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Risk Management for Linear and Non-Linear Assets: A Bootstrap Method with Importance Resampling to Evaluate Value-at-Risk

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Abstract Many empirical studies suggest that the distribution of risk factors has heavy tails. One always assumes that the underlying risk factors follow a multivariate normal distribution that is a assumption in conflict with empirical evidence. We consider a multivariate t distribution for capturing the heavy tails and a quadratic function of the changes is generally used in the risk factor for a non-linear asset. Although Monte Carlo analysis is by far the most powerful method to evaluate a portfolio Value-at-Risk (VaR), a major drawback of this method is that it is computationally demanding. In this paper, we first transform the assets into the risk on the returns by using a quadratic approximation for the portfolio. Second, we model the return's risk factors by using a multivariate normal as well as a multivariate t distribution. Then we provide a bootstrap algorithm with importance resampling and develop the Laplace method to improve the efficiency of simulation, to estimate the portfolio loss probability and evaluate the portfolio VaR. It is a very powerful tool that propose importance sampling to reduce the number of random number generators in the bootstrap setting. In

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the simulation study and sensitivity analysis of the bootstrap method, we observe that the estimate for the quantile and tail probability with importance resampling is more efficient than the naive Monte Carlo method. We also note that the estimates of the quantile and the tail probability are not sensitive to the estimated parameters for the multivariate normal and the multivariate t distribution.

Keywords Bootstrap · Heavy-tailed · Importance resampling · Monte Carlo simulation · Multivariate normal distribution · Multivariate t distribution · Quadratic approximation · Value-at-Risk · Variance reduction

1 Introduction

Along with the financial commodity unceasing innovation, as well as economic environment unceasingly globalization at the same time, the investors and the financial organs faced comparatively formerly a bigger risk to its asset allocation or the investment strategy formulation. Accordingly, risk-management has practical experienced a revolution since 1990, when several prosperous institutions failed, as a result of improperly manipulating derivatives (cf. Duffie and Pan 1997; Jorion 2000). Value-at-Risk (VaR) has become the most broadly employed risk measurement instrument and application of VaR have also been widely developed to measure and control risk, to manage risk, and investment management. Many major organizations agree on public-policy issues and become scrutiny of regulators. For instance, the Group of Thirty issued G-30 report that is a landmark on derivatives in 1993. JP Morgan launched its free RiskMetrics service in October 1994 (cf. Morgan 1995), and the Basle Committee amended the Basel Capital Accord to incorporate market risk capital requirements for banks in 1996.

Before 2000, the most common VaR implementations rely exclusively on normal assumptions, in other words, the risk factors are computed from a univariate normal distribution or a multivariate normal distribution (cf. Morgan 1995; Duffie and Pan 1997; Jorion 2000). However, there is much empirical evidence suggesting that risk factors, such as log-returns of US stocks, do not follow a normal distribution (cf. Mandelbrot 1963; Fama 1965; Blattberg and Gonedes 1974; Rachev and Mittnik 2000; Cont 2001). These heavy tails are particularly troublesome because VaR attempts to capture the behavior of the portfolio return in the tail. Since the VaR estimation for high-quantiles is sensitive to the assumed distribution, the hypothesis of normal should be revisited and heavy-tailed distributions that assign higher probabilities to large moves must be considered. In this paper, for practical purposes, we extend the assumption of commonly used multivariate normal distribution to the multivariate t distribution for the portfolio log-returns.

In order to evaluate a portfolio VaR, Monte Carlo analysis is by far the most powerful method, as it can be used to calculate the distribution of the portfolio returns. However, the biggest drawback of this method is that it is computationally demanding when there are hundreds of securities (cf. Jorion 2000; and Hull 2000). Many investigations have proposed using importance sampling to improve the efficiency of evaluating the portfolio VaR or quantile. (cf. Johns 1988; Goffinet and Wallach 1996;

Glasserman et al. 2000, 2002; Glasserman 2003). Importance sampling can reduce the sample size to save computational time with a certain accuracy. The main idea behind importance sampling is the change of measure. That is, we carry out a simulation with an importance tilting measure instead of the original probability measure, to obtain the accurate estimator for the portfolio VaR. (Glasserman et al. 2002, 2003) considered the idea of combining importance testing with stratified sampling for further reduction in variance. The idea of importance sampling can also be used to price the path-dependent and early exercise option (cf. Glasserman et al. 1999a,b; Glasserman 2003).

The bootstrap algorithm has been applied to finance and risk management (cf. Efron 1979; Efron and Tibshirani 1993; Horowitz 2001; Jorion 2000). The main idea of the bootstrap algorithm is to approximate the distribution of the portfolio VaR by using its bootstrap analogue, and then use the analogue to approximate the portfolio VaR. Bootstrap is a statistical resampling method and there are two approaches for the bootstrap algorithm: parametric bootstrap and nonparametric bootstrap (cf. Efron and Tibshirani 1993). The parametric bootstrap are two steps to process. First, it uses the observed data to estimate the unknown parameters of the given distribution. Second, from the estimated distribution constructs the sampling distribution of the VaR by using the bootstrap algorithm. Without any assumption of the underlying distribution, the nonparametric bootstrap uses the observed data to construct the sampling distribution of VaR.

There are two aspects of this paper. First, we provide the parametric bootstrap method to evaluate VaR in financial assets. Next, we introduce an importance resampling technique for more efficient simulation in bootstrap replications, and for greater accuracy in the portfolio VaR estimation. Further extension from the parametric bootstrap to the nonparametric bootstrap with importance resampling can be seen in Fuh and Hu (2004), in the setting of i.i.d. random variables.

The rest of this paper is organized as follows. In Sect. 2, both linear and non-linear portfolio VaR, and the bootstrap method are discussed. Importance resampling within the bootstrap procedure is also proposed in Sect. 2. In Sect. 3, we evaluate the VaR when assuming a normal distribution or t distribution of the linear asset return. Both of them have closed form tilting point. In Sect. 4, we compute the VaR only assuming a t distribution of the non-linear asset return. In Sect. 5, we first report the simulation results, which demonstrate the relative efficiency of naive Monte Carlo and importance sampling. Then, we provide the comparison of the default bootstrap algorithm and the bootstrap algorithm with importance resampling. The sensitivity analysis of the tail probability and the quantile are also given. As an illustration of our proposed methods, in Sect. 6, we present an empirical study based on three stock index returns and three option returns. Concluding remarks and further research are presented in Sect. 7.

2 Value-at-Risk

In the following we use a notation similar to that one commonly used in VaR literature. VaR is defined as the quantile l_p of the loss in portfolio value L during a holding period of a given time horizon t (cf. Glasserman et al. 2002). The VaR is a standard benchmark of the disclosure of financial risk (cf. Duffie and Pan 1997; Jorion 2000). Thus the VaR is the loss in market value over the time horizon t that is exceeded with probability $1 - p$. Our task is to estimate the tail probability p for a given r_p, and the quantile r_p for a given p. Let the portfolio $V(t, \widetilde{S}(t))$ denote the function of risk factor $S(t)$ and time t, where $\widetilde{S}(t) = (S_1(t), \ldots, S_n(t))'$ is the n underlying assets of the portfolio at time t, and the value of the portfolio at time $t + 1$ is $V(t+1, \widetilde{S}(t+1))$. The loss in portfolio value during the holding period is $L = -\Delta V$ where $\Delta V = V(t+1, \widetilde{S}(t+1)) - V(t, \widetilde{S}(t))$, and the VaR, l_p, associated with a given probability p is defined by

$$P(L > l_p) = p. \tag{2.1}$$

Assume the density of the portfolio return is symmetric and the loss distribution is absolutely continuous. Here, we can change the density of the portfolio loss into the density of the portfolio return,

$$P(R(t) > r_p) = p, \tag{2.2}$$

where $r_p = 2\mu - l_p/V$, and μ denotes the mean of the portfolio. Because we model the distribution of the asset return, the aim of the following subsection is to transform the portfolio from the assets into the return in linear and non-linear portfolios.

2.1 Linear Portfolio

The analysis of the linear portfolio, which is the linear combination of investment assets. Let a portfolio weight vector $\tilde{w}(t) = (w_1(t), \ldots, w_n(t))'$ denote the investment assets for the portfolio value, where $w_i(t)$ is an adapted process, i.e., $\mathcal{F}_t$-measurable, and $\tilde{r}(t) = (r_1(t), \ldots, r_n(t))'$ is a vector of the discrete return of the assets, where $r_i(t) = (S_i(t+1) - S_i(t))/S_i(t)$. Then the return of the portfolio at time t is the linear combination of the asset returns multiplied by the portfolio weight vector, denoted as

$$R(t) = \tilde{w}'(t)\tilde{r}(t). \tag{2.3}$$

For Eq. 2.3, we are interested in the event $A = \{\tilde{r}(t) : f(R(t)) = R(t) - r_p = \tilde{w}(t)'\tilde{r}(t) - r_p > 0\}$. Hence, what we are interested in is the return $R(t)$ of the portfolio by modeling the asset return $\tilde{r}(t)$, and investigating p and r_p by the parameter bootstrap algorithm. The linear portfolio is satisfactory for small movements in the underlying asset. A better approximation may be achieved by going to higher order and incorporating the gamma or convexity effect.

2.2 Quadratic Approximation

Glasserman (2003) develops a method for calculating the distribution of the change in portfolio value over a fixed horizon assuming that the changes in underlying risk factors over the horizon are described by a multivariate t distribution. Also, we assume that the change in portfolio value is a quadratic function of the change in the risk factors. By the delta-gamma approximation (quadratic approximation), the change in portfolio value for the non-linear portfolio can be written as

$$V(t+1, \tilde{S}(t+1)) - V(t, \tilde{S}(t)) \approx \frac{\partial V}{\partial t}\Delta t + \delta' \Delta \tilde{S}(t) + \frac{1}{2}\Delta \tilde{S}(t)' \Gamma \Delta \tilde{S}(t),$$

where $\frac{\partial V}{\partial t}$ is the change of the portfolio from t to $t+1$, $\delta_i = \frac{\partial V}{\partial S_i}$ denotes the delta approximation of the portfolio for the change of the asset i, $\delta' = [\delta_1, \ldots, \delta_n]$ is the vector of the delta approximation, $\Gamma_{ij} = \partial^2/\partial S_i \partial S_j$ is the gamma approximation of the portfolio for the change of the asset i and asset j, Γ is the matrix of the gamma approximation, and $\Delta \tilde{S}(t)' = [\Delta S_1(t), \ldots, \Delta S_n(t)]$ denotes the change of the assets. More importantly, the effect of the gamma is to introduce a term that is non-linear in the random component of $\Delta \tilde{S}(t)$. Hence, the loss in portfolio, L, can be rewritten as

$$\begin{aligned} L &\approx a_0 + a' \Delta \tilde{S}(t) + (\Delta \tilde{S}(t))' A \Delta \tilde{S}(t) \\ &= a_0 + a_1' \tilde{r} + \tilde{r}' A_1 \tilde{r}, \end{aligned}$$

where $a_0 = \frac{\partial V}{\partial t}\Delta t$ is a scalar, $a = -\delta$, $A = -\frac{1}{2}\Gamma$ is a $m \times m$ matrix, $a_1' = [-S_1\delta_1, \ldots, -S_n\delta_n]$, $(A_1)_{ij} = -\frac{1}{2}\Gamma_{ij} S_i S_j$, and $\tilde{r}' = [\frac{\Delta S_1(t)}{S_1(t)}, \ldots, \frac{\Delta S_n(t)}{S_n(t)}]$ denotes the vector of the discrete returns in the assets. One obvious conclusion is that positive gamma is good for a portfolio and negative gamma is bad. With a positive gamma the downside is limited, but with a negative gamma the upside is limited.

2.3 Bootstrap Method

The bootstrap method is able to estimate measures of variability and bias. It can be employed in nonparametric or in parametric model. In this research, we consider a parametric bootstrap method. Suppose an empirical data set, $\tilde{r}^F = (\tilde{r}^F(1), \ldots, \tilde{r}^F(T))$ of size T from a distribution F. For instance, a multivariate normal or a multivariate t distribution are considered in this article. The parameters of the distribution are estimated by the method of maximum likelihood estimate or the moment estimate. To obtain the empirical distribution $\hat{F}$, according to the estimators of the parameters of the distribution. For example, in the case of a normal distribution $N(\mu, \sigma)$ with unknown parameters μ and σ, we can use the estimators of the sample mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$, to obtain the empirical distribution $N(\hat{\mu}, \hat{\sigma})$.

