield that $v(x, t)$ is a super-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right) v-K\left(V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)\right)^{q} . \tag{5.6}
\end{equation*}
$$

From Lemma 3.4, $V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ is a sub-solution of Eq. (5.6). Recall that $v_{0}(\cdot) \geqslant$ $V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ on $\mathbb{R}$ and $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for $(x, t)=\left(x_{t}, t\right)$ and $t \geqslant 0$. Therefore, with the aid of the maximum principle, we have $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for $x \geqslant x_{t}$ and $t \geqslant 0$. Together with the fact that $V_{\lambda}^{-}\left(z_{1} ; x_{1}, x_{0}\right)=0$, we conclude that $v(x, t) \geqslant V_{\lambda}^{-}(x-$ $\left.c_{\lambda} t ; x_{1}, x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. The proof of this lemma is thus completed.

### 5.2. The proof of Theorem 2

Now we are in a position to establish Theorem 2. It suffices to show assertions (i), (ii), and (iii) of Theorem 2 since the other three assertions can be proven in a similar way. Now using
condition (1.3), we can find sufficiently large numbers $x_{0}$ and $x_{1}$ with $x_{1}<0$ such that condition (5.2) in Lemma 5.2 holds. Hence Lemma 5.2 and the definitions of $U_{\lambda^{+}}^{-}$and $V_{\lambda^{+}}^{ \pm}$imply that

$$
\begin{gather*}
\max \left\{0, u^{*}-M e^{-\gamma\left(x-c_{\lambda}+t\right)}\right\} \leqslant u(x, t) \leqslant u^{*} \\
\max \left\{0, e^{-\lambda^{+}\left(x-c_{\lambda}+t-x_{1}\right)}-L e^{-\left(\lambda^{+}+\eta\right)\left(x-c_{\lambda}+t\right)}\right\} \leqslant v(x, t) \leqslant e^{-\lambda^{+}\left(x-c_{\lambda}+t-x_{0}\right)} \tag{5.7}
\end{gather*}
$$

for all $(x, t) \in \mathbb{R} \times[0, \infty)$, where $M=M\left(x_{0}\right), L=L\left(x_{1}, x_{0}\right)$ and $\eta$ are positive constants defined in Section 3.2.1. Set $\psi_{\lambda^{+}}^{+}:=\max \left\{0, u^{*}-M e^{-\gamma x}\right\}$ and $\phi_{\lambda^{+}}^{+}(x):=\max \left\{0, e^{-\lambda^{+}\left(x-x_{1}\right)}-\right.$ $\left.L e^{-\left(\lambda^{+}+\eta\right) x}\right\}$. Then assertions (i), (ii), and (iii) of Theorem 2 follows from the inequality (5.7).

## Acknowledgments

The authors also would like to thank the referees for a careful reading of the manuscript and many helpful suggestions. The work is partially supported by National Science Council and National Center of Theoretical Sciences of Taiwan.

## Appendix A

## A.1. A priori estimates for the inhomogeneous linear equation

In this section, we collect some a priori estimates in [11] for solutions of the inhomogeneous linear equation

$$
\begin{equation*}
w^{\prime \prime}(z)+A w^{\prime}(z)+g(z) w(z)=h(z) \tag{A.1}
\end{equation*}
$$

Lemma A.1. (See Lemma 3.2 of [11].) Let A be a positive constant and let $g$ and $h$ be continuous functions on $[a, b]$. Suppose that $w \in C([a, b]) \cap C^{2}((a, b))$ satisfies the differential equation (A.1) in $(a, b)$ and $w(a)=w(b)=0$. If

$$
-C_{1} \leqslant g \leqslant 0 \quad \text { and } \quad|h| \leqslant C_{2} \quad \text { on }[a, b],
$$

for some constants $C_{1}, C_{2}$, then there exists a positive constant $C_{3}$, depending only on $A, C_{1}$, and the length of the interval $[a, b]$, such that

$$
\begin{equation*}
\|w\|_{C([a, b])} \leqslant C_{2} C_{3} . \tag{A.2}
\end{equation*}
$$

Lemma A.2. (See Lemma 3.3 of [11].) Let A, g, and h be as in Lemma A.1. Suppose that $w \in C([a, b]) \cap C^{2}((a, b))$ satisfies (A.1) in $(a, b)$. If $\|w\|_{C([a, b])} \leqslant K_{0}$ for some constant $C_{0}$, then there exists a positive constant $C_{4}$, depending only on $A, C_{0}, C_{1}, C_{2}$, and the length of the interval $[a, b]$, such that

$$
\begin{equation*}
\left\|w^{\prime}\right\|_{C([a, b])} \leqslant C_{4} . \tag{A.3}
\end{equation*}
$$

