PRICING AND HEDGING OF QUANTO RANGE ACCRUAL NOTES UNDER GAUSSIAN HJM WITH CROSS-CURRENCY LEVY PROCESSES

SZU-LANG LIAO*
PAO-PENG HSU

This study analyzes the pricing and hedging problems for quanto range accrual notes (RANs) under the Heath-Jarrow-Morton (HJM) framework with Levy processes for instantaneous domestic and foreign forward interest rates. We consider the effects of jump risk on both interest rates and exchange rates in the pricing of the notes. We first derive the pricing formula for quanto double interest rate digital options and quanto contingent payoff options; then we apply the method proposed by Turnbull (*Journal of Derivatives*, 1995, 3, 92–101) to replicate the quanto RAN by a combination of the quanto double interest rate digital options and the quanto contingent payoff options. Using the pricing formulas derived in this study, we obtain the hedging position for each issue of quanto

We would like to thank the editor of this journal, Robert I. Webb, and an anonymous referee for their very useful comments and suggestions. The remaining errors are solely ours.

*Correspondence author, Department of Money and Banking, National Chengchi University, No. 64, Section 2, ZhiNan Road, Wenshan District, Taipei City 11605, Taiwan. Tel: +886-2-2939-3091x81251, Fax: +886-2-2939-8004, e-mail: liaosl@nccu.edu.tw

Received September 2007; Accepted December 2008

■ *Szu-Lang Liao is a Professor at the Department of Money and Banking, National Chengchi University, Taipei City, Taiwan.*

■ *Pao-Peng Hsu is a Ph.D. Candidate at the Department of Money and Banking, National Chengchi University, Taipei City, Taiwan, and an Adjunct Researcher at the Polaris Research Institute, Taipei City, Taiwan.*

The Journal of Futures Markets, Vol. 29, No. 10, 973–998 (2009)

Published online in Wiley InterScience (www.interscience.wiley.com).
DOI: 10.1002/fut.20396

RANs. In addition, by simulation and assuming the jump risk to follow a compound Poisson process, we further analyze the effects of jump risk and exchange rate risk on the coupons receivable in holding a RAN. Jrl Fut Mark 29:973–998, 2009

INTRODUCTION

A range accrual note (RAN) is a structured product that offers a coupon calculated by the number of days in which the reference interest rate falls inside a corridor times the pre-specified fixed rate (e.g. fixed RAN) or an interest rate specified at the start of each period (e.g. floating range accrual note (FRAN)). The returns from such notes will be higher than traditional fixed-rate deposits if the reference rate (for example, six-month LIBOR) moves within a predetermined range during the lifetime of the note. Hence, in a low-interest-rate environment, RANs are more attractive compared with treasury bonds. The total volume of RAN derivatives has a 15% share of total issuances of structured notes issued around the world as of November 2008.[1]

As investors receive coupons each period for how often the reference interest rate stays within a certain range, investors should be concerned about the risk of the short rate jumping outside the range. However, few studies have considered the jump risk of reference rates while pricing RANs. The pioneering study of Turnbull (1995) explicitly priced each coupon of a FRAN using a one-factor Gaussian model. Then, Navatte and Quittard-Pinon (1999) reevaluated each coupon of a FRAN with the same approach as Turnbull (1995) using a modified numeraire technique. Then, Nunes (2004) generalized the solutions above with a multifactor Gaussian model.

However, the aforementioned three studies did not consider the fact that the geometric Brownian motion (GBM) is generally a misspecification of asset prices (Merton, 1976). He relaxed the assumption about the diffusion part of random processes by introducing a mixture of GBM with randomly (Poisson) arriving rare events to replace GBM. Since Merton, many went to observe whether jump components existed in markets, where in the literature jumps were mostly documented in stock and exchange rate markets, for example, in Bates (1996, 2000).

The fixed income market is also significantly affected by informational shocks (which causes jumps). As cited in Das (2002), a number of researchers have found that the announcements of economic news and other releases of information have impacts on the treasury bond market. Such findings include

[1]Based on data from Bloomberg, as of November 26, 2008, structured notes issued around the world are classified into 30 types; excluding those callable structure notes, RAN derivative issues total to 7741, which has a 0.15 share of total issuances, and RAN derivative types include RANs, spread RANs, dual RANs, multi-RANs, countdown RANs, and digital RANs.

Hardouvelis (1988), Dwyer and Hafer (1989), Naik and Lee (1990), Das (1995), and Heston (1995). These studies suggest that discontinuous processes do a better job of describing the movements of short-term interest rates than the commonly used diffusion processes (such as GBM). Also, some jump diffusion models for interest rates are proposed and analyzed, for example, by Björk, Kabanov, and Runggaldier (1997), Glasserman and Kou (2003), and Shirakawa (1991). As more and more articles include jump characteristics into their random processes, Levy process is the one that has both a jump and a continuous part, which makes it more general than GBM, and numerous studies are starting to assume their underlyings to follow Levy processes. For example, Cont and Tankov (2004), Eberlein and Raible (1999) and Eberlein and Ozkan (2005) further presented a LIBOR forward rate approach with Levy process. Eberlein and Kluge (2006) generalized some results of RAN with a multivariate Levy term structure model but did not consider exchange rate risks. Hence, in this study a RAN will be priced with the forward rate following a Levy process.[2] In addition, an exchange rate process is incorporated and driven by Levy processes as well.

In options on foreign assets, contracts on an underlying that is subject to a fixed exchange rate are known as quantos (Reiner, 1992). Quanto-structured notes allow the investors to gain access to foreign bond markets using their own domestic currency with the foreign exchange rate fixed at the date of entry. Therefore, quanto-structured notes provide a solution for investors seeking protection from changes in exchange rate, that is, excluding exchange rate risks. Huang and Hung (2005) priced foreign equity options under multidimensional Levy process. Koval (2005) provided a cross-currency model in which the combination of foreign exchange and interest rate risk is specified by Levy processes. As the underlyings of RANs are usually interest rates in international financial markets, such as LIBOR, the investors may not receive returns directly in their own domestic currencies. For example, a domestic investor holding a RAN may receive coupons denominated in a foreign currency, which exposes the investor to exchange rate risk. In this study, quanto-structured notes will be analyzed to avoid this risk.

The pricing formula provided in this article can be used to develop a hedging strategy for quanto floating range accrual note (QFRAN) issuers, and the bucket hedging strategy will be adopted. Jarrow and Turnbull (1994) presented the bucket hedging of interest rate instruments, which is sophisticated in that there is more than one factor describing the evolution of the term structure of interest rates when there are multiple term structures underneath. Hence, some derivatives with multiple underlying term structures (e.g. swaptions) are hedged by the bucket hedging strategy (Driessen, Klaassen, & Melenberg, 2000).

[2]The pricing of a fixed RAN is skipped in this study because the pricing of a fixed RAN can be regarded as a special case for the pricing of a FRAN.

Considering an international security market, this study prices the QFRAN under the Gaussian HJM with Levy process. We will derive the closed-form solution of QFRAN and also find its hedging strategies. This study not only generalizes the results in the past literature with GBM diffusion processes but also generalizes the results to cover both situations of with and without exchange rate risk. The effects of jump risk and exchange rate risk are also illustrated by assuming the jump components to be compound Poisson processes, i.e. the distribution of jump size is specified explicitly, during simulation. Our numerical results show that the QFRAN has higher return than traditional FRAN in some states. For instance, situations of increasing volatility in exchange rate, increasing jump intensity, and increasing volatility of foreign zero coupon bond (ZCB) prices will increase the coupons of QFRANs.

The rest of this article is organized as follows. The second section presents the framework for Levy processes in an international security economy and introduces the method of changing numeraires. The third section provides a closed-form solution for the quanto interest rate digital options that would be used to price the QFRANs. We then derive the hedging strategy in the fourth section, which includes the hedging of exchange rate and jump risks. Numerical evaluations under the assumption that the jump component is a compound Poisson process are presented in the fifth section. We will compare between the FRAN under the HJM model and the FRAN under the HJM–compound Poisson model and then compare the FRAN to the QFRAN. The last section concludes the article.

