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The valuation of contingent capital with catastrophe risks

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A B S T R A C T

The Intergovernmental Panel on Climate Change Fourth Assessment Report (2007) indicates that unanticipated catastrophic events could increase with time because of global warming. Therefore, it seems inadequate to assume that arrival process of catastrophic events follows a pure Poisson process adopted by most previous studies (e.g. [Louberge, H., Kellezi, E., Gilli, M., 1999. Using catastrophe-linked securities to diversify insurance risk: A financial analysis of lCAT bonds. J. Risk Insurance 22, 125–146; Lee, J.-P., Yu, M.-T., 2002. Pricing default-risky CAT bonds with moral hazard and basis risk. J. Risk Insurance 69, 25–44; Cox, H., Fairchild, J., Pedersen, H., 2004. Valuation of structured risk management products. Insurance Math. Econom. 34, 259–272; Jaimungal, S., Wang, T., 2006. Catastrophe options with stochastic interest rates and compound Poisson losses. Insurance Math. Econom., 38, 469–483]. In order to overcome this shortcoming, this paper proposes a doubly stochastic Poisson process to model the arrival process for catastrophic events. Furthermore, we generalize the assumption in the last reference mentioned above to define the general loss function presenting that different specific loss would have different impacts on the drop in stock price. Based on modeling the arrival rates for catastrophe risks, the pricing formulas of contingent capital are derived by the Merton measure. Results of empirical experiments of contingent capital prices as well as sensitivity analyses are presented.

1. Introduction

The Intergovernmental Panel on Climate Change (IPCC) Fourth Assessment Report (2007) states that the world's average surface temperature has increased by around 0.74 °C over the past 100 years (1906–2005). A warming of about 0.2 °C is predicted for each of the next two decades. This report also indicates that heat waves will continue to become more frequent with a confidence level of at least 90% and future tropical cyclones (typhoons and hurricanes) will become more intense with a confidence level of at least 66%. Based on the increasing number of natural catastrophes (CATs), the demand of the CAT risk instruments for the (re)insurance company with high CAT risk exposure is predicted to rise in the future.

The types of CAT risk instruments include hedging instruments (such as CAT swaps and CAT bonds) and financing instruments (such as contingent surplus notes issues (CSNs) and catastrophe equity put options (CatEPut)). For the overall financing instrument markets, a number of insurance companies have issued contingent capital for a total of US$8 billion approximately from the mid-1990s. Among the contingent capital markets, there are some CSNs, such as Hannover Reinsurance Company (US$85 million) in 1994, Nationwide Mutual Insurance Company (US$400 million) in 1995, and Arkwright Company (US$100 million) in 1996. The first CatEPut was issued on behalf of RLI Corporation in 1996, giving RLI the right to issue up to $50 million of cumulative preferred shares. Other CatEPut issues include Horace Mann Educators Company ($100 million) in 1997 and La Salle Reinsurance Company ($100 million) in 1997. Since unanticipated catastrophic events increase and change with time, the contingent capital market is expected to grow rapidly, and thus it is absolutely crucial to price contingent capital for insurance companies. However, the previous studies consider little about the valuation of contingent capital, particularly for the valuation of CSNs. In this paper, we provide general formulas for contingent capital, which can derive the formulas for CSNs and CatEPuts.

A Pure Poisson process is used to describe the arrival rate of catastrophic events and is applied for the pricing of CAT insurance products (e.g. Cummins and Geman, 1995; Louberge et al., 1999; Lee and Yu, 2002; Cox et al., 2004; Jaimungal and Wang, 2006). Cox et al. (2004) use a pure Poisson process to model the aggregate CAT loss of an insurance company, and derive the pricing formula of CatEPuts with two important assumptions. One is the constant arrival rates of catastrophic events; the other is the constant impact on the market price of the insurance company's stock in

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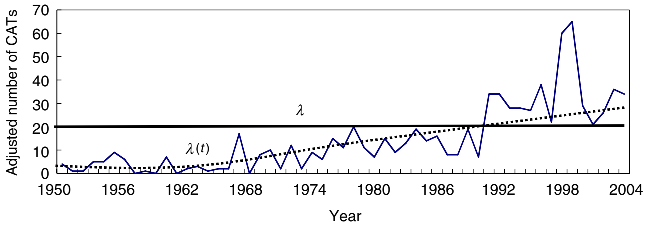

Fig. 1. Adjusted number of natural CATs in the United States from 1950 to 2004. Note that the dotted line represents an increasing exponential trend in the frequency of catastrophic events. The solid line denotes the average constant frequency of catastrophic events.

catastrophic events. Jaimungal and Wang (2006) use a compound Poisson process to describe the dynamic losses accurately, but maintain the assumption of the constant arrival rate of a CAT. Fig. 1 shows the annual number of catastrophic events adjusted by Commerce Census Fixed Weighted Construction Cost index and population in the United States from 1950 to 2004. Rather than the average constant frequency of catastrophic events (solid line), this figure seems to show an increasing exponential trend (dotted line) in the frequency of catastrophic events.[1]

The objectives and main contributions of this study are as follows: (1) to be consistent with the upward exponential trend for Fig. 1, the doubly stochastic Poisson process with lognormal intensity is used to capture the uptrend and randomness. (2) Jaimungal and Wang (2006) denote a constant percentage to represent the drop in the stock price per unit of the loss, and this implies that each specific loss has the same effect on the stock price. However, in practice, different specific losses should have different impacts on the drop in the underlying asset price. This paper generalizes the assumption of Jaimungal and Wang (2006) to define the general loss function, expressing that different specific loss have different impacts on the drop in the underlying asset price. Hence, our closed-form expression can reduce to the pricing formulas of Cox et al. (2004) and Jaimungal and Wang (2006), where the pure Poisson process drives the arrival rate of catastrophic events and the general loss function simplifies to a product of a constant rate and the specific loss. Furthermore, in terms of several specific cases for the general loss function, the pricing formulas of contingent capital are also proposed. (3) We use the data of PCS loss index and the annual number of natural catastrophic events during 1950 to 2004 to test the quality of the fitting under the doubly stochastic Poisson process and the pure Poisson process. The result shows that the doubly stochastic Poisson process is fitter than the pure Poisson process when pricing the CatEPut. Moreover, a numerical example shows that the CatEPut price under the doubly stochastic Poisson process is larger than that under the pure Poisson process when the instantaneous growth rate of catastrophic intensity rises. It also reveals that, based on the numerical example, arrival rate of catastrophic events dominates the instantaneous growth rate of catastrophic intensity and the mean and standard deviations of the loss in determining the CatEPut prices under the doubly stochastic Poisson process.

The remainder of the paper is organized as follows. Section 2 illustrates the contract and the model assumptions. Section 3 shows the equivalent martingale probability measures and derives the pricing formula for contingent capital. Section 4 presents empirical and numerical analyses. Section 5 summarizes the article and presents the conclusions. For ease of exposition, most proofs are in an Appendix.

[1] The data were obtained by Insurance Service Office. The term "natural CAT" includes hurricanes, storms, floods, waves, and earthquakes. A "natural CAT" event denotes a natural disaster that affects many insurers, for which claims are expected to reach a certain dollar threshold. We also find that specific natural catastrophic events, such as wind and thunderstorm, hurricanes and waves events, display a trend of increasing CAT event frequency.