## A.2. The proof for $U(-\infty)=0$

In this section, we show the assertion $U(-\infty)=0$ which is stated in Lemma 4.1 and which is equivalent to the assertion $u_{*}=0$. Again the proof follows the lines of [28, Proposition 4.2]. For the reader's convenience, we sketch it here. For contradiction, we assume that $u_{*}>0$. In view of the fact that $q>1$ and $V(-\infty)=0$, we can choose a $z_{*}<0$ such that $K V^{q-1}(z)-U(z)<0$ for $z \leqslant z_{*}$. Now integrating (1.4b) from $-\infty$ to $z \leqslant z_{*}$, we obtain

$$
V^{\prime}(z)+c V(z)=\int_{-\infty}^{z}\left(K V^{q-1}(\tau)-U(\tau)\right) V(\tau) d \tau<0
$$

for $z \leqslant z_{*}$, which yields $\left(e^{c z} V(z)\right)^{\prime}<0$ for $z \leqslant z_{*}$. Together with $V(-\infty)=0$, this implies that $V(z)<0$ for $z \leqslant z_{*}$, a contradiction. Hence the assertion that $u_{*}=0$ is established.

## A.3. The uniqueness of waves of system (1.2) for $q \in(1,2]$

In this section, we establish the uniqueness of waves of system (1.2) for $q \in(1,2]$. We note that the uniqueness of waves with large wave speed for the case $q=2$ is shown in [28]. Since the argument is similar to that of [28, pp. 543-547], we will only state the necessary ingredients and modification.

To begin with, we denote the problem (1.4)-(1.5) as $P\left[u^{*}, c, K\right]$ to specify the dependence of the problem (1.4)-(1.5) on the parameters $\left(u^{*}, c, K\right)$. Then we have the crucial scaling observation, as stated below.

Lemma A.3. Let $u^{*}, c$, and $K$ be positive numbers. Suppose that the problem $P\left[u^{*}, c, K\right]$ has a unique solution, then the problem $P\left[\epsilon,\left(\left(u^{*}\right)^{-\frac{1}{2}} c\right) \epsilon^{\frac{1}{2}},\left(\left(u^{*}\right)^{q-2} K\right) \epsilon^{2-q}\right]$, where $\epsilon>0$, also has a unique solution and vice versa.

Proof. For $\epsilon>0$, consider the invertible linear transformation $T_{\epsilon}:(U, V, z)^{t} \rightarrow(\hat{U}, \hat{V}, \hat{z})^{t}$ by

$$
\hat{U}=\epsilon\left(u^{*}\right)^{-1} U, \quad \hat{V}=\epsilon\left(u^{*}\right)^{-1} V, \quad \hat{z}=\epsilon^{-\frac{1}{2}}\left(u^{*}\right)^{\frac{1}{2}} z
$$

One can verify that $T_{\epsilon}$ transforms the solution of the problem $P\left[u^{*}, c, K\right]$ into the solution of the problem $P\left[\epsilon,\left(\left(u^{*}\right)^{-\frac{1}{2}} c\right) \epsilon^{\frac{1}{2}},\left(\left(u^{*}\right)^{q-2} K\right) \epsilon^{2-q}\right]$, and vice versa via the transformation $T_{\epsilon}^{-1}$. Hence the assertions of the lemma follows.

Next, by putting

$$
\mathrm{u}:=U, \quad \mathrm{v}:=V, \quad \mathrm{w}:=U_{z}, \quad \mathrm{y}:=V_{z},
$$

we rewrite system (1.4) as the first-order system:

$$
\begin{equation*}
\mathrm{u}_{z}=\mathrm{w}, \quad \mathrm{v}_{z}=\mathrm{y}, \quad \mathrm{w}_{z}=-\frac{c}{\delta} \mathrm{w}+\frac{1}{\delta} \mathrm{uv}, \quad \mathrm{y}_{z}=-c \mathrm{y}-\mathrm{uv}+K \mathrm{v}^{q} . \tag{A.4}
\end{equation*}
$$

Note that each point $(u, 0,0,0)$ with $u \geqslant 0$ is an equilibrium point of system (A.4). With this observation, a nonnegative solution of $P\left[u^{*}, c, K\right]$ is equivalent to a heteroclinic orbit of system (A.4) connecting $(0,0,0,0)^{t}$ to $\left(u^{*}, 0,0,0\right)$ which lies entirely in the region $\{(u, v, w, y): u \geqslant 0, v \geqslant 0\}$. Hence there is a one-to-one correspondence between nonnegative solutions of $P\left[u^{*}, c, K\right]$ and solutions of $Q\left[u^{*}, c, K\right]$, where $Q\left[u^{*}, c, K\right]$ denotes the problem consisting of system (A.4), the constraint that $(\mathrm{u}, \mathrm{v})(z) \geqslant 0$ for $z \in \mathbb{R}$, and the boundary conditions that $(\mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{y})(-\infty)=(0,0,0,0)$ and $(\mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{y})(\infty)=\left(u^{*}, 0,0,0\right)$.