Define a bootstrap sample $\tilde{r}^{F,*} = (\tilde{r}^F(*, 1), \ldots, \tilde{r}^F(*, T))$ as a random sample of size T drawn from $\widehat{F}$. That is,

$$\widehat{F} \longrightarrow \tilde{r}^{F,*} = (\tilde{r}^F(*, 1), \ldots, \tilde{r}^F(*, T)), \tag{2.4}$$

where $\tilde{r}^F(*, t)$, $t = 1, 2, \ldots, T$, are i.i.d. random variables from a multivariate $\widehat{F}$ distribution. In general, we can employ the Monte Carlo simulation to approximate it since the bootstrap distribution is difficult to obtain. We can generate many possible realizations by repeating this simulation to obtain an accurate distribution of all future returns. To be precise, corresponding to a bootstrap dataset $\tilde{r}^{F,*}$, we have a bootstrap replication of $\widehat{F}$. Then the bootstrap estimate of the probability of interest is

$$\hat{p}_*^F = \frac{1}{T}\sum_{t=1}^{T} 1_{A^F(*,t)}, \tag{2.5}$$

where $A^F(*,t) = \{\tilde{r}^F(*,t) : f(R^F(*,t)) = R^F(*,t) - r_p^F = \tilde{w}'(t)\tilde{r}^F(*,t) - r_p^F > 0\}$, where $\tilde{r}^F(*,t)$ is the tth return vector sample of a bootstrap drawn from the multivariate $\widehat{F}$ distribution, $R^F(*,t)$ is a bootstrap dataset of the portfolio return from a vector of the returns $\tilde{r}^F(*,t)$, and r_p^F denotes the quantile or VaR of $R^F(t)$. For bootstrap resampling with B replications, we can estimate the tail probability and the standard error se_p^F of the tail probability estimate by

$$\bar{p}^F = \frac{\sum_{b=1}^{B}\hat{p}_b^F}{B}, \tag{2.6}$$

$$\widehat{se}_p^F = \left(\frac{\sum_{b=1}^{B}(\hat{p}_b^F - \bar{p}^F)^2}{B-1}\right)^{1/2}. \tag{2.7}$$

By using this method we generate a distribution of possible future scenarios based on historical data. This method ensures that we capture any correlation that may exist between assets. The advantage of this method is that it incorporates any correlation between assets and non-Normality in asset price changes. The parametric bootstrap procedure for estimating the probability and the standard error of $\hat{p}$ from the observed data $\tilde{r}(t)$ can be described as follows.

(1) Obtain the empirical distribution $\widehat{F}$. Here, the observed data $\tilde{r}^F$ is assumed to come from distribution F, the unknown parameters of the distribution are estimated by the maximum likelihood estimate or the method of moment.
(2) Select B independent bootstrap samples $\tilde{r}^{F,1}, \tilde{r}^{F,2}, \ldots, \tilde{r}^{F,B}$, each consisting of T data values drawn with replacement from $\widehat{F}$.
(3) Evaluate the bootstrap replication corresponding to each bootstrap sample,

$$\hat{p}_b^F = \frac{1}{T}\sum_{t=1}^{T} 1_{A^F(b,t)} \quad b = 1, 2, \ldots, B, \tag{2.8}$$

where $A^F(b,t) = \{\tilde{r}^F(b,t) : f(R^F(b,t)) = R^F(b,t) - r_p^F = \tilde{w}'(t)\tilde{r}^F(b,t) - r_p^F > 0\}$.
(4) Estimate the probability p and standard error $se_p(\hat{p})$ by using Eqs. 2.6 and 2.7.

Define r_p^F as the solution of the Eq. 2.2,

$$P(R^F(t) \leq r_p^F) = 1 - p, \tag{2.9}$$

where r_p^F is the quantile of the return of the portfolio $R^F(t)$ at time t for the distribution F of the return $\tilde{r}^F(t)$. We consider the problem of estimating the $(1-p)$ quantile of the portfolio return $R^F(t)$, where a vector of the asset returns $\tilde{r}^F(t)$ is assumed come from the multivariate distribution F. In this paper, the vector of the returns of the assets is assumed to be a multivariate normal or a multivariate t distribution. The bootstrap estimate of the quantile r_p^F is the solution $\hat{r}_p^F$ of

$$P(R^F(t) \leq \hat{r}_p^F) = 1 - p. \tag{2.10}$$

For the sake of notations (cf. Hall 1990a,b, 1992), we shall define

$$\hat{r}_p^F = \widehat{H}_F^{-1}(p) = \inf\{x : \widehat{H}_F(x) \geq (1-p)\}, \tag{2.11}$$

where $\widehat{H}_F(x) = P(R(t) \leq x)$. The estimating process of bootstrap algorithm for the quantile $\hat{r}_p^F$, and the standard error of $\hat{se}_p^F$ can be described as follows (cf. Efron and Tibshirani 1993):

(1) Obtain the empirical distribution $\widehat{F}$.
(2) Select B independent bootstrap samples $\tilde{r}^{F,1}, \tilde{r}^{F,2}, \ldots, \tilde{r}^{F,B}$, each consisting of T data values drawn with replacement from $\widehat{F}$.
(3) Evaluate the quantile of the bootstrap sample from Eq. 2.12,

$$\hat{r}_p^F(b) = \widehat{H}_{F,b}^{-1}(p) = \inf\{x : \widehat{H}_{F,b}(x) \geq (1-p)\}, \quad b = 1, 2, \ldots, B, \tag{2.12}$$

where $\hat{r}_p^F(b)$ is the $(1-p)T$th largest value of the bth bootstrap sample.
(4) Estimate the quantile r_p^F and standard error se_p^F as

$$\bar{r}_p^F = \frac{\sum_{b=1}^{B} \hat{r}_p^F(b)}{B}, \tag{2.13}$$

$$\hat{se}_p^F = \left(\frac{\sum_{b=1}^{B} (\hat{r}_p^F(b) - \bar{r}_p^F)}{B-1} \right)^{1/2}. \tag{2.14}$$

2.4 Parametric Bootstrap with Importance Resampling

In this subsection, we discuss the method of parametric bootstrap with importance resampling to evaluate VaR in financial markets. To be precise, the problem is how to simulate a rare event in order to find its probability? This is a difficult problem for simulating large random number generators. As an illustration for this type of problem, the reader can be referred to Bucklew (1990) who consider a sequence x_i of i.i.d.

Bernoulli random variables with probability density

$$P\{x_i = 1\} = p = 1 - P\{x_i = 0\}.$$

Suppose that we wish to estimate from the observed sequence the parameter p, and have at most a β error on p with α confidence. In other words, we must have

$$P\{|p - \hat{p}| \leq \beta p\} = \alpha,$$

where $\hat{p}$ is the estimate of p. Since $\hat{p} \equiv \sum_{i=1}^{T} x_i/T$ is the maximum likelihood estimate of p, and the variance of a Bernoulli(p) random variable is $p(1-p)$. For a rare event as p is very small, the variance of x_i is approximately p. Hence the variance of $\hat{p}$ is

$$\frac{p(1-p)}{T} \approx \frac{p}{T}.$$

By the standard Central Limit Theorem approximation, we have

$$P\{|p - \hat{p}| \leq \beta p\} = P\left\{\left|\frac{1}{T}\sum_{i=1}^{T}\frac{x_i - p}{\sqrt{p}}\right| \leq \beta\sqrt{p}\right\} \approx P\{|Z| \leq \beta\sqrt{pT}\}, \quad (2.15)$$

where Z is the standard normal distribution. For a given standard normal distribution, we have $P\{|Z| \leq z\} = 0.95$, which implies that $z \approx 2$, i.e., two standard deviations about the mean capture 95% of the probability of the standard normal distribution. Hence, if $\beta = 0.2$, then $0.2\sqrt{pT} = 2$ implies that $T = 100/p$. Therefore, if p is somewhere around the order 10^{-6}, we would need $10^8 \approx 2^{27}$ number of samples to estimate it to the desired level of precision. If $\beta = 0.1$ and $p = 10^{-2}$, we require $T = 40,000 \approx 2^{16}$. Typical random number generators have a period of anywhere from 2^{15} to 2^{32}, with the latter number considered a long period. The number of "good" random numbers that one generates is typically much less than the period of the generator. That is the motivation for us to propose importance sampling to reduce the number of random number generators in the bootstrap setting.

3 Portfolios of Linear Assets

We first introduce importance sampling for a multivariate normal and a multivariate t distribution. That is, we need to choose an importance sampling distribution to make the rare event a central event. For general accounts of importance sampling, the reader is referred to Bucklew (1990) and Glasserman (2003) for details.

3.1 Risk Factors with a Multivariate Normal Distribution

In this subsection, we propose a parametric bootstrap process with importance resampling. To estimate p in an n-dimensional Ito process in the continuous financial models. Assume that the vector of the asset price follows an n-dimensional Ito process, then it is also denoted that the vector of the asset returns follows the multivariate normal distribution in discrete time data. Set $r_i^N(t) = \Delta S_i(t)/S_i(t)$, $i = 1, 2, \ldots, n$, then the vector of the asset returns follows

$$\tilde{r}^N(t) = \begin{bmatrix} r_1^N(t) \\ \vdots \\ r_n^N(t) \end{bmatrix} = \begin{bmatrix} \mu_1 + \sigma_1 \varepsilon_1 \\ \vdots \\ \mu_n + \sigma_n \varepsilon_n \end{bmatrix} = \tilde{\mu} + \sigma \tilde{\varepsilon}. \tag{3.1}$$

Our basic approach is to use model (3.1) to approximate the portfolio return loss probability, and then apply it to obtain importance resampling distribution for variance reduction.

The event $A^N(t) = \{\tilde{r}^N(t) : f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N > 0\}$ is interested by risk management. We can rewrite

$$f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N = \tilde{\sigma}_w^{'}\tilde{\varepsilon} + \tilde{w}'(t)\tilde{\mu} - r_p^N \tag{3.2}$$

$$\stackrel{d}{=} KZ + \tilde{w}'(t)\tilde{\mu} - r_p^N, \tag{3.3}$$

where $\tilde{\sigma}_w = \tilde{w}'(t)\sigma = (w_1(t)\sigma_1, w_2(t)\sigma_2, \ldots, w_n(t)\sigma_n)'$, $K = \sqrt{\tilde{\sigma}_w' \Sigma \tilde{\sigma}_w}$, Z is a standard normal distribution, $r_p^N (r_p^N = Kz_p + \tilde{w}'(t)\tilde{\mu})$ is the quantile of the portfolio return with a multivariate normal assumption, $\stackrel{d}{=}$ means equal in distribution, and z_p is the quantile of the standard normal density. By using the Cholesky decomposition for Σ, we have

$$f(R^N(t)) = \tilde{w}'(t)\tilde{r}^N(t) - r_p^N \stackrel{d}{=} \tilde{\sigma}_w' C \widetilde{Z} + \tilde{w}'(t)\tilde{\mu} - r_p^N \tag{3.4}$$

$$\stackrel{d}{=} DZ + \tilde{w}'(t)\tilde{\mu} - r_p^N, \tag{3.5}$$

where $C = [c_{ij}]$ is used by Cholesky decomposition for Σ, and $D = \sqrt{\sum_{j=1}^n (\sum_{i=1}^n w_i(t)\sigma_i c_{ij})^2} = K$.