THE SETTING

Let $T^* > 0$ be a fixed time horizon. All processes considered hereafter are defined on a common probability space $(\Omega, \Im_{T^*}, P)$, endowed with a canonical filtration $(\Im_t)_{0\le t\le T^*}$ associated with a d-dimensional Levy process $(L_t)_{0\le t\le T^*}$. More specifically, $L = (L_1, \ldots, L_d)$ is a process with independent increments and absolutely continuous characteristics, which can be expressed by the following characteristic function, in which $u \in \mathbb{R}^d$ and its transpose is denoted by u':

$$E[\exp(iu'L_t)] = \exp\left(\int_0^t\left(iu'b_s - \frac{1}{2}u'C_su + \int_{\mathbb{R}^d}(\exp(iu'x) - 1 - iu'x)F_s(dx)\right)ds\right).$$

Here F_s with $s \in [0, T^*]$ is a probability distribution on $\mathbb{R}^d$, which satisfies $F_s(\{0\}) = 0$ for all $s \ge 0$ and integrates $(|x|^2 \wedge |x|)$; b_s is a vector in $\mathbb{R}^d$; C_s is a symmetric positive semi-definite $d \times d$ matrix such that

$\int_0^{T^*}\left(|b_s| + \|C_s\| + \int_{\mathbb{R}^d}(|x|^2 \wedge |x|)F_s(dx)\right)ds < \infty$, $\int_0^{T^*}\|C_s\|dt < \infty$, where $|\cdot|$ denotes the Euclidean vector norm and $\|\cdot\|$ denotes any norm on the set of $d \times d$ matrices and c_s is a measure version of the square root of C_s. Assume that there are constants M, $\varepsilon > 0$ such that $\int_0^{T^*}\int_{\{|x|>1\}} \exp(u'x)F_s(dx)\,ds < \infty$ for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$. We denote by $(\Im_s)_{0\le s\le T^*}$ the filtration generated by L and assume $\Im = \Im_{T^*}$. Thus, the canonical representation of L is

$$L_t = \int_0^t b_s ds + \int_0^t c_s dW_s + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^L)(ds, dx)$$

where W_s is a d-dimensional Brownian motion, μ^L is the random measure associated with the jumps of L, and $\nu^L(ds, dx) = F_s(dx)ds$ is L's compensator.

Let φ_s denote the cumulant associated with the infinitely divisible distribution characterized by the Levy–Khinchin triplet (b_s, c_s, F_s), i.e. for $z \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$,

$$\varphi_s(z) = z'b_s + \frac{1}{2}z'C_s z + \int_{\mathbb{R}^d}(\exp(z'x) - 1 - z'x)F_s(dx).$$

Introducing f by $F_s(dx) = \lambda f(dx)$, the Levy–Khinchin formula becomes

$$\varphi_s(z) = z'b_s + \frac{1}{2}z'C_s z + \int_{\mathbb{R}^d}(\exp(z'x) - 1)F_s(dx). \tag{1}$$

We can extend φ_s to complex numbers with $\mathrm{Re}(z_j) \in [-(1+\varepsilon)M, (1+\varepsilon)M]$ for $j \in \{1, \ldots, d\}$ and then write the characteristic function of L_t as

$$E[e^{iu'L_t}] = \exp\left(\int_0^t \varphi_s(iu)ds\right).$$

The multidimensional Levy processes are formally introduced below.[3] The dynamics of the instantaneous domestic/foreign forward rates $0 \le T \le T^*$ under the original P measure are given by

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds - \int_0^t \varsigma(s, T)'dL_s \tag{2}$$

[3]During the process of pricing, all parameter settings are initially under original measure P, and are then transformed to the domestic risk-neutral measure. The conditions that the three random processes, domestic and foreign interest rates and exchange rate, must satisfy when the measure P is transformed to domestic risk-neutral measure Q are provided in Appendix A.

$$f^*(t,T) = f^*(0,T) + \int_0^t \alpha^*(s,T)ds - \int_0^t \varsigma^*(s,T)'dL_s. \tag{3}$$

The initial values $f(0, T)$ and $f^*(0, T)$ are both deterministic and bounded in T. Also, $\alpha(s, T)$, $\alpha^*(s, T)$ and $\varsigma(s, T)$, $\varsigma^*(s, T)$ are deterministic functions with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, and defined on $\{(s, T) \in [0, T^*] \times [0, T^*] : s \leq T\}$. Domestic and foreign ZCB prices can be recovered from the forward rates via $p(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ and $p^*(t,T) = \exp\left(-\int_t^T f^*(t,s)ds\right)$, respectively. The domestic money market account $B(t) = \exp\left(\int_0^t r(s)ds\right)$, where $r(s) = f(s, s)$, denotes the domestic short rate, whereas the foreign money market account $B^*(t) = \exp\left(\int_0^t r^*(s)ds\right)$, where $r^*(s) = f^*(s, s)$, denotes the foreign short rate. Next, we want to transform the original measure P to domestic risk-neutral measure Q (a similar approach can also be found in Koval, 2005). Let $Z(t, T)$ be equal to $p(t, T)/B(t)$. Under Q measure, $Z(t, T)$ is a martingale, that is, the drift term of $dZ(t, T)/Z(t, T)$ equals zero. The condition that $A(t, T)$ in $p(t, T)$ must satisfy when the measure P is transformed to Q measure is given in the following equation[4]:

$$\int_0^t A(s,T)ds = \int_0^t b_s'\Sigma(s,T)ds + \frac{1}{2}\int_0^t \Sigma(s,T)'C_s\Sigma(s,T)ds + \int_0^t\int_{\mathbb{R}^d} (\exp(\Sigma(s,T)'x) - 1)F_s(dx)ds. \tag{4}$$

Therefore, the domestic ZCB price is given by the following representation:

$$p(t,T) = p(0,T)B(t)\exp\left(\int_0^t - A(s,T)ds + \int_0^t \Sigma(s,T)'dL_s^Q\right). \tag{5}$$

Set z in Equation (1) to be equal to $\Sigma(t, T)$. Then Equation (4) can be rewritten as $A(t, T) = \varphi_t(\Sigma(t, T))$.[5] The exchange rate $S(t)$ represents the units of the domestic currency per unit of foreign currency at time t. Under measure Q, the drift term of $dS(t)/S(t)$ equals $r(t) - r^*(t)$. The condition that $m(t)$ in $S(t)$ must satisfy when the measure P is transformed to Q measure is given in the following equation[6]:

$$\int_0^t m(s)ds = \int_0^t b_s'\sigma(s)ds + \frac{1}{2}\int_0^t \sigma(s)'C_s\sigma(s)ds$$

[4]The proof is in Appendix A and Equation (4) is the same as (A4).
[5]The setting $A(t, T) = \varphi_t(\Sigma(t, T))$ guarantees that domestic ZCB discounted by the domestic money market account is a martingale also had been shown in Eberlein and Kluge (2006).
[6]The proof is in Appendix A and Equation (6) is the same as (A7).

$$+ \int_0^t \int_{\mathbb{R}^d} (\exp(\sigma(s)'x) - 1) F_s(dx)ds. \tag{6}$$

Therefore, the exchange rate follows the following representation:

$$S(t) = \frac{S(0)B(t)}{B^*(t)} \exp\left(\int_0^t - m(s)ds + \int_0^t \sigma(s)dL_s^Q \right). \tag{7}$$

Set z in Equation (1) to be equal to $\sigma(t)$. Then Equation (6) can be rewritten as $m(t) = \varphi_t(\sigma(t))$. Finally, the condition that $A^*(t, T)$ in $p^*(t, T)$ must satisfy when P measure is transformed to Q measure is given as follows:

$$\int_0^t A^*(s, T)ds = \int_0^t \Sigma^*(s, T)'b_s ds + \int_0^t \Sigma^*(s, T)'C_s\sigma(s)ds + \frac{1}{2}\int_0^t |\Sigma^*(s, T)'c_s|^2 ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} [e^{\sigma(s)'x}(e^{\Sigma^*(s,T)'x} - 1)]F_s(dx)ds. \tag{8}$$

The drift term $\int_0^t A^*(S, T)ds$ can be obtained from the fact that the ZCB price relative to the money market account is a martingale (see Appendix A for details). Hence, $p^*(t, T)$ can be expressed as

$$p^*(t, T) = p^*(0, T)B^*(t)\exp\left(\int_0^t - A^*(s, T)ds + \int_0^t \Sigma^*(s, T)'dL_s^Q \right). \tag{9}$$

Note that $A^*(t, T)$ is not equal to $\varphi(\Sigma^*(t, T))$. The following equation is another representation of Equation (9) that will be useful later:

$$p^*(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp\left(-\int_0^t A^*(s, t, T)ds + \int_0^t \Sigma^*(s, t, T)'dL_s^Q \right) \tag{10}$$

where we have used the abbreviations

$$A^*(s, t, T) = A^*(s, T) - A^*(s, t) \tag{11}$$

and

$$\Sigma^*(s, t, T) = \Sigma^*(s, T) - \Sigma^*(s, t). \tag{12}$$

Tools for Changing Measures

In order to price digital options and QFRANs, we change the numeraire and switch from the domestic spot martingale measure Q to a domestic forward martingale measure Q_T or some other adequate measure. Notice that all adopted measures are denominated in domestic currencies. In addition, the pricing formulas are expressed in terms of characteristic functions. Therefore, the characteristic functions under each relevant measure will also be derived.