2. The contract and model

2.1. The concept of contingent capital

Contingent capital is an agreement entered into before any natural CAT losses occur, enabling the insurance company to raise cash by selling stock or issuing debt at prearranged terms following a natural CAT loss that exceeds a certain threshold. The insurance company pays a capital commitment fee (premium) to the party that agrees in advance to purchase debt or equity following a loss. Using contingent capital can enhance financial flexibility of the insurance company. With a contingent capital arrangement, the insurance company does not transfer its risk of loss to investors. Instead, after a specifically pre-defined natural CAT occurs, the company receives a capital injection in the form of debt or equity to help it pay the loss. Because the terms of the capital injection are pre-agreed, the insurance company generally receives more favorable terms than it would receive if it were to raise capital after a large CAT loss, when it is likely to be in a weakened financial condition. Contingent capital can be structured in various forms, but in general, two broad classes are considered: (1) the CSNs and (2) the CatEPuts.

Considering the issuers' perspective with the theory of firm's capital structure and financing decisions, the "pecking order theory" developed by Myers (1984) states that companies prioritize financing sources from internal funds, debt to equity. Furthermore, the "information theory" developed by Leland and Pyle (1977) and Myers and Majluf (1984) regards a company's issuing stocks as a sign of the coming of bad news, while debts, good news. According to these theories, the insurance company may prefer issuing CSNs to issuing CatEPuts. CSNs are made available to an insurer through a CSN trust, so the insurer issuing CSNs may incur higher transaction costs, including the commitment fees (premium), the expense of setting up an investment trust, and risk-evaluating cost than those issuing CatEPuts. In light of Generally Accepted Accounting Principles (GAAP), CSNs would appear on the insurer's balance sheet as the liability, whereas GAAP considers CatEPuts as the equity. The funding received by exercising the CatEPuts can increase the capital adequacy ratio of the insurance company. Nevertheless, exercising CatEPuts after a CAT may dilute the value of an insurer's outstanding shares. Investors can earn higher returns by investing in CSNs trusts than those by investing directly in Treasury securities. CatEPuts can provide investors with an equity interest in the insurer in exchange for the capital they provide. As we know, debts are senior to common stock because they have prior claim to the issuer's assets in the event of bankruptcy. Therefore, rather than purchasing CatEPuts, the investors would tend to purchase CSNs.

2.2. Structure of contingent capital

In light to the structure of contingent capital, the insurance company issues surplus notes, common, or preferred shares at a predetermined price, much like a regular put option. However, this right is exercisable only in the event when the accumulated losses, which cause financial distress of the purchaser of protection, exceed a critical coverage limit during the life time of the option. Hence, such a contract can be viewed as a special form of a double trigger put option, where the payoff for the option is a function of underlying asset price and level of insured losses. If no triggering catastrophic event occurs, then the insurance company has no need for additional capital and the facility remains unused. With $C(T)$ as the payoff of contingent capital at maturity time T, it is written mathematically as:

$$C(T) = 1_{\{L(T)>L+L(t_0)\}}(K - V(T))1_{\{V(T)<K\}}, \quad (1)$$

where $V(T)$ denotes the underlying asset price and $L(T) - L(t_0)$ denotes the total loss-percentage rate process over the time period $[t_0, T)$ making the underlying asset price to drop. L is the specified limit of losses, above which the contingent capital becomes in-the-money; while K represents the strike price, at which the issuer is obligated to purchase unit shares in the event that losses exceed L. In the case where $V(T)$ denotes the stock price, Eq. (1) represents the payoff of the CatEPut; in the case where $V(T)$ denotes the surplus notes price, Eq. (1) becomes the payoff of the CNS, where K typically expresses the Treasury note.

2.3. Model

Under the original probability measure P, our modeling assumption is supplied by the following dynamics of the underlying asset $V(t)$, the short interest rate $r(t)$, and the total loss-percentage rate process causing the drop in underlying asset prices $L(t)$:

$$V(t) = V(0)\exp\left\{\int_0^t \left(\mu(u) - \frac{1}{2}\sigma_v^2\right)\mathrm{d}u + \sigma_v W_v(t) - L(t)\right\}, \quad (2)$$

$$L(t) = \sum_{n=1}^{N(t)} h(Y_n, \tau_n), \quad (3)$$

$$\mathrm{d}r(t) = \tilde{k}[\tilde{\theta} - r(t)]\mathrm{d}t + \sigma_r \mathrm{d}W_r(t),$$

where the drift $\mu(u)$ is the instantaneous return of the underlying asset price at time u. $\tilde{k}$ and $\tilde{\theta}$ are the respective constant reversion rate and long-term mean in the drift coefficient of the Vasicek dynamics of the interest rate; σ_v represents the instantaneous volatility of returns of the underlying asset; σ_r expresses the instantaneous volatility of short-term interest rate. Term $W_v(t)$ is the Brownian motion of the returns of the underlying asset price and it can be used to capture the unanticipated instantaneous change of the underlying asset price, which is the reflection of ordinary (non-catastrophic) events. However, the Brownian motion of the returns of the underlying asset price may not work so well for loss amounts in excess of CAT threshold. $W_r(t)$ is the Brownian motion of the short interest rate that satisfies $Cov(W_v(t), W_r(t)) = \rho t$, where ρ is the correlation coefficient of the returns of the underlying asset price and the short interest rate.

$h(Y_n, \tau_n)$ represents the loss function when a specific CAT loss Y_n occurs at time τ_n, with $n \geq 1$, which captures the nth loss ratio that influences the downward jumps in the underlying asset price. Thus, it expresses that different specific losses have different impact on the drop in underlying asset price. $L(t)$ denotes the total loss-percentage rate process that affects the downward underlying asset price, and the range in $[0, \infty]$. $N(t)$ denotes the CAT number in $[0, t]$. For example, Jaimungal and Wang (2006) assume that $V(T)$ is the stock price, $N(t)$ stands for a pure Poisson process, and $h(Y_n, \tau_n) = \alpha Y_n$, where $\alpha \geq 0$ represents the constant rate drops in the stock price per unit of loss. In other words, per unit of each specific loss, $Y_n \geq 0, n \geq 1$, causes the same effect, α, on the stock price. We construct our model on a filtered probability space (Ω, P, F) generated by these three processes; i.e., $V(t)$, $r(t)$ and $L(t)$. The filtration $F = (F_t)_{t\geq 0}$ satisfies $F_t = F_t^W \vee F_t^L$ for any time t, where $F_t^W = \sigma((W_v(u), W_r(u)), 0 \leq u \leq t)$, and $F_t^L = \sigma(L(u), 0 \leq u \leq t)$. Hence, $F_T^W \vee F_T^L$ contains complete information on Brownian motions of the returns of the underlying asset price, the short interest rate, and the total loss-percentage rate process. The interest rate process and the loss process are assumed to be stochastically independent.