Following the same idea as that in Glasserman et al. (1999a), we next describe how to select the tilting measure in risk factors. Standard exponential embedding leads to

$$\frac{d\mathbb{P}^{\theta}_{\tilde{a}^N}}{d\mathbb{P}} = \exp\{\theta f(R^N(t)) - \psi^N(\theta)\}, \tag{3.6}$$

where $d\mathbb{P}$ is the original probability measure, and $d\mathbb{P}^{\theta}_{\tilde{a}^N(\theta)}$ is the tilting measure from the multivariate normal distribution $MN(\tilde{0}, I)$ to the multivariate normal distribution

$MN(\tilde{a}^N(\theta), I)$. Here

$$\psi^N(\theta) = \log E(\exp\{\theta f(R^N(t))\}) = \theta(\tilde{w}'(t)\tilde{\mu} - r_p^N) + \frac{1}{2}\theta^2 K^2. \tag{3.7}$$

Let $A_\theta^N(t) = \{\tilde{r}_\theta^N(t) : f(R_\theta^N(t)) = \tilde{w}'(t)\tilde{r}_\theta^N(t) - r_p^N > 0\}$ be the event to be simulated. Denote

$$\hat{p}^N(\theta) = 1_{A_\theta^N(t)} \exp\{-\theta f(R_\theta^N(t)) + \psi^N(\theta)\}, \tag{3.8}$$

and let $\tilde{r}_\theta^N(t)$ be drawn from the tilting measure $\mathbb{P}^\theta_{\tilde{a}^N(\theta)}$, then the estimator $1_{A^N(t)}$ is unbiased. That is,

$$E(1_{A^N(t)}) = E^\theta(1_{A_\theta^N(t)} \exp\{-\theta f(R_\theta^N(t)) + \psi^N(\theta)\}) = E^\theta(\hat{p}^N(\theta)) = p.$$

Therefore, we only compute the second moment of the estimator for the tail probability

$$\begin{aligned} M_2^N(\theta) &= E^\theta(1_{A_\theta^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R_\theta^N(t))\}) \\ &= \exp\{\psi^N(\theta) - \theta\tilde{w}'(t)\tilde{\mu} + \theta r_p^N + \frac{1}{2}\theta^2 K^2\} \int_{\lambda_1}^{\infty} \phi(z) dz, \end{aligned} \tag{3.9}$$

where $\lambda_1 = \theta K + (r_p^N - \tilde{w}'(t)\tilde{\mu})/K$, and $\phi(z)$ is the standard normal density. Take log and differentiate θ to obtain

$$2\theta K - \frac{\phi(\lambda_1)}{1 - \Phi(\lambda_1)} = 0, \tag{3.10}$$

where Φ is the standard normal cumulative density function. Since it is difficult to find the value of θ by minimizing $M_2^N(\theta)$, we will minimize its upper bound (cf. Glasserman et al. 1999a) as follows,

$$\begin{aligned} M_2^N(\theta) &= E^\theta(1_{A_\theta^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R_\theta^N(t))\}) \\ &\le \exp\{\psi^N(\theta)\}, \end{aligned} \tag{3.11}$$

because $\exp\{-\theta f(R_\theta^N(t))\} \le 1$ ($\because f(\tilde{r}_\theta^N(t)) > 0$ and $\theta > 0$), and $1_{A_\theta^N(t)} \le 1$. Taking log of the bound equation (3.11) and differentiating θ we have

$$\theta_g^N = \frac{r_p^N - \tilde{w}'(t)\tilde{\mu}}{K^2}. \tag{3.12}$$

We can also find the new upper bound of the second moment for the estimate of the tail probability by using the inequality (Durrett 1996), the Laplace method,

$$\int_{\lambda}^{\infty} \exp\{-z^2\}dz \le \lambda^{-1} \exp\{-\lambda^2/2\}. \tag{3.13}$$

Compute the new second moment upper bound,

$$\begin{aligned} M_2^N(\theta) &= E^{\theta}(1_{A_{\theta}^N(t)} \exp\{2\psi^N(\theta) - 2\theta f(R(t))\}) \\ &\le \exp\left\{\psi^N(\theta) - \frac{1}{2}\left(\frac{r_p^N - \tilde{w}'(t)\tilde{\mu}}{K}\right)^2\right\} \frac{1}{\sqrt{2\pi}} \frac{K}{r_p^N - \tilde{w}'(t)\tilde{\mu} + \theta K^2}. \end{aligned} \tag{3.14}$$

Then, taking the new bound equation (3.14) into log and differentiating θ, we obtain the solution of θ_l^N for the second moment of tail probability by the inequality (Durrett 1996) in the multivariate normal distribution,

$$\theta_l^N = \sqrt{\frac{K^2 + (r_p^N - \tilde{w}'(t)\tilde{\mu})^2}{K^4}}. \tag{3.15}$$

Hence, we obtain the nearly optimal tilting probability measure for the multivariate normal distribution θ_g^N (cf. Glasserman et al. 1999a)

$$\begin{aligned} d\mathbb{P}_{\tilde{a}^N}^{\theta} &= \exp\{\theta f(R^N(t)) - \psi^N(\theta)\}d\mathbb{P} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{\widetilde{Z}'_{\tilde{a}^N(\theta)} I \widetilde{Z}_{\tilde{a}^N(\theta)}}{2}\right\}, \end{aligned} \tag{3.16}$$

where I is the identity matrix, $\widetilde{Z}_{\tilde{a}^N(\theta)} = (Z_1 - \theta \sum_{i=1}^n \sigma_i w_i(t) c_{i1}, \ldots, Z_n - \theta \sum_{i=1}^n \sigma_i w_i(t) c_{in})'$. That is, $\widetilde{Z}_{\tilde{a}^N(\theta)} \sim MN(\tilde{a}^N(\theta), I)$, where $\tilde{a}^N(\theta) = (\theta \sum_{i=1}^n \sigma_i w_i(t) c_{i1}, \ldots, \theta \sum_{i=1}^n \sigma_i w_i(t) c_{in})'$. For simplicity, we use $d\mathbb{P} = \phi(Z)$ in Eq. 3.6 as the original measure, then the tilting measure is

$$\begin{aligned} d\mathbb{P}_{a^N(\theta)}^{\theta} &= \exp\{\theta f(R^N(t)) - \psi^N(\theta)\}d\mathbb{P} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(Z - \theta K)^2}{2}\right\}, \end{aligned} \tag{3.17}$$

where $a^N(\theta) = \theta K$. Therefore, we may generate portfolio samples under $d\mathbb{P}_{\tilde{a}^N(\theta)}^{\theta}$ and estimate $P(f(R_{\theta}^N(t)) > 0)$ using the expression (3.8) by substituting θ_g^N for θ in Eq. 3.12.

Now, we present the parametric bootstrap algorithm with importance resampling for a multivariate normal distribution as follows:

(1) Obtain the empirical distribution $MN(\hat{\tilde{\mu}}, \hat{\Sigma})$, where $\hat{\tilde{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)'$. The observed data $\tilde{r}^N$ is assumed to come from the multivariate normal distribution $MN(\tilde{\mu}, \Sigma)$. Unknown parameters $\hat{\mu}_i$, $\hat{\sigma}_i$, and $\hat{\rho}_{ij}$ are estimated by using the method of moment estimates

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_i(t), \tag{3.18}$$

$$\hat{\sigma}_i = \left(\frac{1}{T-1}\sum_{t=1}^{T}(r_i(t) - \hat{\mu}_i)^2\right)^{1/2}, \tag{3.19}$$

$$\hat{\rho}_{ij} = \frac{\sum_{t=1}^{T}(r_i(t) - \hat{\mu}_i)(r_j(t) - \hat{\mu}_j)}{(\sum_{t=1}^{T}(r_i(t) - \hat{\mu}_i)^2 \sum_{t=1}^{T}(r_j(t) - \hat{\mu}_j)^2)^{1/2}}, \tag{3.20}$$

and

$$\hat{\Sigma} = \begin{bmatrix} 1 & \hat{\rho}_{12} & \hat{\rho}_{13} & \ldots & \hat{\rho}_{1d} \\ \hat{\rho}_{21} & 1 & \hat{\rho}_{23} & \ldots & \hat{\rho}_{2d} \\ \hat{\rho}_{31} & \hat{\rho}_{32} & 1 & \ldots & \hat{\rho}_{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\rho}_{n1} & \hat{\rho}_{n2} & \hat{\rho}_{n3} & \ldots & 1 \end{bmatrix}.$$

(2) Generate $\widetilde{Z}_{\tilde{a}^N(\theta)}$ from the multivariate normal distribution $MN(\tilde{a}^N(\theta), I)$, use the tilting measure to obtain B independent bootstrap samples $\tilde{r}^{N,1}, \tilde{r}^{N,2}, \ldots, \tilde{r}^{N,B}$, each consisting of T data values drawn by the tilting probability measure.

(3) Evaluate the bootstrap replication corresponding to each bootstrap sample,

$$\hat{p}_b^N(\theta) = \frac{1}{T}\sum_{t=1}^{T} 1_{A_\theta^N(b,t)} \exp\{-\theta f(R_\theta^N(b,t)) + \psi^N(\theta)\} \quad b = 1, 2, \cdots, B, \tag{3.21}$$

where $A_\theta^N(b,t) = \{\tilde{r}_\theta^N(b,t) : f(R_\theta^N(b,t)) = R_\theta^N(t) - r_p^N = \tilde{w}'(t)\tilde{r}_\theta^N(b,t) - r_p^N > 0\}$, and substitute θ_g^N into θ, or θ_l^N into θ.

(4) Estimate the probability p and standard error se_p^N by using B replications as

$$\bar{p}^N(\theta) = \frac{\sum_{b=1}^{B}\hat{p}_b^N}{B}, \tag{3.22}$$

$$\widehat{se}_p^N(\theta) = \left(\frac{\sum_{b=1}^{B}(\hat{p}_b^N(\theta) - \bar{p}^N(\theta))^2}{B-1}\right)^{1/2}. \tag{3.23}$$

3.2 Risk Factors with a Multivariate t Distribution

Although the multivariate normal distribution assumption is commonly used in the literature, many empirical studies suggest that the distribution has heavy tails. One of the most pervasive features observed across equity, foreign exchange, and interest rate markets is that they have kurtosis excess, so the distribution of the asset has leptokurtic features. That means, compared to a normal distribution with the same mean and standard deviation, the true distribution assigns greater probability to extreme market moves. Clearly, extreme moves are of paramount importance in risk management and should be modeled accurately to calculate VaR.

Now, we model the changes in risk factors by using a multivariate t distribution. We are mainly interested in values of ν roughly in the range of 3–7, since this seems to be the level of heaviness typical of market data per Glasserman et al. (2002). As ν tends to infinity, the t distribution converges to the normal distribution, so the normal may be viewed as a special or benchmark, limiting case of the t distribution. Because it is characterized by the matrix Σ, the multivariate t shares some attractive properties with the multivariate normal while possessing heavy tails. The linear portfolio case with multivariate t distribution can be solved by similar method, obtaining

$$\theta_g^T = \frac{r_p^T - \tilde{w}'(t)\tilde{\mu}}{K^2}, \tag{3.24}$$

and

$$\theta_l^T = \frac{(r_p^T - \tilde{w}'(t)\tilde{\mu}) + \sqrt{(r_p^T - \tilde{w}'(t)\tilde{\mu})^2(\nu^2+\nu+1) + K^2\nu^2}}{(\nu+1)K^2}. \tag{3.25}$$

Compared with (3.12), (3.24) is the tilting formula for the multivariate t distribution. The formulations of (3.12) and (3.24) are quite similar, but with different $r_p^i, i = N, T$. Note that as ν tends to infinity, r_p^T converges to r_p^N and the tilting points are the same. By using the same argument, it is easy to see that (3.25) will converge to (3.15), as ν tends to infinity.

4 Portfolios of Non-Linear Assets

We use the structure of the multivariate t distribution to develop efficient methods for calculating portfolio loss probability, capturing heavy tails. At the end of Sect. 3, we compare the limit result for the t distribution that it will converge to the normal distribution. Here, we consider only the more complicated event with quadratic approximation and multivariate t distribution.

4.1 Risk Factors with a Multivariate t Distribution

Assume the return of the assets is equal to the mean $\tilde{u}$, and a multivariate t distribution with the degree of freedom ν as follows:

$$\tilde{r} = \tilde{u} + t_\nu,$$

where $t_\nu = \frac{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)}{\sqrt{Y/\nu}} = \frac{\tilde{\varepsilon}}{\sqrt{Y/\nu}}$ has a multivariate t distribution with the degree of freedom ν, $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ denotes a multivariate normal distribution with zero mean vector and covariance matrix Σ. Let $CC' = \Sigma$, and $C' A_1 C = \Lambda$. Then

$$\begin{aligned} L &= a_0 + a_1' \tilde{r} + \tilde{r}' A_1 \tilde{r} \\ &= b_0 + a_1' t_\nu + (t_\nu)' A_1 (t_\nu) \end{aligned}$$

where $b_0 = a_0 + a_1' \tilde{u} + \tilde{u}' A_1 \tilde{u}$. Thus, let $Q \equiv L - b_0$, then

$$\begin{aligned} Q &= a_1' t_\nu + (t_\nu)' A_1 (t_\nu) \overset{d}{=} a_1' C X + X' C' A_1 C X \\ &= b' X + X' \Lambda X = \sum_{j=1}^{n} b_j X_j + \lambda_j X_j^2 \end{aligned} \tag{4.1}$$

where $b' = a_1' C$. Let $t_\nu \overset{d}{=} CX = C \frac{\tilde{Z}}{\sqrt{Y/\nu}}$, where $\tilde{Z}$ has a multivariate normal distribution with zero mean vector and identity covariance matrix I, and

$$X = (X_1, \ldots, X_n) = \frac{\tilde{Z}}{\sqrt{Y/\nu}},$$

where $X_j = \frac{\tilde{Z}_j}{\sqrt{Y/\nu}}$ are independent.