Define $F(t, s, T)$ to be the forward price denominated in domestic currency, i.e.

$$F(t, s, T) = \frac{S(t)p^*(t, s)}{S(t)p^*(t, T)}. \tag{13}$$

The domestic forward martingale measure for the settlement date T, denoted by Q_T, is defined by the Radon–Nikodym derivative

$$\frac{dQ_T}{dQ} = \frac{S(T)p^*(T, T)/B(T)}{S(0)p^*(0, T)/B(0)} = \frac{S(T)}{S(0)B(T)p^*(0, T)}$$

$$\frac{dQ_T}{dQ}|\Im_t = E_Q\left[\frac{S(T)}{S(0)B(T)p^*(0, T)}\middle|\Im_t\right] = \frac{S(t)p^*(t, T)}{S(0)B(t)p^*(0, T)}.$$

Usually, this measure is defined on $(\Omega, \Im_{T^*})$. Q and Q_T are equivalent and from (7) and (9) we have the explicit expression

$$\frac{dQ_T}{dQ} = \exp\left(\int_0^T (-m(s) - A^*(s, T))ds + \int_0^T (\sigma(s)' + \Sigma^*(s, T)')dL_s^Q\right).$$

We need the result of $E_{Q_{T+\theta}}\left[\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]$ when we price a QFRAN. Therefore, let us define $M_{T+\theta}^x(z) = E_{Q_{T+\theta}}\left[\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]$. We have Proposition 1.

Proposition 1: $\varphi_s(u)$ satisfies (1). We then have an explicit expression for $M_{T+\theta}^x$[7]:

$$M_{T+\theta}^x(z) = H(0, T+\theta; T+\theta)H_z(t, T; T+\theta)$$

where

$$H(0, T; T) = \exp\left(\int_0^T - A^*(s, T)ds\right)E_Q\left[\exp\left(\int_0^T \Sigma^*(s, T)'dL_s^Q\right)\right]$$

and

[7]Even though $M_{T+\theta}^x$ could not be expressed as an analytic form and could only be represented by a form of expected function, the expected function under measure Q is sufficient to be calculated.

$$H_z(t, T; T+\theta) = E_Q\left[\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right].$$

Proof:

$$\begin{aligned}
M^x_{T+\theta}(z, t, T; T+\theta) &= E_{Q_{T+\theta}}\left[\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]\\
&= E_Q\left[\exp\left(\int_0^{T+\theta}(-m(s) - A^*(s, T+\theta))ds\right)\right.\\
&\quad \left.\times \exp\left(\int_0^{T+\theta}(\sigma(s)' + \Sigma^*(s, T+\theta)')dL_s^Q + z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]\\
&= \exp\left(\int_0^{T+\theta} -A^*(s, T+\theta)ds\right)\\
&\quad \times E_Q\left[\exp\left(\int_0^{T+\theta}\Sigma^*(s, T+\theta)'dL_s^Q + z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]. \quad (14)
\end{aligned}$$

For $T < T+\theta$ we define the adjusted forward measure $Q_{T,T+\theta}$ on $(\Omega, \Im_{T^*})$ via

$$\frac{dQ_{T,T+\theta}}{dQ_{T+\theta}} := \frac{F(T, T, T+\theta)}{F(0, T, T+\theta)} = \frac{p^*(0, T+\theta)}{p^*(0, T)p^*(T, T+\theta)}$$

$$\left.\frac{dQ_{T,T+\theta}}{dQ_{T+\theta}}\right|\Im_t = \frac{F(t, T, T+\theta)}{F(0, T, T+\theta)} = \frac{p^*(0, T+\theta)p^*(t, T)}{p^*(0, T)p^*(t, T+\theta)}.$$

According to the result of Eberlein and Kluge (2006), we divide the time period into two segments. For $0 \le t \le T$, restricting this density to $\Im_t$, we have

$$\begin{aligned}
\left.\frac{dQ_{T+\theta}}{dQ}\frac{dQ_{T,T+\theta}}{dQ_{T+\theta}}\right|\Im_t &= \left.\frac{dQ_{T,T+\theta}}{dQ}\right|\Im_t\\
&= \frac{S(t)p^*(t, T+\theta)}{S(0)B(t)p^*(0, T+\theta)}\frac{p^*(0, T+\theta)p^*(t, T)}{p^*(0, T)p^*(t, T+\theta)}\\
&= \frac{S(t)p^*(t, T)}{S(0)B(t)p^*(0, T)} \quad (15)
\end{aligned}$$

$$\left.\frac{dQ_{T,T+\theta}}{dQ}\right|\Im_t = \exp\left(\int_0^t(-m(s) - A^*(s, T))ds + \int_0^t(\sigma(s)' + \Sigma^*(s, T)')dL_s^Q\right).$$

For $T \le t \le T+\theta$, restricting this density to $\Im_t$, we have

$$\left.\frac{dQ_{T,T+\theta}}{dQ}\right|\Im_t = \frac{S(t)p^*(t, T+\theta)}{S(T)p^*(T, T+\theta)B(t)p^*(0,T)}$$
$$= \exp\left(\int_T^t -m(s)ds + \int_T^t \sigma(s)'dL_s^Q\right)$$
$$\times \exp\left(-\int_0^T A^*(s,T)ds + \int_0^T \Sigma^*(s,T)'dL_s^Q\right)$$
$$\times \exp\left(-\int_T^t A^*(s, T+\theta)ds \int_T^t \Sigma^*(s, T+\theta)'dL_s^Q\right). \quad (16)$$

Again, we find $\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)$ under delayed measure $Q_{T,T+\theta}$ and we have Proposition 2.

Proposition 2: Set $M^x_{T,T+\theta}(z) = E_{Q_{T,T+\theta}}\left[\exp\left(z\int_t^T \Sigma^*(s, T, T+\theta)'dL_s^Q\right)\right]$. Then $M^x_{T,T+\theta}(z) = H(0,T;T)H_z(t,T;T+\theta)[1_{\{0\le s\le T\}} + 1_{\{T\le s\le T+\theta\}}H(T, T+\theta;\ T+\theta)]$, where $1_{\{\cdot\}}$ is an indicator function.

Proof: Similar to the proof of Proposition 1.

DIGITAL OPTIONS

According to Turnbull (1995), a FRAN includes a series of interest rate digital options. The payoff of a QFRAN can be expressed in terms of the payoffs from a digital call spread: long one digital call option with strike price equal to the lower bound and short one digital call option with strike price equal to the upper bound. We can now price a QFRAN by summing up its digital spreads.

First, we discuss the pricing of interest rate digital options and then introduce the delayed digital option (see Navatte & Quittard-Pinon, 1999). The time T price of a standard European interest rate digital call option with strike rate m is given by

$$SD_T(L^*(T, T+\theta); m; T) = 1_{\{L^*(T,T+\theta)>m\}},$$
$$L^*(T, T+\theta) = \frac{1}{\theta}\left[\frac{1}{p^*(T, T+\theta)} - 1\right]. \quad (17)$$

A delayed interest rate digital option is the case that the option maturity T and payment date T_1, e.g. $T_1 = (T+\theta)$, are different with $T_1 > T$. The time T_1 price of a delayed digital option is given by

$$DD_{T_1}(L^*(T, T + \theta); m; T_1) = 1_{\{L^*(T, T+\theta) > m\}}. \tag{18}$$

The price of a European quanto range digital option on the LIBOR rate $L^*(T, T + \theta)$, with the upper bound M, the lower bound m, and the maturity at time $T_1(>T)$, is equal to

$$QDV_{T_1}[L^*(T, T + \theta); m, M; T_1] = 1_{\{L^*(T,T+\theta) \in [m,M]\}}. \tag{19}$$

Hence,

$$QDV_{T_1}[L^*(T, T + \theta); m, M; T_1] = DD_{T_1}(L^*(T, T + \theta); m; T_1) - DD_{T_1}(L^*(T, T + \theta); M; T_1). \tag{20}$$

From (18), we can also obtain

$$\begin{aligned} DD_t[L^*(T, T + \theta); m; T_1] &= B(t)E_Q\left[\frac{1}{B(T_1)}1_{\{L^*(T,T+\theta)>m\}}\middle|\Im_t\right] \\ &= p(t, T_1)E_{Q_{T_1}}[1_{\{L^*(T,T+\theta)>m\}}|\Im_t] \\ &= p(t, T_1)E_{Q_{T_1}}[1_{\{p^*(T,T+\theta)<(1+\theta m)^{-1}\}}|\Im_t] \\ &= p(t, T_1)E_{Q_{T_1}}[1_{\{(p^*(t,T+\theta)/p^*(t,T))\exp(-\int_t^T A^*(s,T,T+\theta)ds + \int_t^T \Sigma^*(s,T,T+\theta)' dL_s) < (1+\theta m)^{-1}\}}|\Im_t] \end{aligned} \tag{21}$$

where Q_{T_1} is the forward measure using $p(t, T_1)$ as the numeraire.