2.4. Doubly stochastic Poisson process

A doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Hence, the doubly stochastic Poisson process can be viewed as a two-step randomization procedure. A process $\lambda_d(t)$ is used to generate another process $\Phi(t)$ by acting as its arrival rate. Accordingly, we have:

$$P(\Phi(t) = m|\lambda_d(u), 0 \leq u \leq t) = \frac{\left(\int_0^t \lambda_d(u)\mathrm{d}u\right)^m}{m!}\exp\left[-\int_0^t \lambda_d(u)\mathrm{d}u\right] \quad P\text{-a.s.}$$

The equation above means that $\Phi(t)$ is P-almost surely equal to the distribution of a Poisson distribution given arrival rates of catastrophic events. Then the total loss-percentage rate process causing the drop in underlying asset prices,

$$L(t) = \sum_{n=1}^{\Phi(t)} h(Y_n),$$

is referred to as the compound doubly stochastic Poisson process. $\Phi(t)$ stands for the doubly stochastic Poisson process with the arrival rate of catastrophic events $\lambda_d(t)$. And Y_n, where $n = 1, 2, \ldots$ are identical incremental distribution random variables, also independent from $\Phi(t)$.

Due to global warming, the climate change over time will become more serious, and unanticipated catastrophic events also increase over time. A homogeneous Poisson process or a non-homogeneous Poisson process does not adequately explain this phenomenon of CATs. Nevertheless, using of the doubly stochastic Poisson process can represent the phenomenon that the arrival rate of catastrophic events is random and is associated with time. Referring to Fig. 1, the catastrophic event frequency seems to have a trend of exponential increase over time. Thus, we assume that the number of catastrophic events stands for the doubly stochastic Poisson process with lognormal intensity, and the process of the stochastic intensity could be expressed as

$$\lambda_d(t) = \lambda_d(0)\exp\left[\mu_\lambda t - \frac{1}{2}\sigma_\lambda^2 t + \sigma_\lambda W_\lambda(t)\right],$$

where μ_λ and σ_λ represent the instantaneous change rate and the volatilities of change rate of the arrival rate of catastrophic events, respectively. $W_\lambda(t)$ is the Brownian motion of the change rate of the arrival rate of catastrophic events. The Brownian motion for the arrival rate is assumed to be independent of the Brownian motion for the underlying asset price and the Brownian motion of the interest rate.

3. Pricing contingent capital

This section illustrates the martingale probability measure when the arrival rate of catastrophic events follows the doubly stochastic Poisson process. There are several choices of equivalent martingale probability measures to price contingent capital when the market is incomplete. The popular approaches for the risk neutral martingale measures are the Merton (1976) measure and the Esscher transform adopted from Gerber and Shiu (1994). The Merton measure, introduced by Cox et al. (2004) and Jaimungal and Wang (2006), assumes that the CAT risk presents non-systematic and diversifiable risk. On the other hand, the Esscher transform allows for the CAT risk, regarding which as systematic and non-diversifiable. This paper is not to decide which measure is more appropriate. Rather, we use the Merton measure to define the Radon–Nikodym derivative in incomplete market situations such that contingent capital formula is achieved.

3.1. Equivalent martingale probability measures

Here we compute contingent capital under a risk neutral probability measure. If a liquid market for contingent capital exists, then standard derivative pricing theory implies that an equivalent probability measure Q exists with respect to the real probability measure P, where the equivalent probability measure is not necessarily unique, so that underlying asset prices discounted at the risk-free rate are Q martingales denoted as the following.

Lemma 1. *Let $\beta_1(t)$ denote the Radon–Nikodym process for the doubly stochastic Poisson process as follows:*

$$\begin{aligned}\log\beta_1(t) = &-\frac{1}{2}\int_0^t[\eta_1^2(u,r(u))+\eta_2^2(u,r(u))]\mathrm{d}u\\&+\int_0^t\left[\eta_1(u,r(u))-\frac{\rho}{\sqrt{1-\rho^2}}\eta_2(u,r(u))\right]\mathrm{d}W_r(u)\\&+\int_0^t\frac{\eta_2(u,r(u))}{\sqrt{1-\rho^2}}\mathrm{d}W_v(u)+\left[\sum_{n=1}^{N_c(t)}\log\varphi(Y_n,\tau_n)\right.\\&\left.+\int_0^t\int_0^\infty[1-\varphi(y,u)]\lambda_d(u)f(y)\mathrm{d}y\mathrm{d}u\right],\end{aligned}$$

where $\int_0^t\eta_i^2(u,r(u))\mathrm{d}u<\infty, i=1,2$, and $\int_0^t\int_0^\infty\varphi(y,u)\lambda_d(u)f(y)\mathrm{d}y\mathrm{d}u<\infty$, there exists a probability measure Q for $dQ=\beta_1(t)dP$ such that the new Wiener process of the interest rate $W_r^Q(t)$ and the underlying asset price $W_v^Q(t)$ under the risk neutral probability measure are defined by $W_r^Q(t)=W_r(t)-\int_0^t\eta_1(u,r(u))\mathrm{d}u$, $W_v^Q(t)=W_v(t)-\int_0^t\eta_2(u,r(u))\mathrm{d}u$, the new arrival rate under the doubly stochastic Poisson process at time u, $u\in[0,t]$ becomes $\lambda_d^Q(u)=\varphi(y,u)\lambda_d(u)f(y)\mathrm{d}y$.

$\eta_1(u,r(u))$ and $\eta_2(u,r(u))$ may be interpreted as the market prices of risk associated with the component of the Brownian motion for the interest rate and underlying asset price at time u. $\varphi(y,u)$ denotes the CAT risk premium connected with the arrival of loss y at time u.[2] The processes $\eta_i(u,r(u))$, $i=1,2$ and $\varphi(y,u)$ need to be determined when the martingale condition for discount underlying asset price is satisfied and the martingale condition under the doubly stochastic Poisson process can be expressed as:

$$\begin{aligned}&\mu(u)-r(u)+\sigma_v\rho\eta_1(u,r(u))+\sigma_v\sqrt{1-\rho^2}\eta_2(u,r(u)),\\&-\int_0^\infty[1-\mathrm{e}^{-h(y,u)}]\varphi(y,t)\lambda_d(u)f(y)\mathrm{d}y=0,\quad\forall u.\end{aligned}$$

Note that as the CAT risk presents non-systematic and diversifiable risk, and then the Merton measure is used. It follows that the risk premium for the CAT risk is $\varphi(y,u)=1$, meaning that the investors receive a zero premium for the CAT risk and the catastrophic arrival rate and distribution are unaffected by the measure change. Furthermore, the market prices of risk associated with the component of the Brownian motion for the interest rate and the underlying asset price are determined respectively as:

$$\begin{aligned}\eta_1(u,r(u))&=\frac{1}{\sigma_r}[k\theta-\tilde{k}\tilde{\theta}+(\tilde{k}-k)r(u)],\\\eta_2(u,r(u))&=-\frac{[\mu(u)-r(u)]}{\sigma_v\sqrt{1-\rho^2}}-\frac{\rho}{\sqrt{1-\rho^2}}\eta_1(u,r(u)).\end{aligned}$$

The Esscher transform is adapted when the CAT risks are systematic and are non-diversifiable risks, thus $\varphi(y,u)=\exp(wy)$, is the risk premium of the CAT risk and the new catastrophic arrival rate is $\lambda_d^Q(\mathrm{d}y,u)=\exp(wy)\lambda_d(\mathrm{d}y,u)$, $u\in[0,t]$, where w is a real number.