An attempt to apply similar ideas to a multivariate t distribution seems doomed by the failure of (4.1) to generalize to the heavy-tailed setting. In any model in which the risk factors are heavy tailed, one cannot define an exponential change of measure based on Q because $E[\exp(\theta Q)]$ is infinite for any $\theta > 0$; while most successful applications of importance sampling are based on an exponential change of measure. Instead, we use an indirect transform analysis through which we are able to compute the distribution of interest. Similarly, we are interested in event A, and through the indirect approach, we have

$$P(Q > 0) = P\left(\frac{Y}{\nu} Q > 0\right). \tag{4.2}$$

It is interesting for us to calculate loss probabilities $P(L > x)$ assuming equality in (4.1). Therefore, the problem becomes calculating VaR, that is, to find a quantile l_{1-p}

for which $P(L > l_{1-p}) = p$. The probability of the loss can be rewritten

$$\begin{aligned} P(L > l_{1-p}) = p &= P(b_0 + Q > l_{1-p}) \\ &= P(\tfrac{Y}{\nu}(Q - x) > 0) \\ &= P(Q_x > 0), \end{aligned}$$

where $x = l_{1-p} - b_0$ and $Q_x = \frac{Y}{\nu}(Q - x)$.

Then, use the exponential change of measure (Glasserman et al. 2000, 2002)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\{\theta Q_x - \psi(\theta)\}, \tag{4.3}$$

where $\psi(\theta) = \log E(\exp(\theta Q_x))$. In Glasserman et al. (2002), they obtain the moment generating function of Q_x as follows:

$$\phi(\theta) = \left(1 + \frac{2\theta x}{\nu} - \sum_{j=1}^{n} \frac{\theta^2 b_j^2}{(1 - 2\theta\lambda_j)\nu}\right)^{-\frac{\nu}{2}} \prod_{j=1}^{n} \frac{1}{\sqrt{1 - 2\theta\lambda_j}}, \tag{4.4}$$

and

$$\begin{aligned} \psi(\theta) &= \log\phi(\theta) = -\frac{\nu}{2}\log\left(1 + \frac{2\theta x}{\nu} - \sum_{j=1}^{n} \frac{\theta^2 b_j^2}{(1 - 2\theta\lambda_j)\nu}\right) \\ &\quad + \sum_{j=1}^{n} \log\frac{1}{\sqrt{1 - 2\theta\lambda_j}}. \end{aligned} \tag{4.5}$$

Let $A_\theta(t) = \{\tilde{r}_\theta(t) : Q_x > 0\}$ be the event of interest to be simulated for the tail probability, and let the estimator of the tail probability be

$$\hat{p} = e^{-\theta Q_x + \psi(\theta)} 1_{\{Q_x > 0\}}.$$

The estimator $\hat{p}$ is unbiased in the sense that

$$\begin{aligned} E_\theta(\hat{p}) &= E(1_{\{Q_x > 0\}}) \\ &= P(Q_x > 0) = p. \end{aligned}$$

Therefore, we compute the second moment of estimator for the tail probability

$$E_\theta(\hat{p}^2) = \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \sum_{j=1}^{n} \frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\}$$
$$\times \prod_{j=1}^{n} \frac{1}{\sqrt{1+2\lambda_j\theta}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n$$
$$\times \exp\left\{-\frac{\sum_{j=1}^{n}(z_j^*)^2}{2}\right\} 1_{\{\sum_{j=1}^{n} c_j(z_j^*-d_j)^2 > g\}} dz_1^*, \ldots, dz_n^* f_Y(y)dy, \tag{4.6}$$

where

$$z_j^* = \frac{z_j + \frac{b_j\theta}{1+2\lambda_j\theta}\sqrt{\frac{y}{\nu}}}{\frac{1}{\sqrt{1+2\lambda_j\theta}}}, c_j = \frac{\lambda_j}{1+2\lambda_j\theta}, d_j = \left[\frac{b_j\theta}{\sqrt{1+2\lambda_j\theta}} - \frac{\sqrt{1+2\lambda_j\theta}b_j}{2\lambda_j}\right]$$
$$\times\sqrt{\frac{y}{\nu}}, \text{ and } g = \frac{y}{\nu}x + \sum_{j=1}^{n} \frac{b_j^2 y}{4\lambda_j\nu}.$$

Then, by the exponential change of measure, we have

$$\frac{d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta),h(\theta)}}{d\mathbb{P}_{\tilde{0},2}} = \exp\{\theta Q(R^T(t)) - \psi^T(\theta)\}, \tag{4.7}$$

where $d\mathbb{P}_{\tilde{0},2}$ is the original multivariate t distribution with two independent components $\widetilde{Z} \sim MN(\tilde{0}, I)$, and $Y \sim G(\nu/2, 2)$, $d\mathbb{P}_{\tilde{a}^T(\theta),h(\theta)}$ is the new multivariate t distribution with two independent components $\widetilde{Z} \sim MN(\tilde{a}^T(\theta), I)$, and $Y \sim G(\nu/2, h(\theta))$. We use the estimator

$$\hat{p}^T(\theta) = 1_{A_\theta^T(t)} \exp\{-\theta Q(R_\theta^T(t)) + \psi^T(\theta)\}, \tag{4.8}$$

where $A_\theta^T(t) = \{\tilde{r}_\theta^T(t) : f(R_\theta^T(t)) = \tilde{w}'(t)\tilde{r}_\theta^T(t) - r_p^T > 0\}$ and

$$\psi^T(\theta) = \log E(\exp\{\theta Q(R^T(t))\})$$
$$= \log\left(1 - \frac{2\theta(\tilde{w}^{'}(t)\tilde{\mu} - r_p^T) + \theta^2 K^2}{\nu}\right)^{-\nu/2}. \tag{4.9}$$

Note that the estimator 1_A is unbiased in the sense that

$$E(1_A) = E^\theta(1_{A_\theta^T(t)} \exp\{-\theta Q(R_\theta^T(t)) + \psi^T(\theta)\}) = E^\theta(\tilde{p}^T(\theta)) = p.$$

Note that $Q(R^T(t))$ is not heavy-tailed and so, unlike $f(R^T(t))$, its moment generating function exists. Therefore, we can compute the second moment as follows,

$$M_2^T(\theta) = E^{\theta}(1_{A_\theta^T(t)} \exp\{2\psi^T(\theta) - 2\theta Q(R_\theta^T(t))\})$$

$$= E(1_{A_\theta^T(t)} \exp\{\psi^T(\theta) - \theta Q(R_\theta^T(t))\})$$

$$\leq \exp\{\psi^T(\theta)\}, \tag{4.10}$$

because $\exp\{-\theta Q(R_\theta^T(t))\} \leq 1$ ($\because Q(\tilde{r}_\theta^T(t)) > 0$ and $\theta > 0$), and $1_{A_\theta^T(t)} \leq 1$. While finding the value of θ to minimize $M_2^T(\theta)$ is difficult, it is a simple matter to minimize the upper bound in equation 4.8(cf. Glasserman et al. 2002). Hence we can obtain θ_g^T,

$$\frac{\partial \psi^T(\theta)}{\partial \theta}|_{\theta=\theta_g^T} = 0, \tag{4.11}$$

by differentiating θ with the bound equation (4.10). Or we suggest to use Laplace method to get new θ in Appendix 1. Although the solution of θ for the multivariate normal and t distribution are the same, the new tilting measures are different. Therefore, to obtain the tilting measure, we first twist the original gamma distribution $Y \sim G(\nu/2, 2)$ to the new gamma density $Y_{h(\theta)} \sim G(\nu/2, h(\theta))$, where $h(\theta) = 2\nu/(\nu - 2\tilde{w}'(t)\tilde{\mu}\theta + 2r_p^T\theta - \theta^2 K^2)$. Under the given $Y_{h(\theta)}$, we twist the original multivariate normal density $\widetilde{Z}$ that follows $MN(\tilde{0}, I)$ to the new multivariate normal density $\widetilde{Z}_{\tilde{a}^T(\theta)}$ to follow $MN(\tilde{a}^T(\theta), I)$. Appendix 2 provides the details.

When ν tends to infinity, we have

$$h(\theta) \to 2, \tag{4.12}$$

and

$$\frac{Y_{h(\theta)}}{\nu} \to 1 \text{ in probability.} \tag{4.13}$$

Therefore, as $\nu \to \infty$,

$$\widetilde{X}_{\tilde{a}^T(\theta),h(\theta)} = \frac{\widetilde{Z}_{\tilde{a}^T(\theta)}}{\sqrt{\frac{Y_{h(\theta)}}{\nu}}} \to \widetilde{Z}_{\tilde{a}^N(\theta)}. \tag{4.14}$$

That is, the new tilting measure for the multivariate t distribution converges to the new multivariate normal distribution as ν tends to infinity.

To estimate the tail probability, the parametric bootstrap algorithm with importance resampling for the multivariate t distribution can be described as follows:

(1) Obtain the empirical distribution $\hat{\tilde{\mu}} + \hat{\sigma}\widetilde{X}$ with the degree of freedom ν. The observed data $\tilde{r}^T$ are taken from a multivariate t distribution $\tilde{\mu} + \sigma\widetilde{X}$ with the

degree of freedom ν. Unknown parameters $\hat{\mu}_i$, $\hat{\sigma}_i$, and $\hat{\rho}_{ij}$ are estimated by the method of moment estimates as follows,

$$\hat{\mu}_i = \frac{1}{T}\sum_{t=1}^{T} r_i^T(t), \tag{4.15}$$

$$\hat{\sigma}_i = \left(\frac{(\nu-2)\sum_{t=1}^{T}(r_i^T(t)-\hat{\mu}_i)^2}{\nu T}\right)^{1/2}, \tag{4.16}$$

$$\hat{\rho}_{ij} = \frac{\sum_{t=1}^{T}(r_i^T(t)-\hat{\mu}_i)(r_j^T(t)-\hat{\mu}_j)}{(\sum_{t=1}^{T}(r_i^T(t)-\hat{\mu}_i)^2\sum_{t=1}^{T}(r_j^T(t)-\hat{\mu}_j)^2)^{1/2}}. \tag{4.17}$$

(2) Generate B independent bootstrap samples $Y_{h(\theta)}^1, Y_{h(\theta)}^2, \ldots, Y_{h(\theta)}^B$ from the gamma distribution $\Gamma(\frac{\nu}{2}, h(\theta))$, and given $Y_{h(\theta)}^1, Y_{h(\theta)}^2, \ldots, Y_{h(\theta)}^B$, generate B independent bootstrap samples $\widetilde{Z}_{\tilde{a}^T(\theta)}^1, \widetilde{Z}_{\tilde{a}^T(\theta)}^2, \ldots, \widetilde{Z}_{\tilde{a}^T(\theta)}^B$ from the multivariate normal distribution $MN(\tilde{a}^T(\theta), I)$, then we obtain B independent bootstrap samples $\tilde{r}_\theta^{T,1}, \tilde{r}_\theta^{T,2}, \ldots, \tilde{r}_\theta^{T,B}$, by $\widetilde{X}_{\tilde{a}^T(\theta),h(\theta)}^b = \frac{\widetilde{Z}_{\tilde{a}^T(\theta)}^b}{\sqrt{\frac{Y_{h(\theta)}^b}{\nu}}}$ and $\tilde{r}_\theta^{T,b} = \hat{\tilde{\mu}} + \hat{\sigma}\widetilde{X}_{\tilde{a}^T(\theta),h(\theta)}^b$, each consisting of T data values drawn from the tilting measure.

(3) Evaluate the bootstrap replication according to each bootstrap sample,

$$\hat{p}_b^T(\theta) = \frac{1}{T}\sum_{t=1}^{T} 1_{A_\theta^T(b,t)} \exp\{-\theta Q(R_\theta^T(b,t)) + \psi^T(\theta)\},$$
$$b = 1, 2, \ldots, B, \tag{4.18}$$

where $Q(R_\theta^T(b,t)) = \frac{Y_{h(\theta)}^b}{\nu} f(R_\theta^T(b,t))$, and $A_\theta^T(b,t) = \{\tilde{r}_\theta^T(b,t) : Q(R_\theta^T(b,t)) = \frac{Y_{h(\theta)}^b}{\nu}(R_\theta^T(b,t) - r_p^T) = \frac{Y_h^b}{\nu}(\tilde{w}'(t)\tilde{r}_\theta^T(b,t) - r_p^T) > 0\}$ for $b = 1, 2, \ldots, B$, by substituting θ_g^T into θ.