Formula (20) is the payoff of the quanto range digital option, which is how much coupon investors receive at the next payment date, based on how often the underlying stays within the range $[m, M]$. Next, we will derive the delayed digital option in Theorem 1.

Theorem 1: Choose an $R < 0$ such that $M^x_{T_1}(-R) < \infty$. The value at time $t < T$ of a delayed digital option with strike m and expiry date at time T_1 under the Gaussian HJM with cross-currency Levy process is

$$DD_t(L^*(T, T + \theta); m; T_1) = \frac{1}{\pi}p(t, T_1)\int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T)}{p^*(t, T + \theta)}K\right)^{R+iu}\frac{1}{R + iu}M^x_{T_1}(-R - iu)\right]du$$

with

$$K = \frac{1}{\theta m + 1}\exp\left(\int_t^T A^*(s, T, T + \theta)ds\right).$$

Proof: Let $L[v](z)$ be the bilateral Laplace transform of v for a complex number z defined by

$$L[v](z) = \int_{-\infty}^{\infty} e^{-zx} v(x)dx.$$

According to Theorem 3.2 in Raible (2000), the initial price of an option $V(\varsigma)$ can be obtained by

$$V(\varsigma) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{\varsigma z} L[v](z)dz. \tag{22}$$

Let $y = p^*(t, T + \theta)/p^*(t, T)$ and $E_{QT_1}[\cdot]$ in Equation (21) be $h(y)$. Our goal is to calculate $h(y)$ and express it in the form of Fourier conversion $V(\varsigma)$. Next, we want to find $v(x)$, which corresponds to $h(y)$. Let $X = \int_t^T \Sigma^*(s, T, T + \theta)' dL_s$, $K = (1/\theta m + 1)\exp\left(\int_t^T A^*(s, T, T + \theta)ds\right)$, and $P_{T_1}^x$ denote the distribution of X under measure P_{T_1}. Therefore,

$$h(y) = \int 1_{\{e^X < K/y\}} dP_{T_1} = \int 1_{\{e^X < K/y\}} dP_{T_1}^x(x).$$

If this distribution possesses a Lebesgue density ϕ then

$$h(y) = \int 1_{\{e^X < K/y\}} \phi(x)dx = \int f_y(-x)\phi(x)dx$$

with

$$f_y(x) = 1_{\{e^{-x} < K/y\}}(x).$$

We have $v = f_y^* \phi$, with the convolution denoted by $*$. Substitute $f_y^* \phi$ into Equation (22). The bilateral Laplace transform of f_y and ϕ can be described as

$$L[\phi](z) = M_{T+\theta}^x(-R - iu)$$

$$L[1_{\{e^x < K/y\}}](R + iu) = \frac{1}{R + iu}\left(\frac{K}{y}\right)^{R+iu}.$$

The proof has been done.

Valuation of Quanto Floating Range Accrual Notes

We derive the formula for pricing QFRANs under the Gaussian HJM with cross currency in this section. Suppose the current time is $t(T_0 \leq t \leq T)$. Set $T_j' = T_j + \theta_j$, $T_{ji} = T_j + i$, and $T_{ji}' = T_{ji} + \theta_j$. We denote by m_j and M_j the lower and upper bounds, respectively, for the ith day of the $(j + 1)$th period. $H(T_j, T_{j+1}) =$

$\sum_{i=1}^{n_j} 1_{\{m_j \leq L^*(T_{ji}, T_{ji}+\theta_{ji}) \leq M_j\}}$ denotes the number of days in the $(j+1)$th period with $j = 0, \ldots, N-1$. The value of the $(j+1)$th coupon at time T_{j+1} is equal to

$$V_{j+1}(T_{j+1}) = \frac{L^*(T_j, T_j + \theta_j) + \Delta_j}{D_j} H(T_j, T_{j+1})$$

where Δ_j represents the spread over the reference interest rate paid by the bond during the $(j+1)$th compounding period, whereas D_j indicates the number of days in a year.

To derive the closed-form solution of the QFRAN, we separate the first period $[T_0, T_1]$ from the following period $[T_j, T_{j+1}]$, and n_0 represents the number of days between t and T_1. In fact, during the first period, the value of the reference interest rate is known at t. This will not be the case for the future periods. The value of this note $V_1(t)$ at the first reset date T_1 can be expressed as follows:

$$\begin{aligned}
V_1(t) &= P(t, T_1) E_{Q_{T_1}}\left[\frac{L^*(T_0, T_0 + \theta_0) + \Delta_0}{D_0} H(T_0, T_1) \middle| \Im_t \right] \\
&= \frac{L^*(T_0, T_0 + \theta_0) + \Delta_0}{D_0} \\
&\quad \times \left\{ P(t, T_1) H(T_0, t) + P(t, T_1) E_{Q_{T_1}}\left[\sum_{i=1}^{n_0} 1_{\{m(T_{0t+i}) \leq L^*(t+i, t+i+\theta_0) \leq M(T_{0t+i})\}} \middle| \Im_t \right] \right\} \\
&= \frac{L^*(T_0, T_0 + \theta_0) + \Delta_0}{D_0} \\
&\quad \times \left\{ p(t, T_1) H(T_0, t) + \sum_{i=1}^{n_0} QDV_t[L^*(t+i, t+i+\theta_0); m(T_{0t+i}), M(T_{0t+i}); T_1] \right\}.
\end{aligned} \tag{23}$$

For any future coupon $V_{j+1}(t)(j = 1, \ldots, N-1)$, $V_{j+1}(t)$ is as follows:

$$\begin{aligned}
V_{j+1}(t) &= p(t_1, T_{j+1}) E_{Q_{T_{j+1}}}\left[\frac{L^*(T_j, T_j + \theta_j) + \Delta_j}{D_j} \sum_{i=1}^{n_j} 1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} \middle| \Im_t \right] \\
&= \sum_{i=1}^{n_j} \frac{\Delta_j}{D_j} p(t, T_{j+1}) E_{Q_{T_{j+1}}}[1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t] \\
&\quad + \sum_{i=1}^{n_j} \frac{p(t, T_{j+1})}{D_j} E_{Q_{T_{j+1}}}[L^*(T_j, T_j + \theta_j) 1_{\{m(T_{ij}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ij})\}} | \Im_t] \\
&= \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j} \right) \sum_{i=1}^{n_j} p(t, T_{j+1}) E_{Q_{T_{j+1}}}[1_{\{m(T_{ij}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ij})\}} | \Im_t] \\
&\quad + \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) E_{Q_{T_{j+1}}}[p^*(T_j, T_j + \theta_j)^{-1} 1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t]
\end{aligned}$$

$$= v_{j+1}^1(t) + v_{j+1}^2(t) \tag{24}$$

where

$$v_{j+1}^1(t) \equiv \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j}\right) \sum_{i=1}^{n_j} p(t, T_{j+1}) E_{Q_{T_{j+1}}}[1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t]$$

$$v_{j+1}^2(t) \equiv \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) E_{Q_{T_{j+1}}}[p^*(T_j, T_j + \theta_j)^{-1} 1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t].$$