Then, with the Merton measure, the underlying asset price and interest rate process under Q can be written as:

$$\begin{aligned}V(t)&=V(0)\exp\left\{\int_0^t\left(r(u)-\frac{1}{2}\sigma_v^2\right)\mathrm{d}u+\sigma_vW_v^Q(t)\right.\\&\qquad\left.-\left[L(t)-\int_0^t\int_0^\infty(1-\mathrm{e}^{-h(y,u)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u\right]\right\},\quad(4)\\\mathrm{d}r(t)&=k[\theta-r(t)]\mathrm{d}t+\sigma_rdW_r^Q(t),\end{aligned}$$

where $Cov(W_v^Q(t),W_r^Q(t))=\rho t$.

In the presence of stochastic interest rates, a similar factorization can be obtained by performing a measure change to the forward neutral measure Q^T. This measure is defined by choosing the T-maturity zero coupon bond as the numeraire asset. We assume that the interest rate follows Vasicek model, thus the price of the T-maturity zero coupon bond, $B(t,T)$, in the Vasicek model is given by:

$$B(t,T)=\exp(A(t,T)-U(t,T)r(t)),$$

where

$$\begin{aligned}A(t,T)&=\left(\theta-\frac{\sigma_r^2}{2k^2}\right)(B(t,T)-(T-t))-\frac{\sigma_r^2}{4k}B^2(t,T),\\U(t,T)&=\frac{1}{k}[1-\exp(-k(T-t))],\end{aligned}$$

and therefore the bond price process satisfies

$$\frac{\mathrm{d}B(t,T)}{B(t,T)}=r(t)\mathrm{d}t-\sigma_rU(t,T)\mathrm{d}W_r^Q(t).$$

Let $\beta_2(t)$ denote the Radon–Nikodym process as follows:

$$\begin{aligned}\beta_2(t)=\left(\frac{\mathrm{d}Q^T}{\mathrm{d}Q}\right)_t&=\exp\left\{-\frac{1}{2}\int_0^t\sigma_r^2U^2(u,T)\mathrm{d}u\right.\\&\qquad\left.-\int_0^t\sigma_rU(u,T)\mathrm{d}W_r^Q(u)\right\},\end{aligned}$$

where $\beta_2(0)=1$, and suppose $E^Q[\beta_2(t)]=1$ for all t. There exists a forward neutral probability measure Q^T with $\mathrm{d}Q^T=\beta_2(t)\mathrm{d}Q$ so that, under Q^T, Girsanov's theorem of $\tilde{W}_v(t)$ and $\tilde{W}_r(t)$ are defined by

$$\begin{aligned}\tilde{W}_v(t)&=W_v^Q(t)+\int_0^t\rho\sigma_rU(u,T)\mathrm{d}u,\\\tilde{W}_r(t)&=W_r^Q(t)+\int_0^t\sigma_rU(u,T)\mathrm{d}u,\end{aligned}$$

and

$$\mathrm{d}(\tilde{W}_v(t),\tilde{W}_r(t))=\rho\mathrm{d}t.$$

Under Q^T, the forward price of the underlying asset under the doubly stochastic Poisson process is calculated as:

$$\begin{aligned}\frac{V(T)}{B(T,T)}=&\frac{V(t)}{B(t,T)}\exp\left[-\frac{1}{2}\int_t^T\tilde{\sigma}^2(u,T)\mathrm{d}u+\int_t^T\sigma_vd\tilde{W}_v(u)\right.\\&+\int_t^T\sigma_rB(u,T)d\tilde{W}_r(u)-((L(T)-L(t))\\&\left.-\int_t^T\int_0^\infty(1-\mathrm{e}^{-h(y,u)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u)\right],\qquad(5)\end{aligned}$$

where $\tilde{\sigma}^2(u,T)=\sigma_r^2B^2(u,t)+2\rho\sigma_v\sigma_rB(u,t)+\sigma_v^2$.

[2] Bjork et al. (1997) and Glasserman and Kou (2003) refer to the Radon–Nikodym process of the marked point process which $\varphi(y,u)$ is the jump risk premium of the forward rate process at time u.

$$P_d(t;t_0) = E^{Q_L^T}\left[1_{\{L(T)-L(t)>\tilde{L}\}}\left\{KB(t,T)\mathbb{N}(-d_2^d) - V(t)\exp(-[L(T) - L(t) - \int_t^T\int_0^\infty (1-e^{-h(y,u)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u])\mathbb{N}(-d_1^d)\right\}|F_t^L\right],$$

where

$$d_{1,2}^d = \frac{\ln(V(t)/KB(t,T)) \pm \frac{1}{2}\tilde{\sigma}^2(t,T) - (L(T)-L(t) - \int_t^T\int_0^\infty(1-e^{-h(y,u)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u)}{\tilde{\sigma}(t,T)},$$

$$\tilde{\sigma}^2(t,T) \equiv \int_t^T \tilde{\sigma}^2(u,T)\mathrm{d}u = \sigma_v^2(T-t) + \frac{2k\rho\sigma_r\sigma_v + \sigma_r^2}{k^2}[(T-t) - U(t,T)] - \frac{\sigma_r^2}{2k}U^2(t,T).$$

Box I.

3.2. Valuation of contingent capital

Once the equivalent martingale probability measure is decided, contingent capital can be valued by the discounted expectation of its various payoffs in a risk neutral world. Let $P(t;t_0)$ represent the value of the option at time t, which was signed at time $t_0 < t$ and matures at time T, we have

$$P(t;t_0) = E^{Q^T}[B(t,T)1_{\{L(T)>L+L(t_0)\}}(K-V(T))1_{\{V(T)<K\}}].$$

Based on the forward price of the underlying asset of Eq. (5), the value of contingent capital, $P_d(t;t_0)$ at time t, which was signed at time $t_0 < t$, is given as in Box I. $\mathbb{N}(\cdot)$ denotes the cumulative distribution function of a standard normal random variable and $\tilde{L} = L + L(t_0) - L(t)$.

Equation in Box I is viewed as the expectation of put options conditional on that the total loss-percentage rate process exceeds specified losses under two random variables: the intensity of catastrophic events and the loss size. The detailed proof is shown in Appendix A.

Corollary 3.1. *Suppose $h(Y_n) = \alpha Y_n$, $\alpha \geq 0$. Then equation in Box I could boil down to the following pricing formula:*

$$P_{ds}(t;t_0) = \int_0^\infty\int_{\tilde{L}}^\infty \sum_{m=1}^\infty \frac{e^{-\bar{\lambda}(T-t)}(\bar{\lambda}(T-t))^m}{m!} f_L^m(x) \times [KB(t,T)\mathbb{N}(-d_2^{ds}) - V(t)\exp(-\alpha x + \bar{\lambda}(T-t)\kappa) \times \mathbb{N}(-d_1^{ds})]h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}x\mathrm{d}\bar{\lambda}, \quad (6)$$

where $\bar{\lambda}$ expresses the mean arrival rate of catastrophic events from time t to maturity T defined by $\bar{\lambda} = \frac{\int_t^T \lambda_d^Q(u)\mathrm{d}u}{T-t}$. $h(\bar{\lambda}|\lambda_d^Q(t))$ represents the conditional distribution density function of $\bar{\lambda}$ given $\lambda_d^Q(t)$, and $f_L^m(x)$ denotes the m-fold loss distribution density function. $\kappa = \int_0^\infty (1-e^{-\alpha Y})f(y)\mathrm{d}y$, $d_{1,2}^{ds} = \frac{\ln(V(t)/KB(t,T)) \pm \frac{1}{2}\tilde{\sigma}^2(t,T) - \alpha x + \bar{\lambda}(T-t)\kappa}{\tilde{\sigma}(t,T)}$.