(4) Estimate the probability $p^T(\theta)$ and standard error $se_p^T(\theta)$ by using B replications as

$$\bar{p}^T(\theta) = \frac{\sum_{b=1}^{B}\hat{p}_b^T(\theta)}{B}, \tag{4.19}$$

$$\widehat{se}_p^T(\theta) = \left(\frac{\sum_{b=1}^{B}(\hat{p}_b^T(\theta) - \bar{p}^T(\theta))^2}{B-1}\right)^{1/2}. \tag{4.20}$$

4.2 Quantile Estimation for the Portfolio Return

According to the titling measure of the estimating probability, the quantile estimation would be obtained by the order statistics (Johns 1988). For all values of θ, the importance sampling of probability estimators $\hat{p}_\theta^N$ and $\hat{p}_\theta^T$ are unbiased in a multivariate normal distribution and a multivariate t distribution. Under the titling measure $\mathbb{P}^\theta$, the order of the portfolio return is

$$(R^*_{(1)}, \ldots, R^*_{(T)}), \tag{4.21}$$

where $R^*_{(1)}, \ldots, R^*_{(T)}$ are the order statistics of the sample $\{R^*_1, \ldots, R^*_T\}$. Therefore, an estimate of the quantile R^*_{1-p} for the portfolio return is

$$R^*_{(1-p)} := (1-s)R^*_{(j)} + sR^*_{(j+1)}. \tag{4.22}$$

where j is defined by

$$\frac{1}{T}\sum_{k=1}^{j} 1_{\{R_{(k)}<r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^\theta(R^*_{(k)})} < 1-p, \tag{4.23}$$

$$\frac{1}{T}\sum_{k=1}^{j+1} 1_{\{R_{(k)}<r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^\theta(R^*_{(k)})} > 1-p, \tag{4.24}$$

and s is defined by

$$s = \left(1 - p - \frac{1}{T}\sum_{k=1}^{j} 1_{\{R_{(k)}<r_p\}} \frac{d\mathbb{P}(R^*_{(k)})}{d\mathbb{P}^\theta(R^*_{(k)})}\right) \frac{d\mathbb{P}^\theta(R^*_{(j+1)})}{d\mathbb{P}(R^*_{(j+1)})}.$$

The asymptotic properties of the unbiased estimator for order statistics guarantee that as $T \to \infty$

$$\sqrt{T}(\hat{r}_p - r_p) \to N\left(0, \frac{p(1-p)}{f^2(r_p)}\right),$$

where $f = F'$ exists and is continuous at r_p. (cf. Hall 1990a, b; Johns 1988; Goffinet and Wallach 1996). The variance reduction of the classical estimator in place of the importance sampling estimator of the quantile will be the same as for p. Therefore, we analysis analyze the estimation of the tail probability by the numerical analysis for the linear assets. There are same results in the estimation of the quantile for the non-linear assets.

5 Monte Carlo, Numerical and Sensitivity Analysis

Based on the estimate of the tail probability, there are four aims in this section. First, the relative efficiency is computed by the naive Monte Carlo simulation with respect to importance sampling with Glasserman and Laplace method, for a multivariate normal and a multivariate t distribution. Second, we also compute the efficiency of using importance resampling in the bootstrap algorithm. The efficiency is defined as the bootstrap replication for computing the standard error of the tail probability. The comparison is based on $B = 200$ for bootstrap algorithm with importance resampling and $B = 1{,}000$ for bootstrap algorithm with naive resampling. Third, the sensitivity analysis is used to compare the estimation of tail probability effected by the true parameter, where they are estimated in different periods from $T = 250$ to $T = 500$. Finally, we compare the new titling point method with that in Glasserman et al. (2002) in the cases of one and two assets.

5.1 Monte Carlo Simulation

Consider a linear portfolio with two assets for the parameters $\mu_1 = 0.01$, $\mu_2 = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.8$, $\rho_{12} = 0.3$, $\nu = 5$, the sample size $T = 500$ and the Monte Carlo size $M = 10{,}000$. For simplicity set $w_i(t) = 1$ for all i and t. Table 1 lists the relative efficiency of p using Monte Carlo simulations, importance sampling, and importance sampling for new bound in multivariate normal distribution. The relative efficiency of $\hat{p}(\theta_1)$ relative to $\hat{p}(\theta_2)$ is defined (cf. Hall 1991) as

$$\text{eff}(\hat{p}(\theta_1), \hat{p}(\theta_2)) = \frac{Var(\hat{p}(\theta_2))}{Var(\hat{p}(\theta_1))} \tag{5.1}$$

where $Var(\hat{p}(\theta_i))$ is the variance of the tail probability estimator $p(\theta_i)$ with the parameter θ_i. In Table 1, the estimate of the tail probability with importance sampling of Glasserman method is more efficiency than the Monte Carlo in 10,000 simulations when p is smaller. Although the of the tail probability with importance sampling of Laplace method is more efficiency than Glasserman method, the value of the efficiency decay in the decreasing of p. Therefore, when p is very small, we can use importance sampling with Glasserman method to improve the efficiency, otherwise, we use importance sampling with Laplace method to improve efficiency.

In Table 2, similar results are obtained in the multivariate t distribution. In addition to similar results, if the distribution of the portfolio has heavy, the relative efficiency of the estimating the tail probability with importance sampling relative to Monte Carlo in the multivariate normal distribution is higher than the multivariate t distribution in when p is smaller. That is to say, the estimate of the tail probability with importance sampling in a heavy distribution is more efficiency than a thin distribution.

Table 1 The relative efficiency of the tail probability estimation with a multivariate normal distribution in Monte Carlo simulation

r_p^N	0.5219	0.8013	1.1889	1.5091	2.1093	2.7822
p	0.3000	0.2000	0.1000	0.0500	0.0100	0.0010
$\hat{p}^N$	0.3003	0.1998	0.1000	0.0499	0.0100	0.0010
$\widehat{se}_N$	2.04E−02	1.79E−02	1.33E−02	9.64E−03	4.42E−03	1.41E−03
$\hat{p}^N(\theta_g^N)$	0.3000	0.2001	0.1000	0.0500	0.0100	0.0010
$\widehat{se}_p^N(\theta_g^N)$	1.43E−02	1.02E−02	5.77E−03	3.15E−03	7.31E−04	8.41E−05
$\hat{p}^N(\theta_l^N)$	0.2998	0.2001	0.1000	0.0500	0.0100	0.0010
$\widehat{se}_p^N(\theta_l^N)$	1.32E−02	9.74E−03	5.60E−03	3.15E−03	7.27E−04	8.28E−05
$\text{eff}(\hat{p}^N(\theta_g^N), \hat{p}^N)$	2.0175	3.0510	5.3574	9.3511	36.6857	280.4859
$\text{eff}(\hat{p}^N(\theta_l^N), \hat{p}^N)$	2.3655	3.3970	5.7040	9.9458	37.0562	289.1810
$\text{eff}(\hat{p}^N(\theta_l^N), \hat{p}^N(\theta_g^N))$	1.1725	1.1134	1.0647	1.0636	1.0101	1.0310

p presents the true tail probability, r_p^N denotes the quantile of p, $\hat{p}^N$, and $\widehat{se}_p^N$ are the mean and standard error of the tail probability estimator by Monte Carlo, $\hat{p}^N(\theta_g^N)$ is the mean of the tail probability estimator by importance sampling with Glasserman method, $\hat{p}^N(\theta_l)$ is the mean of the tail probability estimator by importance sampling with Laplace method, $\text{eff}(\hat{p}^N(\theta_g^N), \hat{p}^N)$ is the relative efficiency of $\hat{p}(\theta_g^N)$ relative to $\hat{p}$, $\text{eff}(\hat{p}^N(\theta_l^N), \hat{p}^N)$ is the relative efficiency of $\hat{p}^N(\theta_l^N)$ relative to $\hat{p}$ and $\text{eff}(\hat{p}^N(\theta_l^N), \hat{p}^N(\theta_g^N))$ is the relative efficiency of $\hat{p}^N(\theta_l^N)$ relative to $\hat{p}(\theta_g^N)$ in a multivariate normal distribution

Table 2 The relative efficiency of the tail probability estimation with a multivariate t distribution in Monte Carlo simulation

r_p^T	0.5527	0.8698	1.3590	1.8350	3.0235	5.2450
p	0.3000	0.2000	0.1000	0.0500	0.0100	0.0010
$\hat{p}^T$	0.3001	0.1997	0.0999	0.0500	0.0100	0.0010
$\widehat{se}_T$	2.08E−02	1.79E−02	1.35E−02	9.89E−03	4.39E−03	1.41E−03
$\hat{p}^T(\theta_g^T)$	0.3001	0.2001	0.1001	0.0501	0.0100	0.0010
$\widehat{se}_p^T(\theta_g^T)$	1.46E−02	1.04E−02	5.69E−03	3.01E−03	6.45E−04	6.83E−05
$\hat{p}^T(\theta_l^T)$	0.2999	0.2002	0.1000	0.0500	0.0100	0.0010
$\widehat{se}_p^T(\theta_l^T)$	1.36E−02	9.87E−03	5.47E−03	2.91E−03	6.24E−04	6.77E−05
$\text{eff}(\hat{p}^T(\theta_g^T), \hat{p}^T)$	2.0104	2.9592	5.6207	10.7837	46.3205	428.4190
$\text{eff}(\hat{p}_g^T, \hat{p}^T(\theta_l^T))$	2.3388	3.2924	6.0711	11.5365	49.4667	435.9926
$\text{eff}(\hat{p}^T(\theta_l^T), \hat{p}^T(\theta_g^T))$	1.1633	1.1126	1.0801	1.0698	1.0679	1.0177

p presents the true tail probability, r_p^T denotes the quantile of p, $\hat{p}^T$, and $\widehat{se}_p^N$ are the mean and standard error of the tail probability estimator by Monte Carlo, $\hat{p}^T(\theta_g)$ is the mean of the tail probability estimator $\hat{p}$ by importance sampling, $\hat{p}^T(\theta_l^T)$ is the mean of the tail probability estimator by $\hat{p}^T(\theta_g^T)$ importance sampling with new bound of the second moment, $\text{eff}(\hat{p}^T(\theta_g^T), \hat{p}^T)$ is the relative efficiency of $\hat{p}^T(\theta_g)$ relative to $\hat{p}^T$, $\text{eff}(\hat{p}^T(\theta_l^T), \hat{p}^T)$ is the relative efficiency of $\hat{p}^T(\theta_l^T)$ relative to $\hat{p}^T$, and $\text{eff}(\hat{p}^T(\theta_g^T), \hat{p}^T(\theta_l^T))$ is the relative efficiency of $\hat{p}^T(\theta_l^T)$ relative to $\hat{p}^T(\theta_g^T)$ in a multivariate t distribution

5.2 Numerical Analysis

By the algorithm developed in Sect. 3, we study the parametric bootstrap method with importance resampling. The same parameters is assumed as in Sect. 5.1, with the bootstrap replications $B = 1{,}000$ for the parametric bootstrap with naive Monte Carlo simulation, and the bootstrap replications $B = 200$ for the parametric bootstrap with importance resampling in Table 3. When the estimation of the tail probability under the three methods are almost the same, the standard error under the bootstrap method is larger than those from the multivariate normal and t distribution. That is, the bootstrap with importance resampling with bootstrap replications $B = 200$ is more efficient than the bootstrap method with naive Monte Carlo simulation with replications $B = 1{,}000$ (Table 3).

5.3 Sensitivity Analysis

In the parametric bootstrap algorithm, the procedure has estimation errors due to sampling, based on the parameters estimation of the return model (cf. Lin et al. 2004). Therefore, to calculate sensitivity analysis, we obtain the estimate of the tail probability with the dataset, and compare it to the estimate effected by the true parameter. The same parameters are used as in Sect. 5.1, the bootstrap replications $B = 200$ and Monte Carlo simulation size is 10,000. Table 4 shows the results of sensitivity analysis for the multivariate normal and the multivariate t distribution. Sensitivity is defined as

$$\frac{p - \hat{p}}{p} \tag{5.2}$$

for tail probability estimation.

From Table 4, the sensitivities are quite small for tail probability. This implies that the estimate of the parameters do not affect tail probability estimation in an obvious way when compared with true values for the multivariate normal and the multivariate t distribution.

5.4 New Tilting Point versus GHS Method

In Sect. 4.1, we propose a method to approximate the tilting point. In Tables 5 and 6, we set the specific parameters to compute the titling points for the single asset and two assets case. We report the estimator and variance by using naive simulation, the new titling point importance sampling, and the Glasserman, Heidelberger and Shahabuddin (GHS) method. At the end, we show the variance ratios of relative efficiency using the GHS method divided by the new method. Obviously, the two importance sampling methods are more efficient than naive simulation. Besides, when it comes to the optimal titling point, the new titling point is much closer to it than the GHS method in moderate deviation.