Then $v_{j+1}^1(t)$ becomes the following using Theorem 1:

$$v_{j+1}^1(t) = \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j}\right) \sum_{i=1}^{n_j} QDV_t[L^*(T_{ji}, T'_{ji}); m(T_{ji}), M(T_{ji}); T_{j+1}]$$

$$v_{j+1}^2(t) = \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) \frac{p^*(0, T_j)}{p^*(0, T_{j+1})} E_{Q_{T_{j+1}}}\left[\frac{F(T_j, T_j, T_{j+1})}{F(0, T_j, T_{j+1})} 1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} \middle| \Im_t\right]$$

$$= \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) \frac{p^*(0, T_j)}{p^*(0, T_{j+1})} \frac{p^*(0, T_{j+1}) p^*(t, T_j)}{p^*(0, T_j) p^*(t, T_{j+1})} E_{Q_{T_j, T_{j+1}}}[1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t]$$

$$= \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) \frac{p^*(t, T_j)}{p^*(t, T_{j+1})} E_{Q_{T_j, T_{j+1}}}[1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t].$$

The summations on the right-hand side look very similar to the time t value of a range digital option, the only difference is that the expectation is taken under the adjusted forward measure. We can proceed on $v_{j+1}^2(t)$ in the same way as we did for digital options and use the independence of the increments of L to obtain

$$v_{j+1}^2(t) = \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} p(t, T_{j+1}) \frac{p^*(t, T_j)}{p^*(t, T_{j+1})} D_t^{ji}. \tag{25}$$

Here,

$$D_t^{ji} = E_{Q_{T_j, T_{j+1}}}[1_{\{m(T_{ji}) \leq L^*(T_{ji}, T'_{ji}) \leq M(T_{ji})\}} | \Im_t] = E_{Q_{T_j, T_{j+1}}}[1_{\{1/1+\theta M(T_{ji}) \leq p^*(T_{ji}, T'_{ji}) \leq 1/1+\theta m(T_{ji})\}} | \Im_t]$$

$$= E_{Q_{T_j, T_{j+1}}}[1_{\{K_M^{ji} \leq p^*(t, T'_{ji})/p^*(t, T_{ji}) \leq K_M^{ji}\}} | \Im_t] = h^{ji}\left(\frac{p^*(t, T'_{ji})}{p^*(t, T_{ji})}\right)$$

with $h^{ji}: \mathbb{R} \to [0, 1]$ given by

$$h^{ji}(y) = \int 1_{\{(1/y)K_M^{ji} \leq e^x \leq (1/y)K_M^{ji}\}} dP_{Q_{T_j, T_{j+1}}}^{X^{ji}}(x)$$

and where

$$X^{ji} = \int_t^{T_{ji}} \Sigma^*(s, T_{ji}, T'_{ji}) dL_s, \quad K_M^{ji} = \frac{1}{\theta m(T_{ji}) + 1} \exp\left(\int_t^{T_{ji}} A^*(s, T_{ji}, T'_{ji}) ds\right)$$

$$K_M^{ji} = \frac{1}{\theta M(T_{ji}) + 1} \exp\left(\int_t^{T_{ji}} A^*(s, T_{ji}, T'_{ji}) ds \right)$$

and $P_{Q_{T_j,T_{j+1}}}^{X^{ji}}$ denotes the distribution of X^{ji} with respect to $Q_{T_j,T_{j+1}}$.

There are some other dynamics implied in (24). The "floating" indicates that the coupon not only depends on the underlying rates but also depends on other dynamics. The following theorem is from formulas (24) and (25).

Theorem 2: Under the HJM with cross-currency Levy process, the time t price of a QFRAN, with its last coupon paid at time T_0 ($\leq t$), and N future coupons $V_{j+1}(t)$ paid at times T_{j+1} (>t), $j = 0, \ldots, N - 1$, is equal to

$$QV(t) = p(t, T_N) + V_1(t) + \sum_{j=1}^{N-1} V_{j+1}(t)$$

with

$$V_1(t) = \frac{L^*(T_0, T_0 + \theta_0) + \Delta_0}{D_0} \times \left\{ p(t, T_1) H(T_0, t) + \sum_{i=1}^{n_0} QDV_t[L^*(t + i, t + i + \theta_0); m(T_{0t+i}), M(T_{0t+i}); T_1] \right\}$$

and

$$V_{j+1}(t) = \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j} \right) \sum_{i=1}^{n_j} QDV_t[L^*(T_{ji}, T'_{ji}); m(T_{ji}), M(T_{ji}); T_{j+1}] + \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} P(t, T_{j+1}) \frac{p^*(t, T_j)}{p^*(t, T_j + \theta_j)} D_t^{ji}$$

where

$$D_t^{ji} = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T_{ji})}{p^*(t, T_{ji} + \theta)} K_m^{ji} \right)^{R+iu} \frac{1}{R + iu} M_{T_j, T_j + \theta_j}^X(-R - iu) \right] du - \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T_{ji})}{p^*(t, T_{ji} + \theta)} K_M^{ji} \right)^{R+iu} \frac{1}{R + iu} M_{T_j, T_j + \theta_j}^X(-R - iu) \right] du.$$

In Theorem 2, the results for the first period can be derived from Theorem 1 if the value of the reference interest rate is known at t. $V_j(t)$ is the jth coupon payment received at T_{j+1} and $QV(t)$ includes both the coupon payments and the principal.

HEDGING QUANTO FLOATING RANGE ACCRUAL NOTES

In this section we will obtain the delta of QFRANs. The delta parameter can be used to dynamically hedge the market risk component of the QFRAN. Therefore, we implement bucket hedging as suggested by Jarrow and Turnbull (1994). Suppose we divide the life of a QFRAN into N buckets $[T_0, T_1]$, $[T_1, T_2]$, . . . , $[T_{N-1}, T_N]$. When using the bucket approach in hedging, cash flows of a portfolio must be assigned to a particular bucket. Consider a portfolio with N cash flows. Cash flow a_j or b_j occurs at date T_j, $j = 1, \ldots, N$. The value of this portfolio (t) is

$$\surd(t) = -QV(t) + \sum_{j=1}^{N} a_j p(t, T_j) + \sum_{j=1}^{N} b_j x(t) p^*(t, T_j).$$

The QFRAN price $QV(t)$ satisfies the following stochastic differential equation under domestic measure Q:

$$d\surd(t) = -dQV(t) + r(t)QVdt - \left[\sum_{j=1}^{N} \frac{\partial QV}{\partial p(t, T_j)} p(t, T_j) S_d(t, T_j) + \sum_{j=1}^{N} \frac{\partial QV}{\partial p^*(t, T_j)} p^*(t, T_j) S_f(t, T_j) \right] dW^Q(t) \tag{26}$$

where $S_d(t, T) = \int_t^T \Sigma(t, s)ds$ and $S_f(t, T) = \int_t^T \Sigma^*(t, s)ds$.

Equation (26) shows that the hedging ratio can be directly used to hedge the QFRAN with N domestic ZCBs and foreign ZCBs with maturities T_1, T_2, . . . , T_N as

$$dQV(t) = \ldots dt - \sum_{j=1}^{N} \frac{\partial QV}{\partial p(t, T_j)} dp(t, T_j) - \sum_{j=1}^{N} \frac{\partial QV}{\partial p^*(t, T_j)} dp^*(t, T_j).$$

This is exactly the hedging strategy of bucket hedging and the a_js and b_js are deltas of delta hedging. The delta of the portfolio can be decomposed into two parts. The delta of the domestic ZCB in the jth bracket is

$$a_1 = \frac{L^*(T_0, T_0 + \theta_0) + \Delta_0}{D_0} H(T_0, t) + \sum_{i=1}^{n_0} \Lambda_{T_1}(t + i, t + i + \theta_0; m(T_{0t+i}), M(T_{0t+i}))x$$

$$a_j = \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j}\right) \sum_{i=1}^{n_j} \Lambda_{T_j}(T_{ji}, T'_{ji}; m(T_{ji}), M(T_{ji}))$$

$$+ \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} \Psi_{T_j}(T_{ji}, T_{ji}{}'; m(T_{ji}), M(T_{ji})), j = 2, \ldots, N-1$$

$$a_N = 1 + \left(\frac{\Delta_N}{D_N} - \frac{1}{\theta_N D_N}\right) \sum_{i=1}^{n_N} \Lambda_{T_N}(T_{Ni}, T'_{Ni}; m(T_{Ni}), M(T_{Ni}))$$

$$+ \frac{1}{\theta_N D_N} \sum_{i=1}^{n_N} \Psi_{T_N}(T_{Ni}, T'_{Ni}; m(T_{Ni}), M(T_{Ni}))$$

where

$$\Lambda_{T_j}(T_j, T_j + \theta_j; m, M) =$$

$$\frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T_j)}{p^*(t, T_j + \theta_j)} K\right)^{R+iu} \frac{1}{R+iu} M^x_{T_1}(-R - iu)\right] du$$

$$\Psi_{T_j}(T_j, T_j + \theta_j; m, M) = \frac{p^*(t, T_j)}{p^*(t, T_j + \theta_j)} D_t^{ji}.$$