The detailed proof is shown in Appendix B. Eq. (6) indicates that the value of contingent capital under the doubly stochastic Poisson process is regarded as the double integration of the distribution of arrival rate, and the distribution of loss size for the put option with the limitation of the total loss-percentage rate process of the insured exceeds specified losses. If the Esscher transform allowing for the systematic and non-diversifiable CAT risk is used, it follows that $\varphi(y,u) = \exp(-w\alpha y)$, such that the doubly stochastic Poisson process has the new arrival rate $\exp(-w\alpha y)\lambda_d(u)f(y)\mathrm{d}y$, where w is a real number.

Remark 3.2. For $\alpha \to \infty$, which implies that the impact of the level of CAT losses on drops in the stock price is so huge that the insurance company goes bankruptcy (the stock price goes down to zero), thus Corollary 3.1 could reduce to the following pricing formula:

$$P_{dd}(t;t_0) = \int_0^\infty\int_{\tilde{L}}^\infty \sum_{m=1}^\infty \frac{e^{-\bar{\lambda}(T-t)}(\bar{\lambda}(T-t))^m}{m!} f_L^m(x)[KB(t,T) - V(t)\exp(\bar{\lambda}(T-t))]h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}x\mathrm{d}\bar{\lambda}. \quad (7)$$

Remark 3.3. Let $V(T)$ denotes the stock price and let $\mu_\lambda = \sigma_\lambda = 0$, the stochastic arrival rate under the doubly stochastic Poisson process reduces to the constant arrival rate of the pure Poisson process. In this case, the result of Corollary 3.1 becomes the pricing formula of Jaimungal and Wang (2006). Additionally, if $\mu_\lambda = \sigma_\lambda = 0$, $Y = 1$, and the interest rate is deterministic, then the result of Corollary 3.1 reduces to a result similar to Cox et al. (2004).

Corollary 3.4. *Assume $h(Y_n) = \ln Y_n$ and the loss Y_n is drawn from the lognormal distribution[3] with mean μ_y and variance σ_y^2 ($Y_n \sim \log N(\mu_y, \sigma_y^2)$). And let the volatility of change rate of the arrival rate of catastrophic events $\sigma_\lambda = 0$, then equation in Box I could reduce to the following pricing formula of CatEPut, which the underlying asset price $V(t)$ is the stock price, given by*

$$P_{DS}(t;t_0) = \sum_{m=1}^\infty \frac{e^{-\tilde{\lambda}_d^Q(T)(1-\tilde{\kappa})}(\tilde{\lambda}_d^Q(T)(1-\tilde{\kappa}))^m}{m!}\{KB(t,T) \times \exp(-\tilde{\lambda}_d^Q(T)\tilde{\kappa}(T-t) - \ln m(1-\tilde{\kappa})) \times \mathbb{N}_2(-f_1, g_1, \rho) - V(t)\mathbb{N}_2(-f_1, g_2, \rho)\} \quad (8)$$

where

$$\tilde{\kappa} = \int_0^\infty (1-e^{-\ln Y})f(y)\mathrm{d}y,$$

$$\tilde{\lambda}_d^Q(T) = \int_t^T \lambda_d^Q(t)e^{\mu_\lambda(u-t)}\mathrm{d}u = \frac{\lambda_d^Q(t)}{\mu_\lambda}[e^{\mu_\lambda(T-t)} - 1],$$

$$f_1 = \frac{\tilde{L} - m\mu_y}{\sqrt{m}\sigma_y},$$

$$g_1 = \frac{\ln(KB(t,T)/V(t)) + \frac{1}{2}\tilde{\sigma}^2(t,T) + m\mu_y - \tilde{\lambda}_d^Q(T)\tilde{\kappa}}{\sqrt{\tilde{\sigma}^2(t,T) + m\sigma_y^2}},$$

$$\rho = \frac{-m\sigma_y^2}{\sqrt{m}\sigma_y\sqrt{\tilde{\sigma}^2(t,T) + m\sigma_y^2}},$$

[3] Most previous articles, such as Louberge et al. (1999) and Lee and Yu (2002), assume that CAT loss follows a mutually independent, identical, and lognormal distribution. In their empirical studies, Cummins et al. (1999) and Burnecki et al. (2000) show empirical results that the lognormal distribution seems to give a better fit for PCS indices.

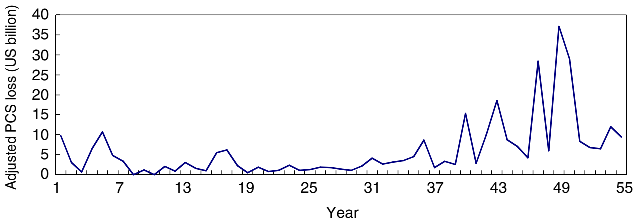

Fig. 2. Adjusted PCS loss for catastrophic events in the United States during 1950 to 2004.

$$g_2 = \frac{\ln(KB(t,T)/V(t)) - \frac{1}{2}\tilde{\sigma}^2(t,T) + m(\mu_y - \sigma_y^2) - \tilde{\lambda}_d^Q(T)\tilde{\kappa}}{\sqrt{\tilde{\sigma}^2(t,T) + m\sigma_y^2}},$$

and $\mathbb{N}_2(\cdot)$ *is the cumulative distribution function of bivariate normal random variables with correlation coefficient* ρ.

Remark 3.5. Suppose $Y \to \infty$, which implies that the level of CAT losses is so high that the insurance company might not have sufficient capital to cover and even goes bankruptcy (the stock price goes down to zero). For example, Hurricane Andrew hit Florida caused more than $25 billion losses in 1992 and led 11 insurance companies to go bankrupt. In view of such a case, Corollary 3.4 could reduce to Eq. (8).

4. Empirical experiments and numerical analyses

4.1. Data description

In this section, we consider that an issuer company issues a CatEPut contract of which the loss is linked to the Property Claim Services (PCS) loss in the United States.[4] The data adapted comes from the Insurance Service Office. This section focuses on the national natural losses, including all natural catastrophic events in the United States. Prior to estimating the parameters of the frequency and severity distributions, two adjustments affecting the frequency and severity of CATs are made to the PCS data. We follow Cummins et al. (1999) to make changes in construction costs by using the United States Department of Commerce Census Fixed Weighted Construction Cost Index and population obtained from the United States Census Bureau. The adjusted PCS loss for catastrophic events during 1950 to 2004 is displayed in Fig. 2.