Table 3 The estimation of tail probability with the multivariate normal and t distribution

Multivariate normal distribution	$p = 5\%$		$p = 1\%$	
	$B = 1,000$	$B = 200$	$B = 1,000$	$B = 200$
	$\bar{p}^N$	$\bar{p}^N(\theta_g^N)$	$\bar{p}^N$	$\bar{p}^N(\theta_g^N)$
p^N	0.0498	0.0501	0.0101	0.0101
se_p^N	1.01E−02	3.13E−03	4.42E−03	7.49E−04
Multivariate t distribution	$p = 5\%$		$p = 1\%$	
	$B = 1,000$	$B = 200$	$B = 1,000$	$B = 200$
	$\bar{p}^T$	$\bar{p}^T(\theta_g^T)$	$\bar{p}^T$	$\bar{p}^T(\theta_g^T)$
p^T	0.0507	0.0507	0.0102	0.0103
se_p^T	9.67E−03	3.04E−03	4.21E−03	6.47E−04

Table 4 The sensitivity of tail probability with multivariate normal and t distribution

Multivariate normal distribution				
	$p = 5\%$			
	$T = 250, B = 200$		$T = 500, B = 200$	
	$\hat{p}^N$	$\hat{p}^N(\theta_g^N)$	$\hat{p}^N$	$\hat{p}^N(\theta_g^N)$
p^N	0.0500	0.0500	0.0500	0.0500
se_p^N	9.7E−03	3.2E−03	1.4E−02	4.5E−03
Sensitivity	3.0E−04	−2.0E−05	4.4E−05	−7.0E−06
Multivariate t distribution				
	$p = 5\%$			
	$T = 250, B = 200$		$T = 500, B = 200$	
	$\hat{p}^T$	$\hat{p}^T(\theta_g^T)$	$\hat{p}^T$	$\hat{p}^T(\theta_g^T)$
p^T	0.0500	0.0500	0.0500	0.0500
se_p^T	1.4E−02	4.2E−03	9.7E−03	3.0E−03
Sensitivity	−5.2E−06	6.7E−05	−1.6E−04	4.4E−05

Table 5 Quadratic approximation function compared with the GHS method. The parameters are $\nu = 5,\ k = 0,\ b = -1,\ \lambda = 0.5,\ T = 500$, and $M = 10{,}000$

	$x = 1$	$x = 2$	$x = 3$	$x = 5$
true P(A)	2.69E−01	1.47E−01	8.78E−02	3.80E−02
naive	2.69E−01	1.47E−01	8.77E−02	3.80E−02
variance	3.94E−04	2.52E−04	1.62E−04	7.30E−05
new I.S. $\hat{p}$	2.69E−01	1.47E−01	8.78E−02	3.80E−02
variance	2.24E−04	9.70E−05	4.26E−05	9.85E−06
θ	4.56E−01	5.30E−01	5.83E−01	6.48E−01
GHS $\hat{p}$	2.69E−01	1.47E−01	8.78E−02	3.80E−02
variance	2.56E−04	1.04E−04	4.33E−05	9.92E−06
θ	2.16E−01	4.08E−01	5.00E−01	5.93E−01
relative efficiency	1.14	1.07	1.02	1.01

In theory, we provide this new tilting point that it is more closer the original second moment than the GHS method. Because the inequality of the GHS method is chosen more wider than our method. We can see the relation of upper bounded of second moment in Fig. 1. The solid line is true upper bounded of second moment for importance sampling. The others dash lines are upper bounded of second moment for the GHS method and new tilting point. It can observe that the line of the GHS method is away from true second moment and the shape is also seemingly different. The line of

Table 6 Quadratic approximation function compared with the GHS method. The parameters are $\nu = 5,\ k = 0,\ b_1 = 0,\ b_2 = -1.183,\ \lambda_1 = 0.247,\ \lambda_2 = 0.147,\ T = 500$, and $M = 10{,}000$

	$x = 1$	$x = 2$	$x = 3$	$x = 5$
true P(A)	3.16E−01	1.56E−01	8.06E−02	2.71E−02
naive $\hat{p}$	3.16E−01	1.56E−01	8.05E−02	2.70E−02
variance	4.22E−04	2.69E−04	1.48E−04	5.26E−05
new I.S. $\hat{p}$	3.16E−01	1.56E−01	8.06E−02	2.70E−02
variance	2.06E−04	7.38E−05	2.48E−05	3.49E−06
θ	5.47E−01	8.94E−01	1.14E+00	1.37E+00
GHS $\hat{p}$	3.16E−01	1.56E−01	8.07E−02	2.70E−02
variance	2.49E−04	7.91E−05	2.53E−05	3.51E−06
θ	3.11E−01	6.48E−01	8.64E−01	1.12E+00
relative efficiency	1.21	1.07	1.02	1.00

Table 7 The statistics of “IBM,” “DELL,” and “SUN MICROSYSTEMS INC”

Company	Mean	Standard Deviation	Skewness	Kurtosis
IBM	−0.0151%	1.0294%	−0.8506	9.1515
DELL	−0.0160%	1.3413%	0.0378	9.0213
SUN MICROSYSTEMS INC	0.0195%	2.5622%	0.9769	9.2649

Table 8 The statistics of “MSFT(call),” “T(put),” and “LU(call)”

Company	Mean	Standard Deviation	Skewness	Kurtosis
MSFT(call)	−0.0264%	2.7943%	−1.3797	5.9770
T(put)	−0.0980%	2.9305%	1.8324	6.1437
LU(call)	0.1333%	7.0605%	0.7929	6.0522

new tilting point is between the others. However, there is no difference in the case of large deviation. The new method will be complicated in the case of more assets and its efficiency improves less in large deviation. Consequently, we suggest computing the titling point by using the GHS method in the case of multi-assets and large deviation, although it will be a little less efficient.

6 Empirical Study

To illustrate our proposed bootstrap algorithm with importance resampling, in this section we analyze stock index returns on the NYSE from the Center for Research in Security Prices (CRSP), including the “IBM” index return, “DELL” index return and

Table 9 The summary of comparison of VaR with naive bootstrap, the GHS method, new method

Panel A: Portfolio of stocks		
	estimate VaR	Standard Deviation
True VaR of portfolio	0.0770	
Naive bootstrap ($B = 1{,}000$)	0.0774	2.17E−03
GHS method	0.0765	4.66E−03
New method ($B = 200$)	0.0769	1.92E−03
Panel B: Portfolio of derivatives		
Naive bootstrap ($B = 1{,}000$)	0.1537	4.25E−03
GHS method	0.1558	8.12E−03
New method ($B = 200$)	0.1528	3.87E−03

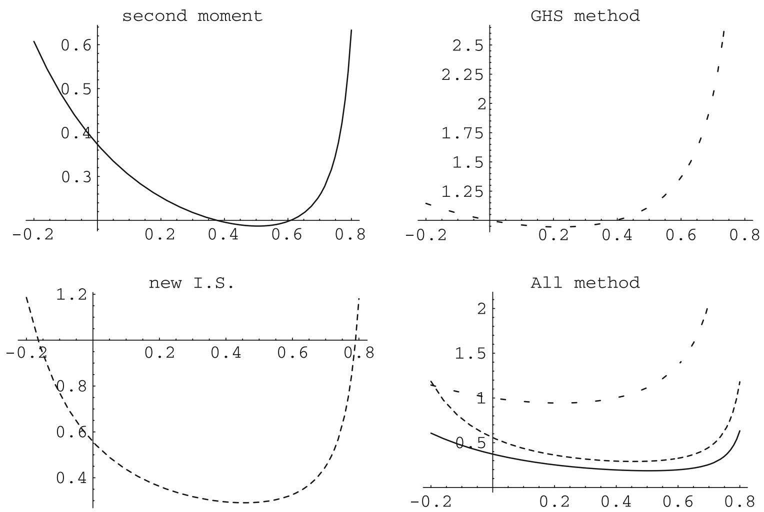

Fig. 1 The relation of upper bounded of second moment by all method

"SUN MICROSYSTEMS INC" index return. The sample period is drawn from January 2, 2004, to December 30, 2005, and includes 504 observations. There is a summary of return means and standard deviations in Table 7. The skewness is −0.8506, 0.0378 and 0.9769, and the kurtosis is 9.1515, 9.0213 and 9.2649, respectively. Note that all of the kurtosis exceed 3, and hence each of the stock index returns have leptokurtic features. We assume that the vector of the returns is a multivariate t distribution, and the correlation coefficient is $\rho_{12} = 0.084$, $\rho_{13} = 0.786$, and $\rho_{23} = 0.125$.

The kurtosis of the three asset returns are close to 9, so the returns are not normally distributed. We suppose the returns follow a multivariate t distribution. The kurtosis excess of each return is $\frac{6}{\nu-4}$, and we let the degree of freedom $\nu = 5$. In addition, we assume $w_i(t) = 1$, for all i and t, and obtain $r_p^T = \widehat{K} t_{0.05}(\nu) + \hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3 = 0.077$, where $\widehat{K} = 0.0383$. The parameter θ_g^T of the change of measure is 52.6043 in Eq. 4.11, so $h(\theta_g^T) = 1.1046$. We change the measure from the original gamma measure $Y \sim G(2.5, 2)$ to the new measure $Y_{h(\theta)} \sim G(2.5, 1.1046)$, then given $Y_{h(\theta)}$, twist the original normal density $Z \sim N(0, 1)$ to the new normal density $Z_{a^T(\theta)} \sim N(a^T(\theta), 1)$, where $a^T(\theta) = 2.015\sqrt{\frac{Y_{h(\theta)}}{5}}$. Hence the event $Q(R^T(t)) > 0$ can be computed in Eq. 4.2, so the estimate and standard error of the tail probability $p = 0.05$ using the bootstrap algorithm with naive resampling for replications $B = 1{,}000$ are 0.0774 and 2.17E−03. Finally, the estimate and standard error of the VaR are 0.0769 and 1.92E−03 using the bootstrap algorithm with importance resampling for replications $B = 200$. There are two advantages in this work; one is the decrease in computational time from bootstrap replications $B = 1{,}000$ to 200, the other is the gain in accuracy by using importance resampling. We summarize and compare the VaR of portfolio of stocks with naive bootstrap, the GHS method, new method at the Panel A in Table 9. The accuracy and efficiency are approached the simulation results in Sect. 5.

We also analyze option returns from Ivy DB OptionMetrics, including the call option returns of "MICROSOFT CORP(MSFT)," put option returns of "AT& T(T)," and call option returns of "LUCENT TECHNOLOGIES INC(LU)." The sample period is drawn from January 2, 2004, to January 21, 2005, and includes 266 observations. All excise dates are the same on January 22, 2005. We summarize return means and standard deviations in Table 8. We assume the returns follow a multivariate t distribution with the degree of freedom $\nu = 6$ and use the quadratic approximation method with standard sensitivities (delta, gamma). All three assets have positive gamma, which is good for the portfolio. Hence the event $Q(R^T(t)) > 0$ can be computed in Eq. 4.2, given the estimate and standard error of the tail probability $p = 0.05$, and the bootstrap algorithm with importance resampling for replications $B = 200$, the VaR are 0.1528 and 3.87E−03. We also summarize and compare the VaR of portfolio of derivatives with naive bootstrap, the GHS method, new method at the Panel B in Table 9. The true VaR is not sure in quadratic approximation method. But, efficiency is also approached the simulation results in Sect. 5.

7 Conclusion

We combine a bootstrap algorithm with importance resampling to improve the efficient of the bootstrap method, and propose a parametric bootstrap algorithm with importance resampling to compute the VaR of a portfolio. Based on linear assets and nonlinear assets, some results are contributed in this paper. First, we develop a new efficient computational procedure for estimating the portfolio loss probability, called importance sampling with Laplace method. According the structure of the multivariate t distribution to develop efficient methods for calculating portfolio loss probability, capturing heavy tails in the joint distribution of market risk factors. For comparison with the classical setting of geometric Brownian motion, we establish the common setting of the risk factors having a multivariate normal distribution, and then consider the risk factors to have a multivariate t distribution. Second, we propose and compare importance sampling titling measures for a multivariate normal distribution and a multivariate t distribution. For a wide range of portfolios, importance sampling is more efficient than other variance reduction tools (cf. Glasserman 2003). Third, we also propose a accurate method to approximate the titling point by using the Laplace method. The variance ratios of moderate events are relatively more effective than those obtained using the GHS method. It is because that the GHS method of upper bounded of second moment is away from true boundary. We also found that the variance decreases and the bootstrap algorithm with importance resampling is more efficient than the naive bootstrap method by using computer simulations as well as analytic studies. According sensitivity analysis, the estimate of parameters does not significantly affect the loss probability when compared with the true parameters.