The delta of the foreign ZCB in the jth bracket is

$$b_1 = \sum_{i=1}^{n_0} \tilde{\Lambda}_{T_1}(t + i, t + i + \theta_0; m(T_{0t+i}), M(T_{0t+i}))$$

$$b_j = \left(\frac{\Delta_j}{D_j} - \frac{1}{\theta_j D_j}\right) \sum_{i=1}^{n_j} \tilde{\Lambda}_{T_j}(T_{ji}, T'_{ji}; m(T_{ji}), M(T_{ji}))$$

$$+ \frac{1}{\theta_j D_j} \sum_{i=1}^{n_j} \tilde{\Psi}_{T_j}(T_{ji}, T'_{ji}; m(T_{ji}), M(T_{ji}))$$

$$b_N = 1 + \left(\frac{\Delta_N}{D_N} - \frac{1}{\theta_N D_N}\right) \sum_{i=1}^{n_N} \tilde{\Lambda}_{T_N}(T_{Ni}, T'_{Ni}; m(T_{Ni}), M(T_{Ni}))$$

$$+ \frac{1}{\theta_N D_N} \sum_{i=1}^{n_N} \tilde{\Psi}_{T_N}(T_{Ni}, T'_{Ni}; m(T_{Ni}), M(T_{Ni}))$$

where

$$\tilde{\Lambda}_{T_j}(T, T + \theta; m, M) = \frac{1}{\pi} p(t, T_j) \int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T_j)}{p^*(t, T_j + \theta_j)} K\right)^{R+iu} \frac{M^x_{T_1}(-R - iu)}{p^*(t, T_j + \theta_j)}\right] du$$

$$\tilde{\Psi}_{T_j}(T, T + \theta; m, M) = P(t, T_{j+1}) \frac{-p^*(t, T_j)}{p^*(t, T_j + \theta_j)^2} D_t^{ji}$$

$$+ \frac{1}{\pi} P(t, T_{j+1}) \frac{p^*(t, T_j)}{p^*(t, T_j + \theta_j)} \int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t, T_{ji})}{p^*(t, T_{ji} + \theta)} K_m^{ji}\right)^{R+iu} \frac{M^x_{T_1}(-R - iu)}{p^*(t, T_{ji} + \theta)}\right] du$$

$$-\frac{1}{\pi}P(t,T_{j+1})\frac{p^*(t,T_j)}{p^*(t,T_j+\theta_j)}\int_0^\infty \mathrm{Re}\left[\left(\frac{p^*(t,T_{ji})}{p^*(t,T_{ji}+\theta)}K_M^{ji}\right)^{R+iu}\frac{M_{T_1}^x(-R-iu)}{p^*(t,T_{ji}+\theta)}\right]du.$$

Consider a financial institution that has written a QFRAN with start date t and end date T_N. Corresponding to the QFRAN, the long position of the hedging portfolio is composed of domestic ZCBs with weights a_j and foreign ZCBs with weights b_j, for $j = 1, \ldots, N$.

NUMERICAL ANALYSIS

Our pricing formulas are applied to QFRANs with jump processes, and we now investigate how the value of QFRAN changes when parameter values change. We use the method "fast Fourier transform (FFT)" proposed by Carr and Madan (1999) to evaluate the formulas in Theorem 2. We assume that: (1) the short-term interest rate follows the Vasicek (1977) model, which is a special case of the one-factor Gaussian HJM model, i.e. $d = 1$, (2) the jump component is a compound Poisson process with jump size normally distributed with mean γ and variance δ^2. The Levy density of this jump component then becomes

$$f(x) = \lambda\frac{1}{\sqrt{2\pi\delta}}\exp\left(\frac{-(x-\gamma)^2}{2\delta^2}\right).$$

The jump component of the characteristic function of compound Poisson is $\exp\{\lambda T(e^{z\gamma T-(1/2)z^2\delta^2 T})\}$, where z is a complex number.

In Table I, each coupon of a FRAN is obtained, which can serve as a benchmark for us to compare against results for QFRAN without jump.

TABLE I

The Coupon Payments of Floating Range Accrual Note and Quanto Floating Range Accrual Note Without Jump

m–M (%)	*First Coupon*	*Second Coupon*	*Third Coupon*	*Fourth Coupon*
Floating range accrual note				
3–6	0.018267	0.014026	0.011798	0.010143
2–7	0.020901	0.017977	0.015306	0.014516
1–8	0.022885	0.021795	0.019278	0.017811
Quanto floating range accrual note				
3–6	0.010284	0.010047	0.009277	0.009205
2–7	0.016769	0.014552	0.013188	0.012910
1–8	0.022701	0.021516	0.018904	0.017337

Note. We set $(\beta_d, B_d, \kappa_d) = (0.1, 0.04, 0.02)$ and $(\beta_f, B_f, \kappa_f) = (0.1, 0.045, 0.3)$, $\sigma = 0.5$, and the initial domestic interest rate and the initial foreign interest rate are 0.034.

TABLE II
The Coupon Payments of Quanto Floating Range Accrual Note Under HJM–Compound Poisson Jump Model

m–M (%)	*First Coupon*	*Second Coupon*	*Third Coupon*	*Fourth Coupon*
Quanto floating range accrual note with jump				
3–6	0.010678	0.010535	0.009886	0.009422
2–7	0.017350	0.015141	0.013891	0.013790
1–8	0.023400	0.022328	0.019811	0.018353

Note. We set $(\beta_d, B_d, \kappa_d) = (0.1, 0.04, 0.02)$ and $(\beta_f, B_f, \kappa_f) = (0.1, 0.045, 0.3)$, $\sigma = 0.3$, and the initial domestic interest rate and the initial foreign interest rate are 0.034.

TABLE III
The Impact of Jump Intensity, Volatility of Exchange Rate, and Volatility of Foreign ZCB Under HJM–Compound Poisson Jump Model

m–M (%)	*First Coupon*	*Second Coupon*	*Third Coupon*	*Fourth Coupon*
$\lambda = 0.3$				
3–6	0.011048	0.010975	0.010425	0.010053
2–7	0.017891	0.015668	0.014507	0.013297
1–8	0.024047	0.023055	0.020602	0.019222
$\sigma = 0.6$				
3–6	0.012019	0.011907	0.011769	0.011613
2–7	0.019298	0.016990	0.016020	0.014908
1–8	0.025710	0.024864	0.022527	0.021303
$\kappa_f = 0.6$				
3–6	0.011769	0.011544	0.011519	0.010899
2–7	0.018989	0.016271	0.015548	0.014654
1–8	0.025412	0.024581	0.020798172	0.019447

Note. We set $(\beta_d, B_d, \kappa_d) = (0.1, 0.04, 0.02)$, $(\beta_f, B_f, \kappa_f) = (0.1, 0.045, 0.3)$, $\Delta\lambda = 0.15$, $\Delta\sigma = 0.3$, $\Delta\kappa_f = 0.3$, $(b_s, c_s, \lambda, \delta) = (0.02, 0.2, 0.01, 0.5)$, and the initial domestic interest rate and the initial foreign interest rate are 0.034. ZCB, zero coupon bond.

In Table II, we give the results for Theorem 2 under our setting. In Table III, the effects of various parameters on the QFRAN are provided.

We use the formulas in Theorem 2 and consider a two-year QFRAN with a reset date every 180 days (the number of days in a year is 360); hence, there are four periods before maturity. The face value of the note is assumed to be one dollar. Assume that $m_{ji} = m_j$ and $M_{ji} = M_j$ for every $j = 1,2,3,4$, and for all i in the same j. The spread Δ is assumed to be constant and is equal to 100 basis points (=1%).

The parameter set $(b_s, c_s, \lambda, \gamma, \delta)$ determines the cumulants of ZCB process. The set $(b_s, c_s, \lambda, \gamma, \delta)$ is the same for both domestic and foreign countries.