After adjusting for inflation and population, we estimate the parameters of lognormal distribution for the adjusted PCS loss severity, and then the mean μ_y and the standard deviation σ_y are estimated to be 0.06 and 0.69, i.e., $Y_n \sim \log N(0.06, 0.48)$. The parameters of lognormal stochastic process for the frequency of the catastrophic events are also estimated. Given $\sigma_\lambda = 0$, the instantaneous change rate of the arrival rate of catastrophic events μ_λ and the initial arrival rate of catastrophic events $\lambda_d^Q(0)$ are estimated to be 0.05 and 2.22, i.e., $\lambda_d^Q(t) = 2.22\exp[0.05(t-1)]$, $t = 1, 2, \ldots 55$. We calculate the average arrival rate overall years to be 14.35 and take it as the arrival rate of the pure Poisson process to reflect the frequencies of all catastrophic events per year.

4.2. Global fit error measurements

This subsection considers the quality of the fitting under the doubly stochastic Poisson process and the pure Poisson process when using Eq. (8). In order to compute the value of CatEPut, we give the parametric values: stock price of the insurance company, $V = 25$; exercise price, $K = 80$; trigger level of losses ratio, $L = 0.1$; parameters of the interest rate model are: $r(0) = 2\%$, $k = 0.3$, $\theta = 5\%$, $\rho = -0.1$, $\sigma_r = 15\%$; stock volatility, $\sigma_v = 0.2$; option term, $T = 4$. The infinite summation over m would be truncated at level $m = 250$ so that the respective cumulative Poisson probabilities are very close to 1. The contract parameters and the chosen model parameters follow the assumptions of Jaimungal and Wang (2006). The data of annual number of all natural catastrophic events in the United States from 1950 to 2004 is used to obtain the real CatEPut value (P_R). Furthermore, the average arrival rate of pure Poisson process (14.35) and the arrival rate of the doubly stochastic Poisson process ($\lambda_d^Q(t) = 2.22\exp[0.05(t-1)]$), $t = 1, 2, \ldots 55$, are used to evaluate the theoretical CatEPut value (P_T). For comparison and to judge the goodness (advantage) under the doubly stochastic Poisson process and the pure Poisson process, we compute four global measurements of fit, which are: average percentage error (APE), average absolute error (AAE), the average relative percentage error (ARPE) and relative mean square error (RMSE):

$$APE = \frac{1}{E(P_R)}\sum_{n=1}^{N}\frac{|P_R - P_T|}{N}, \qquad AAE = \sum_{n=1}^{N}\frac{|P_R - P_T|}{N},$$

$$ARPE = \frac{1}{N}\sum_{n=1}^{N}\frac{|P_R - P_T|}{P_R}, \qquad RMSE = \sqrt{\sum_{n=1}^{N}\frac{(P_R - P_T)^2}{N}},$$

where $E(P_R)$ is the mean of the real CatEPut value and N is the total number of observations.

Table 1 gives an overview of these measurements to show that the four measurements of fit under the doubly stochastic Poisson process are all smaller than those under the pure Poisson process. This implies that the doubly stochastic Poisson process is fitter than the pure Poisson process to model the arrival rate of catastrophic events when pricing the CatEPut. In particular, under the doubly stochastic Poisson process, the largest fit improvement is 8.251 when calculating the AAE. Hence the pricing formula we provide could be more accurate than that of Jaimungal and Wang (2006).

The parameters of base valuation are $V = 25$, $K = 80$, $L = 0.1$, $r(0) = 2\%$, $k = 0.3$, $\theta = 5\%$, $\rho = -0.1$, $\sigma_v = 0.2$, $T = 4$, $m = 250$, $\mu_y = 0.06$, $\sigma_y = 0.69$, $N = 55$. The arrival rate of the PP, $\lambda = 14.35$. The arrival rate of the DSPP is ($\lambda_d^Q(t) = 2.22\exp[0.05(t-1)]$), $t = 1, 2, \ldots 55$. DSPP and PP indicate the doubly stochastic Poisson process and the pure Poisson process, respectively.

Table 1
Global fit error measurements.

	DSPP	PP
APE	0.310	0.761
AAE	8.364	16.615
ARPE	0.052	0.311
RMSE	11.747	19.721

Table 2
The value of CatEPut under the doubly stochastic Poisson process.

σ_y	μ_y	$(\lambda_d^Q, \mu_\lambda)$			
		(10, 0)	(10, 0.1)	(20, 0)	(20, 0.1)
0.4	0.05	19.859	20.489	35.561	36.741
	0.1	24.482	25.772	40.616	42.026
0.8	0.05	24.851	26.112	42.419	43.729
	0.1	31.142	33.056	51.274	53.964

4.3. Numerical analysis

This section considers a numerical analysis for the CatEPut price under various changes of parameters using the doubly stochastic Poisson process. Table 2 investigates the value of CatEPut under the doubly stochastic Poisson process with different parameter changes. When $(\lambda_d^Q, \mu_\lambda) = (10, 0)$ and $(20, 0)$, it implies that the arrival rate of catastrophic events is constant in each year and equal to 10 and 20, respectively. Hence, they can represent the

[4] The PCS has catalogued all CAT losses on national, regional, and state basis in the United States. The PCS periodically changes the minimum criterion for catastrophe.

$$
\begin{aligned}
& E^{Q_L}\{1_{\{L(T)-L(t)>L+L(t_0)-L(t)\}}KB(t,T)E^{Q_{\tilde{W}}^T} \\
& \quad \times \{1_{\{\frac{V(t)}{B(t,T)}\exp(-\frac{1}{2}\int_0^t \tilde{\sigma}^2(s,T)\mathrm{d}s+\int_0^t \sigma_v \mathrm{d}\tilde{W}_v(s)+\int_0^t \sigma_r B(s,T)\mathrm{d}\tilde{W}_r(s)-[(L(T)-L(t))-\int_t^T\int_0^\infty (1-\mathrm{e}^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u]<K)\}}|F_T^L \vee F_t^{\tilde{W}}\}|F_t\} \\
& = E^{Q_L}\{1_{\{L(T)-L(t)>\tilde{L}\}}KB(t,T)\times E^{Q_{\tilde{W}}^T}\{1_{\{\tilde{W}<-[\ln(\frac{V(t)}{KB(t,T)})-\frac{1}{2}\tilde{\sigma}^2(t,T)-[(L(T)-L(t))-\int_t^T\int_0^\infty(1-\mathrm{e}^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u]/\tilde{\sigma}(t,T)]\}}|F_T^L \vee F_t^{\tilde{W}}\}|F_t\} \\
& = E^{Q_L}\{[1_{\{L(T)-L(t)>\tilde{L}\}}KB(t,T)\mathbb{N}(-d_2^d)]|F_t\}
\end{aligned}
$$

where

$$\tilde{L} = L + L(t_0) - L(t),$$

$$d_2^d = \frac{\ln(\frac{V(t)}{KB(t,T)}) - \frac{1}{2}\tilde{\sigma}^2(t,T) - [(L(T)-L(t)) - \int_t^T \int_0^\infty (1-\mathrm{e}^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u]}{\tilde{\sigma}(t,T)}.$$

Box II.

cases under the pure Poisson process. Owing to the increasing catastrophic events, one can project that the instantaneous growth rate of catastrophic intensity μ_λ is positive instead of zero. We find that the CatEPut price under the doubly stochastic Poisson process is larger than that under the pure Poisson process in response to positive μ_λ. Moreover, when arrival rate of catastrophic events λ_d^Q increases, it increases the volatility of CatEPut and further increases the value of CatEPut. This table also reveals that both higher mean and standard deviations of the CAT loss result in higher CatEPut price. As well, it shows that arrival rate of catastrophic events dominates the instantaneous growth rate of catastrophic intensity, the mean of the CAT loss, and standard deviation of the CAT loss in determining the CatEPut prices under the doubly stochastic Poisson process. The parameters of base valuation are $V = 25$, $K = 80$, $L = 0.1$, $r(0) = 2\%$, $k = 0.3$, $\theta = 5\%$, $\rho = -0.1$, $\sigma_v = 0.2$, $T = 4$, $m = 250$.