Based on the bootstrap algorithm with importance sampling, there are some open problems on this topic. This paper considered the situation in which the change in risk factors has a multivariate t distribution and a multivariate normal distribution. A possible shortcoming is that the risk factors are not the distribution of the multivariate t distribution and the multivariate normal distribution. The concept of importance resampling can extend to all parametric bootstrap when we capture the distribution of the heavy tails. For the aspect of the degree freedom with a multivariate t distribution, it also is natural to extend the model with multiple degrees of freedom and use a copula to do it (cf. Nelsen 1999; Embrechts et al. 2002), but leave this for further studies.

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Appendix 1: Compute the Second Moment Upper Bound of the Tail Probability for the Multivariate t Distribution

Compute the second moment of the estimator for the tail probability with the multivariate t distribution

$$
\begin{aligned}
& E_\theta(\hat{p}^2) \\
&= E_\theta(e^{-2\theta Q_x + 2\psi(\theta)} 1_{\{Q_x>0\}}) \\
&= E(e^{-\theta Q_x + \psi(\theta)} 1_{\{Q_x>0\}}) \\
&= E\left(\exp\left\{-\theta \frac{Y}{\nu}\left(\sum_{j=1}^{n}(b_j X_j + \lambda_j X_j^2) - x\right) + \psi(\theta)\right\}\right. \\
&\qquad \left. \times 1_{\left\{\frac{Y}{\nu}\left(\sum_{j=1}^{n}\left(b_j X_j + \lambda_j X_j^2\right) - x\right) > 0\right\}}\right)
\end{aligned}
$$

$$= E\left(\exp\left\{-\theta\left(\sum_{j=1}^{n}\left(b_j\sqrt{\frac{Y}{\nu}}Z_j+\lambda_j Z_j^2\right)-\frac{Y}{\nu}x\right)+\psi(\theta)\right\}\right.$$
$$\left.\times 1_{\left\{\sum_{j=1}^{n}\left(b_j\sqrt{\frac{Y}{\nu}}Z_j+\lambda_j Z_j^2\right)-\frac{Y}{\nu}x>0\right\}}\right)$$
$$= E\left(E\left(\exp\left\{-\theta\left(\sum_{j=1}^{n}\left(b_j\sqrt{\frac{Y}{\nu}}Z_j+\lambda_j Z_j^2\right)-\frac{Y}{\nu}x\right)+\psi(\theta)\right\}\right.\right.$$
$$\left.\left.\times 1_{\{\sum_{j=1}^{n}\lambda_j\left(\frac{b_j}{\lambda_j}\sqrt{\frac{Y}{\nu}}Z_j+Z_j^2\right)>\frac{Y}{\nu}x\}|Y}\right)\right)$$
$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\theta\left(\sum_{j=1}^{n}\left(b_j\sqrt{\frac{y}{\nu}}z_j+\lambda_j z_j^2\right)-\frac{y}{\nu}x\right)+\psi(\theta)\right\}$$
$$\times\left(\frac{1}{\sqrt{2\pi}}\right)^n\exp\left\{-\frac{\sum_{j=1}^{n}z_j^2}{2}\right\}$$
$$1_{\{\sum_{j=1}^{n}\lambda_j(z_j+\frac{b_j}{2\lambda_j}\sqrt{\frac{y}{\nu}})^2>\frac{y}{\nu}x+\sum_{j=1}^{n}\frac{b_j^2 y}{4\lambda_j\nu}\}}dz_1,\ldots,dz_n f_y(y)dy$$
$$= \int_{-\infty}^{\infty}\exp\left\{\frac{yx\theta}{\nu}+\psi(\theta)+\sum_{j=1}^{n}\frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\}\prod_{j=1}^{n}\frac{1}{\sqrt{1+2\lambda_j\theta}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}$$
$$\left(\frac{1}{\sqrt{2\pi}}\right)^n\prod_{j=1}^{n}\sqrt{1+2\lambda_j\theta}$$
$$\exp\left\{-\sum_{j=1}^{n}\frac{(z_j+\frac{b_j\theta}{1+2\lambda_j\theta}\sqrt{\frac{y}{\nu}})^2}{2\frac{1}{1+2\lambda_j\theta}}\right\}$$
$$1_{\{\sum_{j=1}^{n}\lambda_j(z_j+\frac{b_j}{2\lambda_j}\sqrt{\frac{y}{\nu}})^2>\frac{y}{\nu}x+\sum_{j=1}^{n}\frac{b_j^2 y}{4\lambda_j\nu}\}}dz_1,\ldots,dz_n f_y(y)dy$$

The third to last equality holds by using the law of iterated expectations. Let $z_j^*=\frac{z_j+\frac{b_j\theta}{1+2\lambda_j\theta}\sqrt{\frac{y}{\nu}}}{\frac{1}{\sqrt{1+2\lambda_j\theta}}}$, then $dz_j^*=\sqrt{1+2\lambda_j\theta}dz_j$, and we can compute the last equation to be

$$= \int_{-\infty}^{\infty}\exp\left\{\frac{yx\theta}{\nu}+\psi(\theta)+\sum_{j=1}^{n}\frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\}\prod_{j=1}^{n}\frac{1}{\sqrt{1+2\lambda_j\theta}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}$$

$$\times\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{\sum_{j=1}^n (z_j^*)^2}{2}\right\}$$
$$1_{\left\{\sum_{j=1}^n \frac{\lambda_j}{1+2\lambda_j\theta}\left(z_j^*-[\frac{b_j\theta}{\sqrt{1+2\lambda_j\theta}}-\frac{\sqrt{1+2\lambda_j\theta}b_j}{2\lambda_j}]\sqrt{\frac{y}{\nu}}\right)^2>\frac{y}{\nu}x+\sum_{j=1}^n\frac{b_j^2y}{4\lambda_j\nu}\right\}} dz_1^*\dots dz_n^* f_Y(y)dy.$$

Let $c_j = \frac{\lambda_j}{1+2\lambda_j\theta}$, $d_j = \left[\frac{b_j\theta}{\sqrt{1+2\lambda_j\theta}} - \frac{\sqrt{1+2\lambda_j\theta}b_j}{2\lambda_j}\right]\sqrt{\frac{y}{\nu}}$, and $g = \frac{y}{\nu}x + \sum_{j=1}^n \frac{b_j^2 y}{4\lambda_j\nu}$, then, we can obtain

$$= \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \sum_{j=1}^n \frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\} \prod_{j=1}^n \frac{1}{\sqrt{1+2\lambda_j\theta}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}$$
$$\times\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{\sum_{j=1}^n (z_j^*)^2}{2}\right\} 1_{\{\sum_{j=1}^n c_j(z_j^*-d_j)^2>g\}} dz_1^*,\dots,dz_n^* f_Y(y)dy.$$

Appendix 2: Tilting Measure for the Multivariate *t* Distribution

To obtain the tilting measure, we first fix the random variable Y,

$$\begin{aligned}
d\mathbb{P}^{\theta}_{\tilde{a}(\theta),h(\theta)}(\widetilde{Z},Y) &= d\mathbb{P}_{\tilde{0},2}(\widetilde{Z},Y)\exp\{\theta Q(R_\theta^T(t)) - \psi^T(\theta)\} \\
&= d\mathbb{P}_{\tilde{a}}(\widetilde{Z}|Y)\exp\{\theta Q(R_\theta^T(t)) - \psi^T(\theta)\} d\mathbb{P}_2(Y) \\
&= \frac{1}{(\sqrt{2\pi})^n}\exp\left\{\widetilde{Z}'_{\tilde{a}^T(\theta)} I \widetilde{Z}_{\tilde{a}^T(\theta)}\right\} \\
&\qquad \times \exp\left\{\frac{Y}{2\nu}\theta^2K^2 + \frac{Y}{\nu}(\tilde{w}'(t)\tilde{\mu}\theta - r_p^T\theta) - \psi^T(\theta)\right\} d\mathbb{P}_2(Y) \\
&= d\mathbb{P}^{\theta}_{\tilde{a}(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y)\exp\left\{\frac{Y}{2\nu}\theta^2K^2 + \frac{Y}{\nu}(\tilde{w}'(t)\tilde{\mu}\theta - r_p^T\theta) - \psi^T(\theta)\right\} \\
&\qquad \times d\mathbb{P}_2(Y),
\end{aligned} \tag{A.1}$$

where $d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta),h(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}, Y)$ denotes the tilting measure with the mean vector $\tilde{a}^T(\theta)$ of the multivariate normal distribution and the scale parameter $h(\theta)$ for the gamma distribution $G(\nu/2, h(\theta))$, $d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y)$ is the measure given the random variable Y with $\tilde{a}^T(\theta)$ for the mean vector of the multivariate normal distribution $MN(\tilde{a}^T(\theta), I)$, where $\tilde{a}^T(\theta) = \sqrt{Y/\nu}\theta\,(\sum_{i=1}^n \sigma_i w_i(t)c_{i1},\dots,\sum_{i=1}^n \sigma_i w_i(t)c_{in})'$, and $d\mathbb{P}_2(Y)$ is the measure of the gamma distribution $G(\nu/2, 2)$ with the scale parameter 2. For simplicity, we can use $d\mathbb{P}_0(Z|Y)$ as the origin measure for Eq. A.1. Then the tilting measure is

$$d\mathbb{P}^{\theta}_{a^T(\theta)}(Z_{a^T(\theta)}|Y) = d\mathbb{P}_0(Z|Y)\exp\left\{\theta K Z - \frac{Y}{2\nu}K^2\theta^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(Z - \sqrt{\frac{Y}{\nu}}K\theta)^2\right\}, \qquad \text{(A.2)}$$

where $a^T(\theta) = \sqrt{\frac{Y}{\nu}}K\theta$. Then, we see the new gamma density for the tilting measure,

$$d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta),h(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}, Y_{h(\theta)})$$

$$= d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y)\exp\left\{\frac{y}{2\nu}\theta^2K^2 + \frac{y}{\nu}(\tilde{w}'(t)\tilde{\mu}\theta - r_p^T\theta) - \psi^T(\theta)\right\}$$

$$\times\frac{y^{(\nu-2)/2}e^{-y/2}}{\Gamma(\nu/2)2^{\nu/2}}$$

$$= d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y)\frac{y^{(\nu-2)/2}}{\Gamma(\nu/2)2^{\nu/2}}\exp\left\{\frac{-y}{\frac{2\nu}{\nu-2\tilde{w}'(t)\tilde{\mu}\theta+2r_p^T\theta-\theta^2K^2}}\right\}\frac{\nu - 2\tilde{w}'(t)\tilde{\mu}\theta + 2r_p^T\theta - \theta^2K^2}{\nu}$$

$$= d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y)\frac{y^{(\nu-2)/2}}{\Gamma(\nu/2)\left(\frac{2\nu}{\nu-2\tilde{w}'(t)\tilde{\mu}\theta+2r_p^T\theta-\theta^2K^2}\right)^{\nu/2}}\exp\left\{\frac{-y}{\frac{2\nu}{\nu-2\tilde{w}'(t)\tilde{\mu}\theta+2r_p^T\theta-\theta^2K^2}}\right\}$$

$$= d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}|Y_{h(\theta)})d\mathbb{P}^{\theta}_{h(\theta)}(Y_{h(\theta)}). \qquad \text{(A.3)}$$

To find the tilting measure $d\mathbb{P}^{\theta}_{\tilde{a}^T(\theta),h(\theta)}(\widetilde{Z}_{\tilde{a}^T(\theta)}, Y_{h(\theta)})$, under the independent assumption with $\widetilde{X}_{\tilde{a}^T(\theta),h(\theta)} = \widetilde{Z}_{\tilde{a}^T(\theta)}/\sqrt{Y_{h(\theta)}/\nu}$, we first twist the original gamma distribution $Y \sim G(\nu/2, 2)$ to the new gamma density $Y_h \sim G(\nu/2, h(\theta))$, where $h(\theta) = 2\nu/(\nu - 2\tilde{w}'(t)\tilde{\mu}\theta + 2r_p^T\theta - \theta^2K^2)$, then, given $Y_{h(\theta)}$, twist the original multivariate normal density $\widetilde{Z} \sim MN(\tilde{0}, I)$ to the new multivariate normal density $\widetilde{Z}_{\tilde{a}^T(\theta)} \sim MN(\tilde{a}^T(\theta), I)$.