Now we explore the dynamical behavior of system (A.4) around the origin. Indeed, the linearization of system (A.4) around the origin reveals that the eigenvalues are $0,0,-c$, and $-c / \delta$. Hence the local behavior of system (A.4) around the origin can be determined by the local dynamics on the centre manifold at the origin. The local form of the centre manifold at the origin can be represented by a surface

$$
\mathcal{W}_{\left(\psi_{1}, \psi_{2}\right)}^{c}(\mathbf{0}):=\left\{\left(u, v, \psi_{1}(u, v), \psi_{2}(u, v)\right) \in \mathbb{R}^{4} \mid u \geqslant 0, v \geqslant 0, \sqrt{u^{2}+v^{2}}<\delta_{0}\right\}
$$

for some smooth functions $\psi_{i}$ with $\psi_{i}(0,0)=\partial_{1} \psi_{i}(0,0)=\partial_{2} \psi_{i}(0,0)=0, i=1,2$, and sufficiently small $\delta_{0}$. Employing the standard centre manifold theory [8], we can derive the asymptotic expansion of $\psi_{1}$ and $\psi_{2}$ as follows:

$$
\begin{aligned}
& \psi_{1}(\mathrm{u}, \mathrm{v})=(1 / c) \mathrm{uv}+\mathcal{O}\left(\mathrm{u}^{3}, \mathrm{v}^{3}\right) \\
& \psi_{2}(\mathrm{u}, \mathrm{v})=-(1 / c) \mathrm{uv}+(K / c) \mathrm{v}^{q}+\mathcal{O}\left(\mathrm{u}^{3}, \mathrm{v}^{3}\right)
\end{aligned}
$$

as $(u, v) \rightarrow(0,0)$. Further, the governing system on the centre manifold $\mathcal{W}_{\left(\psi_{1}, \psi_{2}\right)}^{c}(\mathbf{0})$ is given by

$$
\begin{align*}
& \mathrm{u}_{z}=(1 / c) \mathrm{uv}+\mathcal{O}\left(\mathrm{u}^{3}, \mathrm{v}^{3}\right), \\
& \mathrm{v}_{z}=-(1 / c) \mathrm{uv}+(K / c) \mathrm{v}^{q}+\mathcal{O}\left(\mathrm{u}^{3}, \mathrm{v}^{3}\right) \tag{A.5}
\end{align*}
$$

as $(u, v) \rightarrow(0,0)$. Up to the leading order, the dynamics of the reduced system (A.5) is well understood [28,30]. Recall that the dynamics of system (A.4) can also be deduced via the dynamics of the reduced system (A.5). With the aid of these discussion, one key conclusion which can be drawn from the arguments of $[28,30]$ is stated in the following lemma.

Lemma A.4. (See $[28,30]$.) For each $c^{\sharp}>0$ and $K^{\sharp}>0$, we can find a small $\varepsilon_{0}>0$ such that for each $(\epsilon, c, K) \in\left(0, \varepsilon_{0}\right) \times\left(0, c^{\sharp}\right) \times\left(0, K^{\sharp}\right)$, there exists a unique solution to the problem $Q[\epsilon, c, K]$.

Now we are in a position to show the uniqueness of waves of system (1.2) whose existence is established in Theorem 1. Recall that it suffices to show the uniqueness of a solution of $P\left[u^{*}, c, K\right]$ with $u^{*}>0, c \geqslant 2 \sqrt{u^{*}}$, and $K>0$. For fixed $u^{*}>0, c \geqslant 2 \sqrt{u^{*}}$, and $K>0$, set

$$
c_{\epsilon}:=\left(\left(u^{*}\right)^{-\frac{1}{2}} c\right) \epsilon^{\frac{1}{2}} \in(0, c) \quad \text { and } \quad K_{\epsilon}:=\left(\left(u^{*}\right)^{q-2} K\right) \epsilon^{2-q} \in(0,2 K)
$$

for $\epsilon \in\left(0, u^{*}\right)$. Here we use the assumption $q \in(1,2]$. Then Lemma A. 4 indicates that there exists a small $\varepsilon_{0} \in\left(0, u^{*}\right)$ such that for each $\epsilon \in\left(0, \varepsilon_{0}\right), Q\left[\epsilon, c_{\epsilon}, K_{\epsilon}\right]$ has a unique solution, which together with the one-to-one correspondence between $P\left[\epsilon, c_{\epsilon}, K_{\epsilon}\right]$ and $Q\left[\epsilon, c_{\epsilon}, K_{\epsilon}\right]$, yields that
$P\left[\epsilon, c_{\epsilon}, K_{\epsilon}\right]$ admits a unique nonnegative solution. Finally, with the use of Lemma A.3, we can conclude the uniqueness of the solution of $P\left[u^{*}, c, K\right]$. This completes the proof of the uniqueness of waves of system (1.2).

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[^0]:    * Corresponding author.

    E-mail addresses: fu@nccu.edu.tw (S.-C. Fu), tsaijc.math@gmail.com (J.-C. Tsai).