The volatility is important for the pricing formula, and we assume that $\varsigma(t, T)$ has an exponential form, which is $\varsigma(t, T) = \beta_d B_d \exp(-\kappa_d(T - t))$ and $\varsigma^*(t, T) = \beta_f B_f \exp(-\kappa_f(T - t))$. Hence, we can get domestic and foreign ZCBs by Equation (3.13) in Zhang (2006). The volatility of the exchange rate process is denoted by σ, the initial domestic short rate and the initial foreign short rate are both 0.034, and the initial exchange rate is 1.2.[8] The adjustment speed R in Theorem 2 is -0.9.

From Table I, we first establish the values of the FRAN and the QFRAN without jump. At the same time we assume $\sigma = 0.5$ to strengthen the effect of exchange rate. It shows that present values are a decreasing function of time for both the QFRAN and the FRAN cases. Because $QV(t)$ is an infinite integral in Theorem 2, it produces errors when we perform numerical analysis; therefore, we show the results of a FRAN, which can serve as a benchmark. The coupons of the QFRAN received each period are less than that of the FRAN.

Table II illustrates the coupons of the QFRAN received each period under $(b_s, c_s, \lambda, \gamma, \delta) = (0.02, 0.2, 0.15, 0.01, 0.5)$ and $\sigma = 0.3$. The phenomenon of decreasing present values in time also holds and coupons received each period are lower than the corresponding coupons of the FRAN.

From Table III, the first three rows are the cases under increasing λ, rows 4–6 are the cases under increasing volatility of exchange rate, and rows 7–9 are the cases under increasing κ_f. The impact of increasing κ_f is equivalent to the impact of increasing $\Sigma^*(t, T)$ and reversion rates. To compare the results of Tables II and III, we find that the parameters λ, σ, κ_f positively impact each coupon. The impact of high σ is more relevant than the impact of high λ and high κ_f on coupon. Higher λ and κ_f may decrease the probability that the underlying is in the money.

CONCLUSION

Under the Gaussian HJM with cross-currency Levy processes, we have derived the closed-form pricing formula for QFRANs. The instantaneous domestic forward interest rates, instantaneous foreign forward interest rates, and exchange rate are assumed to follow Levy processes. Based on this specification and using a delayed numeraire, the payoff of a QFRAN can be expressed as a series of contingent payoff option contracts. Hence, we have also derived the pricing formulas of the quanto double interest rate digital options and quanto contingent payoff option contracts.

Furthermore, we can obtain hedging strategies using our pricing formulas. Using the hedging strategy provided in this study, an issuer who has underwritten

[8]To verify our numerical results, we also simulate the prices by Monte Carlo in Appendix B.

QFRANs can understand how to hedge if the underlying faces uncertainty of jumps in the future. Analysis about RAN hedging has yet been discussed in the existing literature, and it is an important issue for the issuer of RANs especially when interest rates indeed jump.

The numerical results show some interesting phenomena. The coupons of a QFRAN are lower than the coupons of a FRAN without jump risk but with exchange rate risk. The impact of increasing volatility of exchange rate, jump intensity, and volatility of foreign ZCB will increase the coupons received for QFRANs. The impact of high volatility on exchange rate is more significant than the impact of higher jump intensity and volatility of foreign ZCB, as higher jump intensity and volatility of foreign ZCB may decrease the probability that the underlying is in the money.

APPENDIX A

Denote $Z(t, T) = p(t, T)/B(t)$ and $Z^*(t, T) = p^*(t, T)S(t)/B(t)$. Under the domestic risk-neutral measure Q, the drift terms of $dZ(t, T)/Z(t, T)$ and $dZ^*(t, T)/Z^*(t, T)$ must be zero and the drift term of $dS(t)/S(t)$ equals $r(t) - r^*(t)$.

First we prove the condition that $A(t, T)$ in the expression of $p(t, T)$ must satisfy when the measure is changed from the original measure P to the domestic risk-neutral measure Q. According to the assumptions of this study,

$$dL_t = b_t dt + c_t dW_t + \int_{\mathbb{R}^d} x(\mu^L - \nu^L)(dt, dx). \tag{A1}$$

Substituting (A1) into $p(t, T)$, we can rewrite $p(t, T)$ as

$$p(t, T) = p(0, T)B(t)\exp\bigg(\int_0^t - A(s, T)ds + \int_0^t \Sigma(s, T)'(b_s ds + c_s dW_s + \int_{\mathbb{R}^d} x(\mu^L - \nu^L)(ds, dx))\bigg). \tag{A2}$$

Using (A2) and $B(t)$ in $Z(t, T)$, then

$$Z(t, T) = Z(0,T)\exp\bigg(\int_0^t - A(s, T)ds + \int_0^t \Sigma(s, T)'b_s ds + \int_0^t \Sigma(s, T)'c_s dW_s + \int_0^t\int_{\mathbb{R}^d} \Sigma(s, T)'x(\mu^L - \nu^L)(ds, dx)\bigg). \tag{A3}$$

Let $D(s, x, T) = \Sigma(s, T)'x$. The differential of (A3) can be written as follows:

$$\frac{dZ(t, T)}{Z(t, T)} = -A(t, T)dt + \Sigma(t, T)'b_t dt + \Sigma(t, T)'c_t dW_t$$

$$+ \int_{\mathbb{R}^d} e^{D(t,x,T)}(\mu^L - \nu^L)(dt, dx)$$

$$+ \frac{1}{2}\Sigma(t, T)' C_t \Sigma(t, T)dt + \int_{\mathbb{R}^d} (e^{D(t,x,T)} - 1)\nu^L(dt, dx).$$

Under Q measure the drift term of $dZ(t, T)/Z(t, T)$ equals zero:

$$A(t, T) = \Sigma(t, T)' b_t + \frac{1}{2}\Sigma(t, T)' C_t \Sigma(t, T) + \int_{\mathbb{R}^d} (e^{D(t,x,T)} - 1)F_t(dx). \quad \text{(A4)}$$

Next, the condition that the expression $m(t)$ in the exchange rate $S(t)$ must satisfy when the original measure P is changed to the domestic risk-neutral measure Q is derived as follows:

$$B(t) = \exp\left(\int_0^t r(s)ds\right) \text{ and } B^*(t) = \exp\left(\int_0^t r^*(s)ds\right). \quad \text{(A5)}$$

Substituting (A5) into $S(t)$, we can rewrite $S(t)$ as

$$S(t) = S(0)\exp\left(\int_0^t (-m(s) + r(s) - r^*(s))ds + \int_0^t \sigma(s)' b_s ds + \int_0^t \sigma(s)' c_s dW_s\right.$$

$$\left. + \int_0^t \int_{\mathbb{R}^d} \sigma(s)' x(\mu^L - \nu^L)(ds, dx)\right). \quad \text{(A6)}$$

Let $g(s, x, T) = \sigma(s)'x$. The differential of (A6) can be expressed as follows:

$$\frac{dS(t)}{S(t)} = -m(t)dt + r(t)dt - r^*(t)dt + \sigma(t)' b_t dt + \sigma(t)' c_t dW_t$$

$$\int_0^t \int_{\mathbb{R}^d} e^{\sigma(s)'x}(\mu^L - \nu^L)(ds, dx)$$

$$+ \frac{1}{2}\sigma(t)' C_t \sigma(t)dt + \int_{\mathbb{R}^d} (e^{g(t,x,T)} - 1)F_t(dx)dt.$$

Under Q measure, the drift term of $dS(t)/S(t)$ equals $r(t) - r^*(t)$. Hence

$$m(t) = \sigma(t)' b_t + \frac{1}{2}\sigma(t)' C_t \sigma(t) + \int_{\mathbb{R}^d} (e^{g(t,x,T)} - 1)F_t(dx). \quad \text{(A7)}$$

Finally, the condition that $A^*(t, T)$ in $p^*(t, T)$ must satisfy when P measure is changed to Q measure is derived as follows:

$$p^*(t,T) = p^*(0,T)B^*(t)\exp\left(\int_0^t - A^*(s,T)ds + \int_0^t \Sigma^*(s,T)'dL_s^Q\right). \quad \text{(A8)}$$