$$E\left(E\left(\exp\left\{-\theta\left(\sum_{j=1}^{n}\left(b_j\sqrt{\frac{Y}{\nu}}Z_j + \lambda_j Z_j^2\right) - \frac{Y}{\nu}x\right) + \psi(\theta)\right\}\right.\right.$$

$$\left.\left.\times 1_{\{\sum_{j=1}^{n}\lambda_j(\frac{b_j}{\lambda_j}\sqrt{\frac{Y}{\nu}}Z_j+Z_j^2)>\frac{Y}{\nu}x\}}|Y\right)\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\theta\left(\sum_{j=1}^{n}\left(b_j\sqrt{\frac{y}{\nu}}z_j + \lambda_j z_j^2\right) - \frac{y}{\nu}x\right) + \psi(\theta)\right\}\left(\frac{1}{\sqrt{2\pi}}\right)^n$$

$$\times \exp\left\{-\frac{\sum_{j=1}^{n} z_j^2}{2}\right\} 1_{\{\sum_{j=1}^{n} \lambda_j (z_j + \frac{b_j}{2\lambda_j}\sqrt{\frac{y}{\nu}})^2 > \frac{y}{\nu}x + \sum_{j=1}^{n} \frac{b_j^2 y}{4\lambda_j \nu}\}} dz_1, \ldots, dz_n f_y(y)dy$$

$$= \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \sum_{j=1}^{n} \frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\}$$

$$\times \prod_{j=1}^{n} \frac{1}{\sqrt{1+2\lambda_j\theta}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{j=1}^{n} \sqrt{1+2\lambda_j\theta}$$

$$\times \exp\left\{-\sum_{j=1}^{n} \frac{(z_j + \frac{b_j\theta}{1+2\lambda_j\theta}\sqrt{\frac{y}{\nu}})^2}{2\frac{1}{1+2\lambda_j\theta}}\right\}$$

$$\times 1_{\{\sum_{j=1}^{n} \lambda_j (z_j + \frac{b_j}{2\lambda_j}\sqrt{\frac{y}{\nu}})^2 > \frac{y}{\nu}x + \sum_{j=1}^{n} \frac{b_j^2 y}{4\lambda_j \nu}\}} dz_1, \ldots, dz_n f_y(y)dy$$

Let $z_j^* = \frac{z_j + \frac{b_j\theta}{1+2\lambda_j\theta}\sqrt{\frac{y}{\nu}}}{\frac{1}{\sqrt{1+2\lambda_j\theta}}}$, then $dz_j^* = \sqrt{1+2\lambda_j\theta}dz_j$, and we can compute the last equation to be

$$= \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \sum_{j=1}^{n} \frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\} \prod_{j=1}^{n} \frac{1}{\sqrt{1+2\lambda_j\theta}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n$$

$$\times \exp\left\{-\frac{\sum_{j=1}^{n}(z_j^*)^2}{2}\right\}$$

$$\times 1_{\{\sum_{j=1}^{n} \frac{\lambda_j}{1+2\lambda_j\theta}(z_j^* - [\frac{b_j\theta}{\sqrt{1+2\lambda_j\theta}} - \frac{\sqrt{1+2\lambda_j\theta}b_j}{2\lambda_j}]\sqrt{\frac{y}{\nu}})^2 > \frac{y}{\nu}x + \sum_{j=1}^{n} \frac{b_j^2 Y}{4\lambda_j \nu}\}} dz_1^*, \ldots, dz_n^* f_y(y)dy$$

Let $c_j = \frac{\lambda_j}{1+2\lambda_j\theta}$, $d_j = \left[\frac{b_j\theta}{\sqrt{1+2\lambda_j\theta}} - \frac{\sqrt{1+2\lambda_j\theta}b_j}{2\lambda_j}\right]\sqrt{\frac{y}{\nu}}$, and $g = \frac{y}{\nu}x + \sum_{j=1}^{n} \frac{b_j^2 y}{4\lambda_j\nu}$, then, we can obtain

$$= \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \sum_{j=1}^{n} \frac{b_j^2\theta^2 y}{2(1+2\lambda_j\theta)\nu}\right\} \prod_{j=1}^{n} \frac{1}{\sqrt{1+2\lambda_j\theta}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right)^n$$

$$\times \exp\left\{-\frac{\sum_{j=1}^{n}(z_j^*)^2}{2}\right\} 1_{\{\sum_{j=1}^{n} c_j(z_j^* - d_j)^2 > g\}} dz_1^*, \ldots, dz_n^* f_y(y)dy$$

Now, if we let $n = 1$, then the second moment of the estimator is equal to

$$E_\theta(\hat{p}^2) = \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \frac{b_1^2\theta^2 y}{2(1+2\lambda_1\theta)\nu}\right\} \frac{1}{\sqrt{1+2\lambda_1\theta}}$$

$$
\begin{aligned}
&\times\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^{*2}}{2}\right\}1_{\{c_1(z_1^*-d_1)^2>g\}}dz_1^*\right]f_y(y)dy\\
&=\int_{-\infty}^{\infty}\exp\left\{\frac{yx\theta}{\nu}+\psi(\theta)+\frac{b_1^2\theta^2y}{2(1+2\lambda_1\theta)\nu}\right\}\frac{1}{\sqrt{1+2\lambda_1\theta}}\\
&\times\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^{*2}}{2}\right\}1_{\{(z_1^*-d_1)^2>\frac{g}{c_1}\}}dz_1^*\right]f_y(y)dy\\
&=\int_{-\infty}^{\infty}\exp\left\{\frac{yx\theta}{\nu}+\psi(\theta)+\frac{b_1^2\theta^2y}{2(1+2\lambda_1\theta)\nu}\right\}\frac{1}{\sqrt{1+2\lambda_1\theta}}\\
&\times\left[\int_{d_1+\sqrt{\frac{g}{c_1}}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^{*2}}{2}\right\}dz_1^*+\int_{-\infty}^{d_1-\sqrt{\frac{g}{c_1}}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z_1^{*2}}{2}\right\}\right.\\
&\left.\times\, dz_1^*\right]f_y(y)dy,
\end{aligned}
$$

where

$$
\begin{aligned}
d_1+\sqrt{\frac{g}{c_1}}&=\left(\frac{b_1\theta}{\sqrt{1+2\lambda_1\theta}}-\frac{\sqrt{1+2\lambda_1\theta}\,b_1}{2\lambda_1}+\frac{\sqrt{x+\frac{b_1^2}{4\lambda_1}}}{\sqrt{\frac{\lambda_1}{1+2\lambda_1\theta}}}\right)\sqrt{\frac{y}{\nu}}\\
&=h_1(\theta)\sqrt{\frac{y}{\nu}}
\end{aligned}
$$

and

$$
\begin{aligned}
d_1-\sqrt{\frac{g}{c_1}}&=\left(-\frac{b_1\theta}{\sqrt{1+2\lambda_1\theta}}+\frac{\sqrt{1+2\lambda_1\theta}\,b_1}{2\lambda_1}+\frac{\sqrt{x+\frac{b_1^2}{4\lambda_1}}}{\sqrt{\frac{\lambda_1}{1+2\lambda_1\theta}}}\right)\sqrt{\frac{y}{\nu}}\\
&=h_2(\theta)\sqrt{\frac{y}{\nu}}.
\end{aligned}
$$

If $d_1-\sqrt{\frac{g}{c_1}}\ll 0$, then the second moment has a bound as follows

$$
\begin{aligned}
E_\theta(\hat{p}^2)\le&\int_{-\infty}^{\infty}\exp\left\{\frac{yx\theta}{\nu}+\psi(\theta)+\frac{b_1^2\theta^2y}{2(1+2\lambda_1\theta)\nu}\right\}\frac{1}{\sqrt{1+2\lambda_1\theta}}\left[\frac{1}{\sqrt{2\pi}(d_1+\sqrt{\frac{g}{c_1}})}\right.\\
&\left.\times\exp\left\{-\frac{(d_1+\sqrt{\frac{g}{c_1}})^2}{2}\right\}+\frac{1}{\sqrt{2\pi}(-d_1+\sqrt{\frac{g}{c_1}})}\exp\left\{-\frac{(-d_1+\sqrt{\frac{g}{c_1}})^2}{2}\right\}\right]\\
&\times f_y(y)dy
\end{aligned}
$$

$$
\begin{aligned}
E_\theta(\hat{p}^2) \le & \int_{-\infty}^{\infty} \exp\left\{\frac{yx\theta}{\nu} + \psi(\theta) + \frac{b_1^2\theta^2 y}{2(1+2\lambda_1\theta)\nu}\right\} \frac{1}{\sqrt{1+2\lambda_1\theta}} \left[\frac{1}{\sqrt{2\pi}(h_1(\theta)\sqrt{\frac{y}{\nu}})}\right. \\
& \times \exp\left\{-\frac{(h_1(\theta))^2\frac{y}{\nu}}{2}\right\} + \frac{1}{\sqrt{2\pi}(h_2(\theta)\sqrt{\frac{y}{\nu}})} \exp\left\{-\frac{(h_1(\theta))^2\frac{y}{\nu}}{2}\right\}\Bigg] \\
& \times \frac{1}{\Gamma(\frac{\nu}{2})\beta^{\nu/2}} y^{\frac{\nu}{2}-1} \exp\left\{-\frac{y}{2}\right\} dy \\
= & \frac{\sqrt{\nu}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu}{2})\beta^{\nu/2}} y^{(\frac{\nu-1}{2}-1)} \\
& \times \exp\left\{-(\tfrac{1}{2} + \tfrac{h_1(\theta)^2}{2\nu} - \tfrac{\theta x}{\nu} - \tfrac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu})y\right\} dy \\
& + \frac{\sqrt{\nu}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu}{2})\beta^{\nu/2}} y^{(\frac{\nu-1}{2}-1)} \\
& \times \exp\left\{-(\tfrac{1}{2} + \tfrac{h_2(\theta)^2}{2\nu} - \tfrac{\theta x}{\nu} - \tfrac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu})y\right\} dy \\
= & I_1(\theta) + I_2(\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
I_1(\theta) &= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu}{2})\beta^{\nu/2}} y^{(\frac{\nu-1}{2}-1)} \\
&\quad \times \exp\left\{-(\tfrac{1}{2} + \tfrac{h_1(\theta)^2}{2\nu} - \tfrac{\theta x}{\nu} - \tfrac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu})y\right\} dy, \\
I_2(\theta) &= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu}{2})\beta^{\nu/2}} y^{(\frac{\nu-1}{2}-1)} \\
&\quad \exp\left\{-(\tfrac{1}{2} + \tfrac{h_2(\theta)^2}{2\nu} - \tfrac{\theta x}{\nu} - \tfrac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu})y\right\} dy, \\
\beta_1(\theta) &= \frac{1}{\frac{1}{2} + \frac{h_1(\theta)^2}{2\nu} - \frac{\theta x}{\nu} - \frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu}}, \\
\beta_2(\theta) &= \frac{1}{\frac{1}{2} + \frac{h_2(\theta)^2}{2\nu} - \frac{\theta x}{\nu} - \frac{b_1^2\theta^2}{2(1+2\lambda_1\theta)\nu}}.
\end{aligned}
$$

Rewrite $I_1(\theta)$ and $I_2(\theta)$

$$
\begin{aligned}
I_1(\theta) &= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}\Gamma(\frac{\nu-1}{2})\beta_1(\theta)^{\frac{\nu-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)\Gamma(\frac{\nu}{2})\beta^{\nu/2}} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu-1}{2})\beta_1(\theta)^{\frac{\nu-1}{2}}} y^{(\frac{\nu-1}{2}-1)} \exp\left\{-\frac{y}{\beta_1(\theta)}\right\} dy \\
&= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}\Gamma(\frac{\nu-1}{2})\beta_1(\theta)^{\frac{\nu-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_1(\theta)\Gamma(\frac{\nu}{2})\beta^{\nu/2}},
\end{aligned}
$$

$$
\begin{aligned}
I_2(\theta) &= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}\Gamma(\frac{\nu-1}{2})\beta_2(\theta)^{\frac{\nu-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)\Gamma(\frac{\nu}{2})\beta^{\nu/2}} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\frac{\nu-1}{2})\beta_2(\theta)^{\frac{\nu-1}{2}}} y^{(\frac{\nu-1}{2}-1)} \exp\left\{-\frac{y}{\beta_2(\theta)}\right\} dy \\
&= \frac{\sqrt{\nu}\exp\{\psi(\theta)\}\Gamma(\frac{\nu-1}{2})\beta_2(\theta)^{\frac{\nu-1}{2}}}{\sqrt{1+2\lambda_1\theta}\sqrt{2\pi}h_2(\theta)\Gamma(\frac{\nu}{2})\beta^{\nu/2}}.
\end{aligned}
$$

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