5. Conclusions

The IPCC Fourth Assessment Report published in 2007 shows that, due to global warming, the changing of the earth's climate in the future would be more serious, with more unanticipated catastrophic events. In this circumstance, the previous articles' assumption that catastrophic events occur in terms of the pure Poisson process seems inappropriate. This paper proposes the doubly stochastic Poisson process to well grasp the arrival process for catastrophic events. In addition, we generalize the assumption of Jaimungal and Wang (2006) to define the general loss function, presenting that different specific losses have different impacts on the drop in stock price. Most previous articles focus on the pricing of hedging instruments rather than on contingent capital. In our paper, the general pricing formula for contingent capital is derived, and our pricing formula could reduce to the results of Cox et al. (2004) or Jaimungal and Wang (2006) when the underlying asset is stock price.

Based on the data from PCS loss index and the annual number of natural catastrophic events during 1950 to 2004, the experiment result shows that the doubly stochastic Poisson process is fitter than the pure Poisson process when pricing the CatEPut. Numerical example shows that the CatEPut price under the doubly stochastic Poisson process is larger than that under the pure Poisson process as the instantaneous growth rate of catastrophic intensity rises. Furthermore, with a higher arrival rate, mean of the loss, and standard deviation of the loss, there is a higher CatEPut price. It also shows that, based on the numerical example, arrival rate of catastrophic events dominates the instantaneous growth rate of catastrophic intensity, the mean of the loss, and standard deviation of the CAT loss in determining the CatEPut prices under the doubly stochastic Poisson process.

Appendix A

The proof of the formula of equation in Box I is sketched. Let $P_d(t;t_0)$ represent the value of the option at time t, which was signed at time $t_0 < t$ and matures at time T, we have the following equation:

$$P_d(t;t_0) = E^{Q^T}[1_{\{L(T)>L+L(t_0)\}}B(t,T)(K-V(T))1_{\{V(T)<K\}}|F_t] \quad \text{(A.1)}$$

where $Q^T = (Q_L^T, Q_{\tilde{W}}^T)$, and $Q_L^T = Q_L$. Using the law of expected iteration, Eq. (8) can be rewritten as:

$$
\begin{aligned}
& E^{Q_L}[1_{\{L(T)>L+L(t_0)\}}E^{Q_{\tilde{W}}^T}(KB(t,T)1_{\{V(T)<K\}}|F_T^L \vee F_t^{\tilde{W}})|F_t] \\
& \quad - E^{Q_L}[1_{\{L(T)>L+L(t_0)\}}E^{Q_{\tilde{W}}^T}(V(T)B(t,T)1_{\{V(T)<K\}}|F_T^L \vee F_t^{\tilde{W}})|F_t]
\end{aligned}
$$

where $F_T^L \vee F_t^{\tilde{W}}$ contains complete information on Brownian motions of the returns of the underlying asset price, the short interest rate, and the total loss-percentage rate process under the forward probability measure. E^{Q_L} denotes the expectation under the risk neutral probability measure Q conditional on the information of $L(t)$, and $E^{Q_{\tilde{W}}^T}$ denotes the expectation under the forward probability measure Q^T conditional on the information of $\tilde{W}(t)$.

Firstly, the equation

$$E^{Q_L}[1_{\{L(T)>L+L(t_0)\}}E^{Q_{\tilde{W}}^T}(KB(t,T)1_{\{V(T)<K\}}|F_T^L \vee F_t^{\tilde{W}})|F_t]$$

is computed. Given the information of $F_T^L \vee F_t^{\tilde{W}}$, we have the equations as in Box II.

Similarly, using the same computing procedures, we can also obtain that

$$
\begin{aligned}
& E^{Q_L}[1_{\{L(T)>L+L(t_0)\}}E^{Q_{\tilde{W}}^T}(V(T)B(t,T)1_{\{V(T)<K\}}|F_T^L \vee F_t^{\tilde{W}})|F_t] \\
& \quad = E^{Q_L}\left[1_{\{L(T)-L(t)>\tilde{L}\}}E^{Q_{\tilde{W}}^T}(V(t)\exp\left\{-\frac{1}{2}\int_0^t \tilde{\sigma}^2(s,T)\mathrm{d}s\right.\right. \\
& \quad + \int_0^t \sigma_v d\tilde{W}_v(s) + \int_0^t \sigma_r B(s,T)d\tilde{W}_r(s) - \Bigg[(L(T)-L(t)) \\
& \quad \left.\left. - \int_t^T \int_0^\infty (1-\mathrm{e}^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u\Bigg]\right\} 1_{\{V(T)<K\}}|F_T^L \vee Ft^{\tilde{W}})|F_t\right].
\end{aligned} \quad \text{(A.2)}
$$

Denote the Radon–Nikodym process for Brownian motion is given by the following formula:

$$
\begin{aligned}
\left(\frac{\mathrm{d}R}{\mathrm{d}Q_{\tilde{W}}^T}\right)_{T-t} &= \exp\left\{\int_t^T \sigma_v(s,T)d\tilde{W}_v(s)\right. \\
& \quad \left. + \int_t^T \sigma_r(s,T)B(s,T)d\tilde{W}_r(s) - \frac{1}{2}\int_t^T \tilde{\sigma}^2(s,T)ds\right\}.
\end{aligned}
$$

Hence, Eq. (A.2) can be rewritten as in Box III.

$$E^{Q_L}\left\{1_{\{L(T)-L(t)>\bar{L}\}}E^{Q_{\tilde{W}}^T}\left(V(t)\exp\left(-\left[(L(T)-L(t))-\int_t^T\int_0^\infty(1-e^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u\right]\right)\right.\right.$$
$$\left.\left.\times 1_{\{\tilde{W}<-[\ln(\frac{V(t)}{KB(t,T)})+\frac{1}{2}\tilde{\sigma}^2(t,T)-[(L(T)-L(t))-\int_t^T\int_0^\infty(1-e^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u]/\tilde{\sigma}(t,T)]\}}|F_T^L\vee F_t^{\tilde{W}}\right)|F_t\right\}$$
$$=E^{Q_L}\left\{1_{\{L(T)-L(t)>\bar{L}\}}V(t)\exp\left(-\left[(L(T)-L(t))-\int_t^T\int_0^\infty(1-e^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u\right]\right)\mathbb{N}(-d_1^d)|F_t\right\}$$

where

$$d_1^d=\frac{\ln(\frac{V(t)}{KB(t,T)})+\frac{1}{2}\tilde{\sigma}^2(t,T)-[(L(T)-L(t))-\int_t^T\int_0^\infty(1-e^{-h(y,s)})\lambda_d^Q(u)f(y)\mathrm{d}y\mathrm{d}u]}{\tilde{\sigma}(t,T)}.$$

Box III.