Substituting (A8) into $Z^*(t, T)$, we get

$$\begin{aligned}
Z^*(t,T) &= \frac{S(t)p^*(0,T)B^*(t)}{B(t)}\exp\left(\int_0^t - A^*(s,T)ds\right) + \int_0^t \Sigma^*(s,T)'dL_s^Q\Big) \\
&= \frac{S(0)B(t)}{B^*(t)}\frac{p^*(0,T)B^*(t)}{B(t)}\exp\left(\int_0^t - m(s)ds + \int_0^t \sigma(s)dL_s^Q\right) \\
&\quad \times \exp\left(\int_0^t - A^*(s,T)ds + \int_0^t \Sigma^*(s,T)'dL_s^Q\right) \\
&= Z^*(0,T)\exp\left(\int_0^t -(A^*(s,T)+m(s))ds + \int_0^t (\Sigma^*(s,T)+\sigma(s))dL_s^Q\right) \\
&= Z^*(0,T)\exp\Bigg(\int_0^t -(A^*(s,T)+m(s))ds \\
&\quad + \int_0^t (\Sigma^*(s,T)'+\sigma(s)')\left(b_s ds + c_s dW_s + \int_{\mathbb{R}^d} x(\mu^L - \nu^L)(ds,dx)\right)\Bigg)
\end{aligned}$$

$$\begin{aligned}
\frac{dZ^*(t,T)}{Z^*(t,T)} &= -A^*(t,T)dt - m(t)\,dt + (\Sigma^*(t,T)' + \sigma(t)')b_t dt + (\Sigma^*(t,T)' \\
&\quad + \sigma(t)')c_t dW_t + \int_{\mathbb{R}^d} e^{(\Sigma^*(t,T)'+\sigma(t)')x}(\mu^L - \nu^L)(dt,dx) \\
&\quad + \frac{1}{2}\left|(\Sigma^*(t,T)' + \sigma(t)')c_t\right|^2 dt + \int_{\mathbb{R}^d} (e^{(D(t,x,T)+g(t,x,T))} - 1)F_t(dx)dt.
\end{aligned}$$

Under Q measure, the drift term is zero:

$$\begin{aligned}
&- A^*(t,T) - m(t) + (\Sigma^*(t,T) + \sigma(t))b_t + \frac{1}{2}\left|(\Sigma^*(t,T)' + \sigma(t)')c_t\right|^2 \\
&+ \int_{\mathbb{R}^d} (e^{(D(t,x,T)+g(t,x,T))} - 1)F_t(dx) = 0. \quad \text{(A9)}
\end{aligned}$$

Substituting (A7) into (A9), we obtain

$$\begin{aligned}
A^*(t,T) &= \Sigma^*(t,T)'b_t + \Sigma^*(t,T)'C_t\sigma(t) + \frac{1}{2}\left|\Sigma^*(t,T)'c_t\right|^2 \\
&\quad + \int_{\mathbb{R}^d} [e^{\sigma(t)'x}(e^{\Sigma^*(t,T)'x} - 1)]F_t(dx). \quad \text{(A10)}
\end{aligned}$$

TABLE IV
The Coupon Payments of Floating Range Accrual Note and Quanto Floating Range Accrual Note Without Jump by Monte Carlo Simulation

m–M (%)	*First Coupon*	*Second Coupon*	*Third Coupon*	*Fourth Coupon*
Floating range accrual note				
3–6	0.018984	0.015260	0.012979	0.010949
2–7	0.021543	0.019413	0.016864	0.015178
1–8	0.023481	0.022166	0.020509	0.019328
Quanto floating range accrual note				
3–6	0.010756	0.010167	0.009559	0.009293
2–7	0.017824	0.016053	0.014074	0.013162
1–8	0.022829	0.021533	0.019891	0.018724

Note. All parameters are set the same as in Table I.

TABLE V
The Coupon Payments of Quanto Floating Range Accrual Note Under HJM–Compound Poisson Jump Model by Monte Carlo Simulation

m–M (%)	*First Coupon*	*Second Coupon*	*Third Coupon*	*Fourth Coupon*
Quanto floating range accrual note with jump				
3–6	0.011985	0.011122	0.010304	0.009786
2–7	0.017782	0.016120	0.014932	0.014328
1–8	0.023481	0.022166	0.020509	0.019328

Note. All parameters are set the same as in Table II.

APPENDIX B

We use the Monte Carlo method to simulate the closed-form solution under the same settings of Tables I and II. First, the prices of foreign ZCBs are simulated by Monte Carlo method using the following equation:

$$\frac{dp^*(t,T)}{p^*(t-,T)} = (r^*(t) - c_t^2\Sigma(t,T)\sigma(t))dt - e^{\sigma(t)}(e^{\Sigma(t,T)} - 1)v + c_t\Sigma(t,T)dW_t + (e^{\Sigma(t,T)} - 1)u. \tag{B1}$$

Equation (B1) corresponds to Equation (9) and it is a differential form of Equation (9). The setting of parameters is consistent with this study, v is a compensator and u is a compound Poisson. After the prices of foreign ZCBs are obtained, the LIBOR rate is also available using bond prices from Monte Carlo simulation using the following equation:

$$L^*(T, T+\theta) = \frac{1}{\theta}\left[\frac{1}{p^*(T, T+\theta)} - 1\right]. \tag{B2}$$

We summarize some results of Monte Carlo simulation in Tables IV and V.

Tables I, II, IV, and V show that the coupons by Monte Carlo simulation are higher than the coupons by FFT, but the maximum difference between the results of FFT and of Monte Carlo simulation is small, which is 0.001558.

BIBLIOGRAPHY

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. Review of Financial Studies, 9, 69–107.

Bates, D. S. (2000). Post-'87 crash fears in the S&P 500 futures option market. Journal of Econometrics, 94, 181–238.

Björk, T., Kabanov, Y., & Runggaldier, W. (1997). Bond market structure in the presence of marked point processes. Mathematical Finance, 7, 211–223.

Carr, P., & Madan, D. B. (1999). Option valuation using the fast Fourier transform. Journal of Computational Finance, 2, 61–73.

Cont, R., & Tankov, P. (2004). Financial modelling with jump processes. Boca Raton, FL: CRC Press.

Das, S. R. (1994). Jump-diffusion processes and the bond markets (working paper). Stern School of Business. New York University.

Das, S. R. (2002). The surprise element: Jumps in interest rates. Journal of Econometrics, 106, 27–65.

Driessen, J., Klaassen, P., & Melenberg, B. (2000). The performance of multi-factor term structure models for pricing and hedging caps and swaptions (working paper). Tilburg University.

Dwyer, G. P., & Hafer, R. W. (1989). Interest rates and economic announcements. Technical report, Federal Reserve Bank of St. Louis.

Eberlein, E., & Kluge, W. (2006). Valuation of floating range notes in Levy term-structure models. Mathematical Finance, 16, 237–254.

Eberlein, E., & Ozkan, F. (2005). The Levy Libor model. Finance and Stochastics, 9, 327–348.

Eberlein, E., & Raible, S. (1999). Term structure models driven by general Levy processes. Mathematical Finance, 9, 31–53.

Glasserman, P., & Kou, S. G. (2003). The term structure of simple forward rates with jump risk. Mathematical Finance, 13, 383–410.

Hardouvelis, G. A. (1988). Economic news, exchange rates and interest rates. Journal of International Money and Finance, 7, 23–35.

Heston, S. L. (1995). A model of discontinuous interest rate behaviour, yield curves and volatility (working paper). University of Maryland.

Huang, S. C., & Hung, M. W. (2005). Pricing foreign equity options under Levy processes. The Journal of Futures Markets, 25, 917–944.

Jarrow, R. A., & Turnbull, S. M. (1994). Delta, gamma and bucket hedging of interest rate derivatives. Applied Mathematical Finance, 1, 21–48.

Koval, N. (2005). Time-inhomogeneous Lévy processes in cross-currency market models. Dissertation, Universität Freiburg.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3, 125–144.

Naik, V., & Lee, M. (1990). General equilibrium pricing of options on the market portfolio with discontinuous returns. Review of Financial Studies, 3, 493–521.

Navatte, P., & Quittard-Pinon, F. (1999). The valuation of interest rate digital options and range notes revisited. European Financial Management, 5, 425–440.

Nunes, J. P. V. (2004). Multifactor valuation of floating range notes. Mathematical Finance, 14, 79–97.

Raible, S. (2000). Lévy processes in finance: Theory, numerics, and empirical Facts. Dissertation, Universität Freiburg.

Reiner, E. (1992). Quanto mechanics. Risk, 5, 59–63.

Shirakawa, H. (1991). Interest rate option pricing with Poisson–Gaussian forward rate curve processes. Mathematical Finance, 1, 77–94.

Turnbull, S. (1995). Interest rate digital options and range notes. Journal of Derivatives, 3, 92–101.

Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of Financial Economics, 5, 177–188.

Zhang, B. (2006). A new Levy based short rate model for the fixed income market and its estimation with particle filter. Dissertation, University of Maryland.