Appendix B

The proof for the closed-form formula of Eq. (6) is sketched. The present value of the expected terminal value of contingent capital discounted at the zero coupon bond under the forward probability measure is shown as follows:

$$\begin{aligned}P_{ds}(t;t_0)&=E^{Q^T}[1_{\{L(T)>L+L(t_0)\}}B(t,T)(K-V(T))1_{\{V(T)<K\}}|F_t]\\&\equiv F\left(V(t),\sum_{n=1}^{\Phi(t)}Y_n,\lambda_d^Q(t)\right)\\&=B(t,T)\int_0^\infty F\left(V(T),\sum_{n=1}^{\Phi(T-t)}Y_n,\lambda_d^Q(T)\right)\\&\quad\times f\left(V(T)|\sum_{n=1}^{\Phi(T-t)}Y_n,\lambda_d^Q(t)\right)\mathrm{d}V(T)\end{aligned}$$

where the condition distribution of $V(T)$, conditional on $\Phi(T-t)=m, Y_1=y_1, Y_2=y_2,\ldots,Y_n=y_n,\lambda_d^Q(t)$, can be shown as:

$$f(V(T)|\Phi(T-t)=m,Y_1=y_1,Y_2=y_2,\ldots,Y_n=y_n,\lambda_d^Q(t))$$
$$=\int_0^\infty f_L^m(x)\mathrm{d}x\sum_{m=0}^\infty\mathrm{Prob}(\Phi(T-t)=m)f(V(T)|\lambda_d^Q(t))\qquad(\mathrm{B.1})$$

where $f_L^m(x)$ denotes m-fold loss probability density function and $\mathrm{Prob}(\Phi(T-t)=m)=\frac{\exp(-\int_t^T\lambda_d^Q(u)\mathrm{d}u)(\int_t^T\lambda_d^Q(u)\mathrm{d}u)^m}{m!}$.

Following the pricing procedure of Hull and White (1987), define that $\bar{\lambda}=\frac{\int_t^T\lambda_d^Q(u)\mathrm{d}u}{T-t}$ is the mean arrival rate over the time $[t,T]$, and make use of the fact that, for any three related random variables x, y, and z, the conditional density functions are related by

$$f(x|y)=\int g(x|z)h(z|y)\mathrm{d}z.$$

Then Eq. (B.1) can be rewritten as

$$\begin{aligned}&\int_0^\infty f_L^m(x)\mathrm{d}x\sum_{m=0}^\infty\mathrm{Prob}(\Phi(T-t)=m)f(V(T)|\lambda_d^Q(t))\\&=\int_0^\infty f_L^m(x)\mathrm{d}x\sum_{m=0}^\infty\mathrm{Prob}(\Phi(T-t)=m)\\&\quad\times\int_0^\infty g(V(T)|\bar{\lambda})h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}\bar{\lambda}\end{aligned}$$

where $h(\bar{\lambda}|\lambda_d^Q(t))$ represents the conditional distribution density function of $\bar{\lambda}$ given $\lambda_d^Q(t)$, and $g(V(T)|\bar{\lambda})$ denotes the conditional distribution density function of $V(T)$ given $\bar{\lambda}$.

Hence, if the loss function simplifies to $h(Y_n)=\alpha Y_n$, the value of contingent capital under the doubly stochastic Poisson process, $P_{ds}(t;t_0)$ can be represented as

$$\begin{aligned}P_{ds}(t;t_0)&=\int_0^\infty 1_{\{L(T)>L+L(t_0)\}}B(t,T)(K-V(T))\\&\quad\times 1_{\{V(T)<K\}}f(V(T)|\sum_{n=1}^{\Phi(T-t)}Y_n,\lambda_d^Q(t))dV(T)\\&=\int_0^\infty B(t,T)(K-V(T))1_{\{V(T)<K\}}\int_{\bar{L}}^\infty f_L^m(x)\mathrm{d}x\\&\quad\times\sum_{m=0}^\infty\mathrm{Prob}(\Phi(T-t)=m)\int_0^\infty g(V(T)|\bar{\lambda})h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}\bar{\lambda}\mathrm{d}V(T)\\&=\int_0^\infty\int_{\bar{L}}^\infty f_L^m(x)\mathrm{d}x\sum_{m=0}^\infty\mathrm{Prob}(\Phi(T-t)=m)\\&\quad\times\left[B(t,T)\int_0^\infty(K-V(T))1_{\{V(T)<K\}}g(V(T)|\bar{\lambda})\mathrm{d}V(T)\right]\\&\quad\times h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}\bar{\lambda}\end{aligned}$$

where the inner term

$$\left[B(t,T)\int_0^\infty(K-V(T))1_{\{V(T)<K\}}g(V(T)|\bar{\lambda})\mathrm{d}V(T)\right]$$

represents the put option price at time t on the underlying asset with a mean arrival rate $\bar{\lambda}$. By the similar pricing procedure of Appendix A, we have:

$$\begin{aligned}&\int_0^\infty\int_{\bar{L}}^\infty\sum_{m=1}^\infty\frac{\exp\left(-\int_t^T\lambda_d^Q(u)\mathrm{d}u\right)\left(\int_t^T\lambda_d^Q(u)\mathrm{d}u\right)^m}{m!}f_L^m(x)\\&[KB(t,T)\mathbb{N}(-d_2^{ds})-V(t)\exp(-\alpha x+\bar{\lambda}(T-t)\kappa)\mathbb{N}(-d_1^{ds})]\\&\quad\times h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}x\mathrm{d}\bar{\lambda}=\int_0^\infty\int_{\bar{L}}^\infty\sum_{m=1}^\infty\frac{e^{-\bar{\lambda}(T-t)}(\bar{\lambda}(T-t))^m}{m!}f_L^m(x)\\&\quad\times[KB(t,T)\mathbb{N}(-d_2^{ds})-V(t)\exp(-\alpha x+\bar{\lambda}(T-t)\kappa)\mathbb{N}(-d_1^{ds})]\\&\quad\times h(\bar{\lambda}|\lambda_d^Q(t))\mathrm{d}x\mathrm{d}\bar{\lambda}\end{aligned}$$

where

$$\bar{\lambda}=\frac{\int_t^T\lambda_d^Q(u)\mathrm{d}u}{T-t},\qquad\kappa=\int_0^\infty(1-e^{-\alpha Y})f(y)\mathrm{d}y,$$
$$d_{1,2}^{ds}=\frac{\ln(V(t)/KB(t,T))\pm\frac{1}{2}\tilde{\sigma}^2(t,T)-\alpha x+\bar{\lambda}(T-t)\kappa}{\tilde{\sigma}(t,T)}.$$

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